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On meager function spaces, network character and meager convergence in topological spaces

TARAS BANAKH, VOLODYMYR MYKHAYLYUK, LYUBOMYR ZDOMSKYY

Abstract. For a non-isolated point x of a topological space X let $\text{nw}_\chi(x)$ be the smallest cardinality of a family \mathcal{N} of infinite subsets of X such that each neighborhood $O(x) \subset X$ of x contains a set $N \in \mathcal{N}$. We prove that

- each infinite compact Hausdorff space X contains a non-isolated point x with $\text{nw}_\chi(x) = \aleph_0$;
- for each point $x \in X$ with $\text{nw}_\chi(x) = \aleph_0$ there is an injective sequence $(x_n)_{n \in \omega}$ in X that \mathcal{F} -converges to x for some meager filter \mathcal{F} on ω ;
- if a functionally Hausdorff space X contains an \mathcal{F} -convergent injective sequence for some meager filter \mathcal{F} , then for every path-connected space Y that contains two non-empty open sets with disjoint closures, the function space $C_p(X, Y)$ is meager.

Also we investigate properties of filters \mathcal{F} admitting an injective \mathcal{F} -convergent sequence in $\beta\omega$.

Keywords: network character, meager convergent sequence, meager filter, meager space, function space

Classification: Primary 54A20, 54C35; Secondary 54E52

This paper was motivated by a question of the second author who asked if the function space $C_p(\omega^*, 2)$ is meager. Here $\omega^* = \beta\omega \setminus \omega$ is the remainder of the Stone-Čech compactification of the discrete space of finite ordinals ω and $2 = \{0, 1\}$ is the doubleton endowed with the discrete topology. According to Theorem 4.1 of [13] this question is closely related to the so-called meager convergence of sequences in ω^* .

A filter \mathcal{F} on ω is *meager* if it is meager (i.e., of the first Baire category) in the power-set $\mathcal{P}(\omega) = 2^\omega$ endowed with the usual compact metrizable topology. The simplest example of a meager filter is the Fréchet filter $\mathfrak{F}r = \{A \subset \omega : \omega \setminus A \text{ is finite}\}$ of all cofinite subsets of ω . By the Talagrand characterization [18], a free filter \mathcal{F} on ω is meager if and only if $\xi(\mathcal{F}) = \mathfrak{F}r$ for some finite-to-one function $\xi : \omega \rightarrow \omega$. A function $\xi : \omega \rightarrow \omega$ is *finite-to-one* if for each point $y \in \omega$ the preimage $\xi^{-1}(y)$ is finite and non-empty. A filter \mathcal{F} on ω is defined to be ξ -*meager* for a surjective function $\xi : \omega \rightarrow \omega$ if $\xi(\mathcal{F}) = \mathfrak{F}r$.

We shall say that for a filter \mathcal{F} on ω , a sequence $(x_n)_{n \in \omega}$ of points of a topological space X \mathcal{F} -converges to a point $x_\infty \in X$ if for each neighborhood

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$O(x_\infty) \subseteq X$ of x_∞ the set $\{n \in \omega : x_n \in O(x_\infty)\}$ belongs to the filter \mathcal{F} . Observe that the usual convergence of sequences coincides with the $\mathfrak{F}r$ -convergence for the Fréchet filter $\mathfrak{F}r$. The filter convergence of sequences has been actively studied both in Analysis [1], [4] and Topology [5]. A sequence $(x_n)_{n \in \omega}$ will be called *meager-convergent* if it is \mathcal{F} -convergent for some meager filter \mathcal{F} on ω . A sequence $(x_n)_{n \in \omega}$ is called *injective* if $x_n \neq x_m$ for all $n \neq m$.

We shall prove that for a zero-dimensional Hausdorff space X the function space $C_p(X, \mathbb{I})$ is meager if X contains an injective meager-convergent sequence. We recall that a topological space X is *functionally Hausdorff* if for any distinct points $x, y \in X$ there is a continuous function $\lambda : X \rightarrow \mathbb{I}$ such that $\lambda(x) \neq \lambda(y)$. Here $\mathbb{I} = [0, 1]$ is the unit interval. For topological spaces X, Y by $C_p(X, Y)$ we denote the space of continuous functions endowed with the topology of pointwise convergence.

Theorem 1. *Let X be a functionally Hausdorff space and let Y be a topological space that contains two open non-empty subsets with disjoint closures. Assume that X is zero-dimensional or Y is path-connected. If X contains an injective meager-convergent sequence, then the function space $C_p(X, Y)$ is meager.*

PROOF: Let $(x_n)_{n \in \omega}$ be a sequence in X that \mathcal{F} -converges to $x_\infty \in X$ for some meager filter \mathcal{F} in ω . Then there is a finite-to-one surjection $\xi : \omega \rightarrow \omega$ such that $\xi(\mathcal{F}) = \mathfrak{F}r$. By our assumption, Y contains two non-empty open subsets W_0, W_1 with disjoint closures. For every $n \in \omega$ consider the subset $\mathcal{C}_n = \{f \in C_p(X, Y) : \forall i \in \{0, 1\} (f(x_\infty) \notin \overline{W}_i \Rightarrow \forall m \geq n \exists k \in \xi^{-1}(m) (f(x_k) \notin \overline{W}_i))\}$.

The fact that $C_p(X, Y)$ is meager will follow as soon as we check that $C_p(X, Y) = \bigcup_{n \in \omega} \mathcal{C}_n$ and each set \mathcal{C}_n is nowhere dense in $C_p(X, Y)$.

To show that $C_p(X, Y) = \bigcup_{n \in \omega} \mathcal{C}_n$, fix any continuous function $f \in C_p(X, Y)$. Since $Y = (Y \setminus \overline{W}_0) \cup (Y \setminus \overline{W}_1)$, there is $i \in \{0, 1\}$ such that $f(x_\infty) \notin \overline{W}_i$. Since (x_n) is \mathcal{F} -convergent to x_∞ and $f^{-1}(Y \setminus \overline{W}_i)$ is an open neighborhood of x_∞ , the set $F = \{n \in \omega : f(x_n) \notin \overline{W}_i\}$ belongs to the filter \mathcal{F} and thus the image $\xi(F)$, being cofinite in ω , contains the set $\{m \in \omega : m \geq n\}$ for some $n \in \omega$. Then $f \in \mathcal{C}_n$ by the definition of the set \mathcal{C}_n .

Next, we show that each set \mathcal{C}_n is nowhere dense in $C_p(X, Y)$. Fix any non-empty open set $\mathcal{U} \subseteq C_p(X, Y)$. Without loss of generality, \mathcal{U} is a basic open set of the following form:

$$\mathcal{U} = \{f \in C_p(X, Y) : \forall z \in Z f(z) \in U_z\}$$

for some finite set $Z \subseteq X$ and non-empty open sets $U_z \subseteq Y, z \in Z$. We can additionally assume that $x_\infty \in Z$. We need to find a non-empty open set $\mathcal{V} \subseteq C_p(X, Y)$ such that $\mathcal{V} \subseteq \mathcal{U} \setminus \mathcal{C}_n$. If $\mathcal{U} \cap \mathcal{C}_n$ is empty, then put $\mathcal{V} = \mathcal{U}$. So we assume that $\mathcal{U} \cap \mathcal{C}_n$ contains some function f_0 . For this function we can find $i \in \{0, 1\}$ such that $f_0(x_\infty) \notin \overline{W}_i$. Since $f_0(x_\infty) \in U_{x_\infty}$, we lose no generality assuming that $U_{x_\infty} \subseteq Y \setminus \overline{W}_i$.

Since the sequence $(x_n)_{n \in \omega}$ is injective, we can find $m \geq n$ such that the set $X_m = \{x_k : k \in \xi^{-1}(m)\}$ does not intersect the finite set Z . Choose any function $g : Z \cup X_m \rightarrow Y$ such that $g(z) = f_0(z)$ for all $z \in Z$ and $g(x) \in W_{1-i}$ for all $x \in X_m$.

We claim that the function g has a continuous extension $\bar{g} : X \rightarrow Y$. By our assumption, X is zero-dimensional or Y path-connected. In the first case we can find a retraction $r : X \rightarrow Z \cup X_m$ and put $\bar{g} = g \circ r$. If Y is path-connected, then take any injective function $\phi : g(Z \cup X_m) \rightarrow \mathbb{I}$ and extend the function $\phi \circ g : Z \cup X_m \rightarrow \mathbb{I}$ to a continuous map $\lambda : X \rightarrow \mathbb{I}$ using the functional Hausdorff property of X . Since Y is path-connected, the map $\phi^{-1} : (\phi \circ g)(Z \cup X_m) \rightarrow Y$ extends to a continuous map $\psi : \mathbb{I} \rightarrow Y$. Then the continuous map $\bar{g} = \psi \circ \lambda : X \rightarrow Y$ is a required continuous extension of g .

In both cases the set

$$\mathcal{V} = \{f \in C_p(X, Y) : \forall z \in Z f(z) \in U_z, \text{ and } \forall x \in X_m f(x) \in W_{1-i}\}$$

is an open neighborhood of \bar{g} that lies in $\mathcal{U} \setminus \mathcal{C}_n$, witnessing that the set \mathcal{C}_n is nowhere dense in $C_p(X, Y)$. □

Theorem 1 motivates the problem of detecting topological spaces that contain injective meager-convergent sequences. This will be done for spaces containing points with countable network character.

A family \mathcal{N} of subsets of a topological space X is called a π -network at a point $x \in X$ if each neighborhood $O(x) \subset X$ of x contains some set $N \in \mathcal{N}$. If each set $N \in \mathcal{N}$ is infinite, then \mathcal{N} will be called an i -network at x . An i -network at x exists if and only if each neighborhood of x in X is infinite. In this case let $\text{nw}_\chi(x; X)$ denote the smallest cardinality $|\mathcal{N}|$ of an i -network \mathcal{N} at x . If some neighborhood of x in X is finite, then let $\text{nw}_\chi(x; X) = 1$. If the space X is clear from the context, then we write $\text{nw}_\chi(x)$ instead of $\text{nw}_\chi(x; X)$ and call this cardinal the *network character* of x in X . If X is a T_1 -space, then $\text{nw}_\chi(x) \geq \aleph_0$ if and only if the point x is not isolated in X . The cardinal $\text{hnw}_\chi(x) = \sup\{\text{nw}_\chi(x; A) : x \in A \subset X\}$ is called the *hereditary network character* at x . Points $x \in X$ with $\text{hnw}_\chi(x) \leq \aleph_0$ are called *Pytkeev points*, see [11].

Theorem 2. *If some point x of a topological space X has $\text{nw}_\chi(x) = \aleph_0$, then for each finite-to-one function $\xi : \omega \rightarrow \omega$ with $\lim_{n \rightarrow \infty} |\xi^{-1}(n)| = \infty$ there is an injective sequence $(x_n)_{n \in \omega}$ in X that \mathcal{F} -converges to x for some ξ -meager filter \mathcal{F} .*

PROOF: Let $(N_i)_{i \in \omega}$ be a countable i -network at x . Since each set N_i is infinite, we can choose an injective sequence $(x_k)_{k \in \omega}$ in X such that for every $n \in \omega$ and $0 \leq i < |\xi^{-1}(n)|$ the set N_i meets the set $\{x_k : k \in \xi^{-1}(n)\}$.

It is clear that the sequence $(x_n)_{n \in \omega}$ \mathcal{F} -converges to x for the filter

$$\mathcal{F} = \{\{n \in \omega : x_n \in O(x)\} : O(x) \text{ is a neighborhood of } x \text{ in } X\}.$$

It remains to check that the filter \mathcal{F} is ξ -meager. Given any neighborhood $O(x) \subset X$ of x we need to find $n \in \omega$ such that for every $m \geq n$ there is $k \in \xi^{-1}(m)$

with $x_k \in O(x)$. Since $(N_i)_{i \in \omega}$ is a network at x , there is $i \in \omega$ such that $N_i \subset O(x)$. Taking into account that $\lim_{n \rightarrow \infty} |\xi^{-1}(n)| = \infty$, find $n \in \omega$ such that $|\xi^{-1}(m)| > i$ for all $m \geq n$. Now the choice of the sequence (x_k) guarantees that for every $m \geq n$ there is $k \in \xi^{-1}(m)$ with $x_k \in N_i \subset O(x)$. \square

Theorem 2 shows that it is important to detect points x with countable network character $\text{nw}_\chi(x)$. Let us recall that the *character* $\chi(x)$ (resp. the π -*character* $\pi\chi(x)$) of a point x in a topological space X is equal to the smallest cardinality of a neighborhood base (resp. a π -base) at x . A π -*base* at x is any π -network at x consisting of non-empty open subsets of X . These definitions imply the following simple:

Proposition 3. *For any non-isolated point x of a T_1 -space X ,*

- (1) $\text{nw}_\chi(x) \leq \chi(x)$;
- (2) $\text{nw}_\chi(x) \leq \pi\chi(x)$ provided that x has a neighborhood containing no isolated point of X ;
- (3) $\text{nw}_\chi(x) = \aleph_0$ if x is the limit of an injective \mathfrak{R} -convergent sequence in X .

The following simple example shows that the usual convergence of the injective sequence in Proposition 3(3) cannot be replaced by the meager convergence. It also shows that Theorem 2 cannot be reversed.

Example 4. Let \mathcal{F} be the meager filter on ω consisting of the sets $F \subset \omega$ such that

$$\lim_{n \rightarrow \infty} \frac{|F \cap [2^n, 2^{n+1})|}{2^n} = 1.$$

On the space $X = \omega \cup \{\infty\}$ consider the topology in which all points $n \in \omega$ are isolated while the sets $F \cup \{\infty\}$, $F \in \mathcal{F}$, are neighborhoods of ∞ . It is clear that the sequence $x_n = n$, $n \in \omega$, \mathcal{F} -converges to ∞ in X . On the other hand, a simple diagonal argument shows that $\text{nw}_\chi(\infty; X) > \aleph_0$.

Theorem 5. *Each infinite compact Hausdorff space X contains a point $x \in X$ with $\text{nw}_\chi(x) = \aleph_0$.*

PROOF: Theorem trivially holds if X contains a non-trivial convergent sequence. So we assume that X contains no non-trivial convergent sequence. Then X contains a closed subset $C \subset X$ that admits a continuous map $g : C \rightarrow \mathbb{I}$ onto the unit interval $\mathbb{I} = [0, 1]$, see [7, p.172]. Replacing C by a smaller subset, we can assume that the map $g : C \rightarrow \mathbb{I}$ is irreducible, which means that $g(C') \neq \mathbb{I}$ for any proper closed subset $C' \subset C$. Fix any countable base \mathcal{B} of the topology of \mathbb{I} . The irreducibility of the map $g : C \rightarrow \mathbb{I}$ implies that the space C has no isolated points. Also the irreducibility of g implies that the countable family $\mathcal{N} = \{g^{-1}(U) : U \in \mathcal{B}\}$ of open infinite subsets of C is an i -network at each point $x \in C$. Consequently, $\text{nw}_\chi(x) = \aleph_0$ for each point $x \in C$. \square

Theorems 1–5 imply:

Corollary 6. *For each infinite zero-dimensional compact Hausdorff space X and each topological space Y containing two non-empty open sets with disjoint closures the function space $C_p(X, Y)$ is meager. In particular, the function space $C_p(\omega^*, 2)$ is meager.*

Also Theorems 2 and 5 imply

Corollary 7. *Let $\xi : \omega \rightarrow \omega$ be a finite-to-one function with $\lim_{n \rightarrow \infty} |\xi^{-1}(n)| = \infty$. Each infinite compact Hausdorff space X contains an injective \mathcal{F} -convergent sequence for some ξ -meager filter \mathcal{F} on ω .*

In fact, the condition $\lim_{n \rightarrow \infty} |\xi^{-1}(n)| = \infty$ in Corollary 7 cannot be weakened.

Let us recall that an infinite subset A is called a *pseudointersection* of a family of sets \mathcal{F} if $A \subseteq^* F$ for all $F \in \mathcal{F}$ where $A \subseteq^* F$ means that $A \setminus F$ is finite. If a sequence $(x_n)_{n \in \omega}$ in a topological space \mathcal{F} -converges to a point x_∞ for some filter \mathcal{F} with infinite pseudointersection $A \subseteq \omega$, then the subsequence $(x_k)_{k \in A}$ converges to x_∞ in the standard sense.

Lemma 8. *Let I be a countable set and $C = \bigcup_{i \in I} C_i$, where the sets C_i are nonempty and mutually disjoint, and $\sup_{i \in I} |C_i| < \omega$. If \mathcal{H} is a filter on C all of whose elements intersect all but finitely many C_i 's, then \mathcal{H} has an infinite pseudointersection.*

PROOF: The proposition will be proved by induction on $n = \sup_{i \in I} |C_i|$. In case $n = 1$ there is nothing to prove. Suppose that it is true for all $k < n$ and let $I, \{C_i : i \in I\}, \mathcal{H}$ be as above with $\max\{|C_i| : i \in I\} = n$. If for every $H \in \mathcal{H}$ the set $\{i \in I : |C_i \cap H| < n\}$ is finite, then C itself is a pseudointersection of \mathcal{H} . So suppose that $J = \{i \in I : |C_i \cap H_0| < n\}$ is infinite for some $H_0 \in \mathcal{H}$. In this case we may use our inductive hypothesis for $J, \{C_i \cap H_0 : i \in J\}, \mathcal{G} = \mathcal{H} \upharpoonright (\bigcup_{i \in J} C_i \cap H_0)$, and $n - 1$. Thus \mathcal{G} has an infinite pseudointersection, and hence so does \mathcal{H} . □

Proposition 9. *If \mathcal{F} is a ξ -meager filter on ω for some surjective function $\xi : \omega \rightarrow \omega$ with $\underline{\lim}_{n \rightarrow \infty} |\xi^{-1}(n)| < \infty$, then any sequence $(x_n)_{n \in \omega}$ in a topological space X that \mathcal{F} -converges to a point $x_\infty \in X$ contains a subsequence $(x_{n_k})_{k \in \omega}$ that converges to x_∞ .*

PROOF: Choose an infinite set $I \subseteq \omega$ such that $\sup_{i \in I} |\xi^{-1}(i)| < \omega$. Let $C_i = \xi^{-1}(i)$ for every $i \in I, C = \bigcup_{i \in I} C_i$ and $\mathcal{H} = \{F \cap C : F \in \mathcal{F}\}$. According to Lemma 8 there exists an infinite set $D \subseteq C$ such that $D \subseteq^* H$ for every $H \in \mathcal{H}$. Then the subsequence $(x_i)_{i \in D}$ converges to x_∞ . □

Now let us compare two facts:

- (1) the compact Hausdorff space $\beta\omega$ contains no injective \mathfrak{F}_r -convergent sequences;
- (2) each infinite compact Hausdorff space X contains an injective \mathcal{F} -convergent sequence for some meager filter \mathcal{F} .

These two facts suggest a problem of finding the borderline between filters \mathcal{F} that admit an injective \mathcal{F} -convergent sequence in $\beta\omega$ and filters that admit no such sequences. We hope that this borderline passes near analytic filters. Let us recall the definitions of some properties of filters.

A filter \mathcal{F} is *analytic* (resp. an F_σ -filter, $F_{\sigma\delta}$ -filter) if \mathcal{F} is an analytic subset (resp. F_σ -subset, $F_{\sigma\delta}$ -subset) of the power-set $\mathcal{P}(\omega) = 2^\omega$ endowed with the natural compact metrizable topology.

A filter \mathcal{F} is *measurable* (resp. *null*) if it is measurable (resp. has measure zero) with respect to the Haar measure on the Cantor cube 2^ω considered as the countable product of 2-element groups. It is well-known that a filter is measurable if and only if it is null. The relations between meager and null filters are not trivial and were investigated in [18] and [2]. Since each analytic filter is meager and null, we get the following chain of properties of filters:

$$F_\sigma \Rightarrow \text{analytic} \Rightarrow \text{meager \& null.}$$

We are going to show that some meager and null filter \mathcal{F} admits an injective \mathcal{F} -convergent sequence in $\beta\omega$ while no F_σ -filter \mathcal{F} admits such a sequence. The latter fact holds more generally for analytic P^+ -filters.

A filter \mathcal{F} on ω is called a P -filter (resp. a P^+ -filter) if each countable subfamily $\mathcal{C} \subset \mathcal{F}$ has a pseudointersection A that belongs to \mathcal{F} (resp. to \mathcal{F}^+). Here

$$\mathcal{F}^+ = \{A \subset \omega : \forall F \in \mathcal{F} \ A \cap F \neq \emptyset\}$$

coincides with the union of all filters that contain \mathcal{F} . It is clear that each P -filter is a P^+ -filter. In particular, the Fréchet filter \mathcal{F} is both a P -filter and P^+ -filter.

For a filter \mathcal{F} on ω by $\chi(\mathcal{F})$ we denote its *character*. It is equal to the smallest cardinality $|\mathcal{B}|$ of the base $\mathcal{B} \subset \mathcal{F}$ that generates \mathcal{F} in the sense that $\mathcal{F} = \{F \subset \omega : \exists B \in \mathcal{B} \ B \subset F\}$. It is well-known that the character of each free ultrafilter on ω is uncountable. The uncountable cardinal $\mathfrak{u} = \min\{\chi(\mathcal{U}) : \mathcal{U} \in \beta\omega \setminus \omega\}$ is called the *ultrafilter number*, see [3], [20]. The *dominating number* \mathfrak{d} is the smallest cardinality $|D|$ of a cofinal subset D in the partially ordered set (ω^ω, \leq) , see [3], [20]. By Ketonen's Theorem [10], *each filter \mathcal{F} on ω with character $\chi(\mathcal{F}) < \mathfrak{d}$ is a P^+ -filter.*

Now we can establish some properties of filters \mathcal{F} admitting injective \mathcal{F} -convergent sequences in $\beta\omega$.

Theorem 10. *Assume that a filter \mathcal{F} admits an injective \mathcal{F} -convergent sequence $(x_n)_{n \in \omega}$ in $\beta\omega$.*

- (1) *If \mathcal{F} is a P^+ -filter, then for some set $A \in \mathcal{F}^+$ the filter $\mathcal{F}|A = \{F \cap A : F \in \mathcal{F}\}$ on A is an ultrafilter.*
- (2) *$\chi(\mathcal{F}) \geq \min\{\mathfrak{d}, \mathfrak{u}\}$;*
- (3) *\mathcal{F} is not an analytic P^+ -filter;*
- (4) *\mathcal{F} is not an F_σ -filter.*

PROOF: 1. Assume that \mathcal{F} is a P^+ -filter. Let x_∞ be the \mathcal{F} -limit of the \mathcal{F} -convergent sequence $(x_n)_{n \in \omega}$ in $\beta\omega$. Since the sequence (x_n) is injective, there is $m \in \omega$ such that for every $n \geq m$ $x_n \neq x_\infty$ and hence we can fix a neighborhood U_n of x_∞ whose closure does not contain the point x_n . Since the sequence (x_k) \mathcal{F} -converges to x_∞ , for every $n \geq m$ the set $F_n = \{k \in \omega : x_k \in U_n\}$ belongs to the filter \mathcal{F} . Since \mathcal{F} is a P^+ -filter, the sequence $(F_n)_{n \geq m}$ has a pseudointersection $A \in \mathcal{F}^+$. It follows from the choice of the neighborhoods U_n that the set $\{x_n\}_{n \in A}$ is discrete in $\beta\omega$ and the sequence $(x_n)_{n \in A}$ is $\mathcal{F}|A$ -convergent to x_∞ . By Rudin's Theorem [16], the map $f : A \rightarrow \beta\omega, f : n \mapsto x_n$, has injective Stone-Ćech extension $\beta f : \beta A \rightarrow \beta\omega$, which implies that the filter $\mathcal{F}|A$ is an ultrafilter.

2. If $\chi(\mathcal{F}) < \min\{\mathfrak{d}, \mathfrak{u}\}$, then $\chi(\mathcal{F}) < \mathfrak{d}$ and by the Ketonen's Theorem [10] \mathcal{F} is a P^+ -filter. By the preceding statement, $\mathcal{F}|A$ is an ultrafilter for some set $A \in \mathcal{F}^+$. Consequently,

$$\mathfrak{u} \leq \chi(\mathcal{F}|A) \leq \chi(\mathcal{F}) < \mathfrak{u}$$

and this is a desired contradiction.

3. If \mathcal{F} is an analytic P^+ -filter, then by the first statement, $\mathcal{F}|A$ is an ultrafilter for some subset $A \in \mathcal{F}^+$. On the other hand, the filter $\mathcal{F}|A$ is analytic being a continuous image of the analytic filter \mathcal{F} . So, $\mathcal{F}|A$ cannot be an ultrafilter.

4. Assume that \mathcal{F} is an F_σ -filter. In order to apply the preceding statement, it suffices to show that \mathcal{F} is a P^+ -filter. This is done in the following lemma. \square

Lemma 11. *Each F_σ -filter \mathcal{F} on ω is a P^+ -filter.*

PROOF: According to a result of Mazur [12] (see also [17]), for the F_σ -filter \mathcal{F} there exists a lower semi-continuous submeasure ϕ on $\mathcal{P}(\omega)$ such that $\mathcal{F} = \{A \subset \omega : \phi(\omega \setminus A) < \infty\}$. Since $\mathcal{F} \neq \mathcal{P}(\omega)$, $\phi(\omega) = \infty$ and the subadditivity of ϕ implies that $\phi(F) = \infty$ for all $F \in \mathcal{F}$. It follows from $\mathcal{F} = \{A \subset \omega : \phi(\omega \setminus A) < \infty\}$ that a set $A \subset \omega$ belongs to \mathcal{F}^+ if and only if $\phi(A) = \infty$.

To show that \mathcal{F} is a P^+ -filter, fix any decreasing sequence of sets $(A_k)_{k \in \omega}$ in \mathcal{F} . Let $n_0 = 0$ and by induction construct an increasing sequence of positive integers $(n_k)_{k \in \omega}$ such that $\phi([n_k, n_{k+1}) \cap A_k) > k$ for every $k \in \omega$. Then the set $A = \bigcup_{k \in \omega} [n_k, n_{k+1}) \cap A_k$ is a pseudointersection of $(A_k)_{k \in \omega}$ and belongs to the family \mathcal{F}^+ as $\phi(A) = \infty$. \square

Let us remark that Lemma 11 cannot be generalized to $F_{\sigma\delta}$ -filters. The following example was suggested to the authors by Jonathan Verner.

Example 12. The filter

$$\mathfrak{F}r \otimes \mathfrak{F}r = \{A \subset \omega \times \omega : \{n \in \omega : \{m \in \omega : (n, m) \in A\} \in \mathfrak{F}r\} \in \mathfrak{F}r\}$$

on $\omega \times \omega$ is an $F_{\sigma\delta}$ but not P^+ .

Looking at Theorem 10, it is natural to ask the following

Question 13. *Does $\beta\omega$ contain an injective \mathcal{F} -convergent sequence for some analytic filter \mathcal{F} ?*

On the other hand, we have the following fact:

Theorem 14. *Each infinite compact Hausdorff space X contains an injective \mathcal{F} -convergent sequence for some meager and null filter \mathcal{F} .*

PROOF: Choose any finite-to-one function $\xi : \omega \rightarrow \omega$ such that

$$\lim_{n \rightarrow \infty} |\xi^{-1}(n)| = \infty \quad \text{and} \quad \prod_{n \in \omega} (1 - 2^{-|\xi^{-1}(n)|}) = 0.$$

By Corollary 7, any infinite compact Hausdorff space X contains an injective \mathcal{F} -convergent sequence for some ξ -meager filter \mathcal{F} . It is clear that \mathcal{F} is meager. It remains to check that \mathcal{F} is null. The filter \mathcal{F} , being ξ -meager, lies in the union $\bigcup_{n \in \omega} \mathcal{F}_n$ where $\mathcal{F}_n = \{A \subset \omega : \forall k \geq n \ A \cap \xi^{-1}(k) \neq \emptyset\}$. It suffices to prove that each set \mathcal{F}_n has Haar measure zero. Observe that the set \mathcal{F}_n can be identified with the product $\prod_{k \geq n} (\mathcal{P}(\varphi^{-1}(k)) \setminus \{\emptyset\})$, which has Haar measure

$$\prod_{k \geq n} \frac{2^{|\varphi^{-1}(k)|} - 1}{2^{|\varphi^{-1}(k)|}} = \prod_{k \geq n} (1 - 2^{-|\varphi^{-1}(k)|}) = 0.$$

□

Remark 15. After writing this paper the authors learned from V. Tkachuk that the meager property of the function space $C_p(\omega^*, 2)$ was also established by E.G. Pytkeev in his Dissertation [15, 3.24]. Game characterizations of topological spaces X with Baire function space $C_p(X, \mathbb{R})$ were given in [9], [19] and [14].

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