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BOUNDEDNESS AND COMPACTNESS OF THE EMBEDDING BETWEEN SPACES WITH MULTIWEIGHTED DERIVATIVES WHEN $1 \leqslant q < p < \infty$

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Abstract. We consider a new Sobolev type function space called the space with multiweighted derivatives $W^n_{p,\overline{\alpha}}$, where $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, and $\|f\|_{W^n_{p,\overline{\alpha}}} = \|D^n_{\overline{\alpha}} f\|_p + \sum\limits_{i=0}^{n-1} |D^i_{\overline{\alpha}} f(1)|$,

$$D^0_{\overline{\alpha}} f(t) = t^{\alpha_0} f(t), \quad D^i_{\overline{\alpha}} f(t) = t^{\alpha_i} \frac{\mathrm{d}}{\mathrm{d}t} D^{i-1}_{\overline{\alpha}} f(t), \ \ i = 1, 2, \ldots, n.$$

We establish necessary and sufficient conditions for the boundedness and compactness of the embedding $W^n_{p,\overline{\alpha}} \hookrightarrow W^m_{q,\overline{\beta}}$, when $1 \leqslant q < p < \infty$, $0 \leqslant m < n$.

Keywords: weighted function space, multiweighted derivative, embedding theorems, compactness.

MSC 2010: 46E35, 46E30

1. Introduction

Let m and n be natural numbers, $\mathbb{R}$ be the set of real numbers, $1 \leqslant p, q < \infty$, $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $|\overline{\alpha}| = \sum\limits_{i=0}^{n} \alpha_i$, $I = (0, 1)$ or $I = (1, +\infty)$ and $1/p + 1/p' = 1$.

Let $f\colon I \to \mathbb{R}$. We define the differential operations $D^i_{\overline{\alpha}} f$ of order i, $0 \leqslant i \leqslant n$, as follows:

$$D^0_{\overline{\alpha}} f(t) = t^{\alpha_0} f(t), \quad D^i_{\overline{\alpha}} f(t) = t^{\alpha_i} \frac{\mathrm{d}}{\mathrm{d}t} D^{i-1}_{\overline{\alpha}} f(t), \ \ i = 1, 2, \ldots, n,$$

where each derivative is defined in the generalized sense (see e.g. [6]). The operation $D^i_{\overline{\alpha}} f$ is called the $\overline{\alpha}$-multiweighted derivative of the function f of order i, $i = 0, 1, \dots, n$.

Let $W^n_{p,\overline{\alpha}} = W^n_{p,\overline{\alpha}}(I)$ be the space of functions $f\colon I \to \mathbb{R}$, which has $\overline{\alpha}$-multiweighted nth order derivatives on the interval I and for which the following norm is finite:

$$\|f\|_{W^n_{p,\overline{\alpha}}} = \|D^n_{\overline{\alpha}} f\|_p + \sum_{i=0}^{n-1} |D^i_{\overline{\alpha}} f(1)|,$$

where $\|\cdot\|_p$ is the usual norm of the space $L_p(I)$, $1 \leqslant p < \infty$.

When $\alpha_i = 0$, $i = 0, 1, \dots, n-1$, and $\alpha_n = \gamma$ the space $W^n_{p,\overline{\alpha}}$ coincides with the usual Kudryavtsev space $L^n_{p,\gamma} = L^n_{p,\gamma}(I)$ with the finite norm $\|f\|_{L^n_{p,\gamma}} = \|t^\gamma f^{(n)}\|_p + \sum\limits_{i=0}^{n-1} |f^{(i)}(1)|$ (see [5]).

Besides $W^n_{p,\overline{\alpha}}$, we will consider the space $W^m_{q,\overline{\beta}}$ and our aim is to obtain necessary and sufficient conditions for boundedness and compactness of the embedding

$$W^n_{p,\overline{\alpha}} \hookrightarrow W^m_{q,\overline{\beta}} \tag{1.1}$$

when $1 \leqslant q < p < \infty$, $\overline{\beta} = (\beta_0, \beta_1, \dots, \beta_m)$, $\beta_i \in \mathbb{R}$, $i = 0, 1, \dots, m$, $0 \leqslant m < n$.

The embedding (1.1) has been considered in [4], but basically only sufficient conditions for boundedness of the embedding (1.1) have been obtained. In [1] necessary and sufficient conditions for boundedness and compactness of the embedding (1.1) have been established when $1 < p \leqslant q < \infty$.

In order not to disturb our proofs of the main results in Sections 3 and 4 we use Section 2 to present some necessary notation and auxiliary results e.g. from the papers [4] and [7]. In Section 4 the embedding theorems from Section 3 for the spaces $W^n_{p,\overline{\alpha}}(0,1)$ have been rewritten to the case of the spaces $W^n_{p,\overline{\alpha}}(1,+\infty)$.

In this paper we use the following *conventions*: If $i > j$, then the sum $\sum\limits_{k=i}^{j}$ is considered to be equal to zero; and the notation $A \ll B$ means that $A \leqslant cB$, where the constant $c > 0$ may depend on unessential parameters.

2. Preliminaries

In [4] the following relation between the α-multiweighted derivative and the β-multiweighted derivative of the function f was proved:

$$D^k_{\overline{\beta}} f(t) = \sum_{i=0}^{k} c_{k,i} t^{\mu_{k,i}} D^i_{\overline{\alpha}} f(t), \quad k = 0, 1, \dots, m, \tag{2.1}$$

where $\mu_{k,i} = \sum\limits_{j=0}^{k} \beta_j - \sum\limits_{j=0}^{i} \alpha_j + i - k$, $i = 0, 1, \ldots, k$, $k = 0, 1, \ldots, m$; and the coefficients $c_{k,i}$, $i = 0, 1, \ldots, k-1$, $k = 0, 1, \ldots, m$, are defined by the recurrent formula:

$$\begin{aligned}
c_{k,k} &= 1,\\
c_{k,0} &= c_{k-1,0}\Bigl(\sum_{j=0}^{k-1} \beta_j - \alpha_0 - k + 1\Bigr),\\
c_{k,i} &= c_{k-1,i-1} + c_{k-1,i}\Bigl(\sum_{j=0}^{k-1} \beta_j - \sum_{j=0}^{i} \alpha_j + i - k + 1\Bigr), \quad i = 1, 2, \ldots, k-1.
\end{aligned}$$

Moreover, in [4] it was proved that

$$D_{\overline{\alpha}}^{k} f(t) = \sum_{j=0}^{k} d_{k,j} t^{\gamma_{k,j}} D_{\overline{\beta}}^{j} f(t), \quad k = 0, 1, \ldots, m, \tag{2.2}$$

where $\gamma_{k,j} = \sum\limits_{i=0}^{k} \alpha_i - \sum\limits_{i=0}^{j} \beta_i + j - k$ and $d_{k,j}$, $0 \leqslant j \leqslant k < m$, are defined analogously as $c_{k,i}$, $0 \leqslant i \leqslant k \leqslant m$.

For $0 < t \leqslant x$ and for $i, j = 0, 1, \ldots, n-1$ we define the following set of functions:

$$\begin{aligned}
K_{i+1,j}(t,x) &\equiv K_{i+1,j}(t,x,\overline{\alpha})\\
&= \int_t^x t_{i+1}^{-\alpha_{i+1}} \int_{t_{i+1}}^x t_{i+2}^{-\alpha_{i+2}} \ldots \int_{t_{j-1}}^x t_j^{-\alpha_j}\, \mathrm{d}t_j\, \mathrm{d}t_{j-1} \ldots \mathrm{d}t_{i+1} \quad \text{when } i < j,\\
K_{i+1,j}(t,x) &\equiv K_{i+1,j}(t,x,\overline{\alpha}) \equiv 1 \quad \text{when } i = j,\\
K_{i+1,j}(t,x) &\equiv K_{i+1,j}(t,x,\overline{\alpha}) \equiv 0 \quad \text{when } i > j.
\end{aligned}$$

By changing variables, when $i < j$ the following properties of homogeneity of the functions $K_{i+1,j}$ can be established:

$$\begin{aligned}
&K_{i+1,j}(zt,zx)\\
&\quad = \int_{zt}^{zx} t_{i+1}^{-\alpha_{i+1}} \int_{t_{i+1}}^{zx} t_{i+2}^{-\alpha_{i+2}} \ldots \int_{t_{j-1}}^{zx} t_j^{-\alpha_j}\, \mathrm{d}t_j\, \mathrm{d}t_{j-1} \ldots \mathrm{d}t_{i+1}\\
&\quad = [t_k = z\tau_k,\ \mathrm{d}t_k = z\, \mathrm{d}\tau_k]\\
&\quad = \int_t^x (z\tau_{i+1})^{-\alpha_{i+1}} \int_{\tau_{i+1}}^x (z\tau_{i+2})^{-\alpha_{i+2}} \ldots \int_{\tau_{j-1}}^x (z\tau_j)^{-\alpha_j} z^{j-i}\, \mathrm{d}\tau_j\, \mathrm{d}\tau_{j-1} \ldots \mathrm{d}\tau_{i+1}\\
&\quad = z^{\sum\limits_{k=i+1}^{j}(1-\alpha_k)} K_{i+1,j}(t,x).
\end{aligned}$$

In particular, when $x = 1$ and $t = 1$, we have that

$$(2.3)\qquad K_{i+1,j}(zt,z) = z^{\sum\limits_{k=i+1}^{j}(1-\alpha_k)} K_{i+1,j}(t,1),$$
$$K_{i+1,j}(z,zx) = z^{\sum\limits_{k=i+1}^{j}(1-\alpha_k)} K_{i+1,j}(1,x),$$

respectively.

The following integral representation of the α-multiweighted derivative of the function $f \in W^n_{p,\overline{\alpha}}$ was proved in [4]:

$$(2.4)\qquad D^i_{\overline{\alpha}} f(t) = \sum_{j=i}^{n-1} (-1)^{j-i} K_{i+1,j}(t,1) D^j_{\overline{\alpha}} f(1) + \int_t^1 x^{-\alpha_n} K_{i+1,n-1}(t,x) D^n_{\overline{\alpha}} f(x)\,\mathrm{d}x, \quad i = 0,1,\ldots,n-1.$$

By inserting (2.4) into (2.1) when $k = m$ we find that

$$(2.5)\qquad D^m_{\overline{\beta}} f(t) = \sum_{i=i_0}^{m} c_{m,i} t^{\mu_{m,i}} \sum_{j=i}^{n-1} (-1)^{j-i} K_{i+1,j}(t,1) D^j_{\overline{\alpha}} f(1) + \sum_{i=i_0}^{m} c_{m,i} t^{\mu_{m,i}} \int_t^1 x^{-\alpha_n} K_{i+1,n-1}(t,x) D^n_{\overline{\alpha}} f(x)\,\mathrm{d}x.$$

For $0 \leqslant i \leqslant j \leqslant n-1$ we define:

$$k_{i,j} = \min\Big\{k\colon\ i \leqslant k \leqslant j,\ \sum_{s=i+1}^{k} \alpha_s - k = \max_{i \leqslant \xi \leqslant j} \Big(\sum_{s=i+1}^{\xi} \alpha_s - \xi\Big)\Big\},$$

and

$$M_{i,j} = \max_{i \leqslant s \leqslant j} \Big(j - s + 1 - \sum_{k=s+1}^{j+1} \alpha_k\Big).$$

For convenience, we denote $k_i \equiv k_{i,n-1}$, $M_i = M_{i,n-1}$. Note that $M_i \geqslant M_{i+1}$ and $M_0 = \max\limits_{0 \leqslant i \leqslant n-1} M_i$.

Furthermore, for the proof of our main result we need the fact, that for the functions $f_s(t) = t^{-\alpha_0} K_{1,s}(t,1,\overline{\alpha})$, $0 \leqslant m \leqslant s \leqslant n$, their multiweighted derivative $D^m_{\overline{\beta}} f_s$ does not vanish, i.e.

$$(2.6)\qquad D^m_{\overline{\beta}} f_s(t) \neq 0, \quad \forall\, t \in (0,1].$$

Indeed, let us assume the opposite, i.e. let $f_s(t) = t^{-\alpha_0} K_{1,s}(t,1,\overline{\alpha})$, $0 \leqslant m \leqslant s \leqslant n$, be the solutions of the equation

$$D^m_{\overline{\beta}} f(t) = 0, \quad \forall\, t \in (0,1]. \tag{2.7}$$

Then they can be written as linear combinations of the fundamental solutions:

$$f_i(t) = t^{-\beta_0} K_{1,i}(t,1,\overline{\beta}), \quad i = 0,1,\dots,m-1,$$

of the homogeneous equation (2.7), i.e.

$$f_s(t) = \sum_{i=0}^{m-1} c_i t^{-\beta_0} K_{1,i}(t,1,\overline{\beta}), \quad \forall\, t \in (0,1], \tag{2.8}$$

where $\sum\limits_{i=0}^{m-1} c_i^2 \neq 0$, $c_i \in \mathbb{R}$, $i = 0,1,\dots,m-1$.

Taking $\overline{\alpha}$-multiweighted derivative of order k, $k = 0,1,\dots,m-1$, from both parts of (2.8), we have that

$$D^k_{\overline{\alpha}} f_s(t) = \sum_{i=0}^{m-1} c_i D^k_{\overline{\alpha}}(t^{-\beta_0} K_{1,i}(t,1,\overline{\beta})), \quad \forall\, t \in (0,1]. \tag{2.9}$$

Using (2.2) and taking into account that $d_{k,k} \equiv 1$, $0 \leqslant j \leqslant k < m$, from (2.9) for k, $0 \leqslant k < m$, we obtain that

$$D^k_{\overline{\alpha}} f_s(t) = \sum_{j=0}^{k} (-1)^j d_{k,j} t^{\gamma_{k,j}} \sum_{i=j}^{m-1} c_i K_{j+1,i}(t,1,\overline{\beta}) = \sum_{j=0}^{k} (-1)^j d_{k,j} c_j t^{\gamma_{k,j}}, \tag{2.10}$$

since $K_{j+1,j}(t,1,\overline{\beta}) = 1$ and $K_{j+1,i}(t,1,\overline{\beta}) = 0$, $i = j+1, j+2, \dots, m-1$.

On the other hand a straightforward calculation shows that

$$\begin{gathered} D^k_{\overline{\alpha}} f_s(t) = D^k_{\overline{\alpha}}(t^{-\alpha_0} K_{1,s}(t,1,\overline{\alpha})) = (-1)^k K_{k+1,s}(t,1,\overline{\alpha}), \\ k = 0,1,\dots,m-1; \quad s = m, m+1, \dots, n. \end{gathered} \tag{2.11}$$

Thus, from (2.10) and (2.11) we obtain that

$$(-1)^k K_{k+1,s}(t,1,\overline{\alpha}) = \sum_{j=0}^{k} (-1)^j d_{k,j} c_j t^{\gamma_{k,j}},$$

$k = 0,1,\dots,m-1$; $s = m, m+1, \dots, n$.

In particular, when $t = 1$ we get the following system of equations of order m:

$$\sum_{j=0}^{k} (-1)^j d_{k,j} c_j = 0, \quad k = 0, 1, \dots, m-1.$$

Solving this system of equations when $k = 0$, we have that $d_{0,0}c_0 = 0$. Since $d_{0,0} = 1$, it yields that $c_0 = 0$. Furthermore, by successively solving the system for $k = 1, 2, \dots, m-1$ (note that $d_{k,k} \neq 0$), we get that $c_k = 0$, $k = 0, 1, \dots, m-1$. However, by our assumption, c_k, $k = 0, 1, \dots, m-1$, can not be equal to zero simultaneously. This contradiction shows that (2.6) holds.

Moreover, we need upper and lower estimates for the functions $K_{i+1,j}(t,1)$ when $0 < t \leqslant 1$ and $K_{i+1,n-1}(1,t)$ when $1 \leqslant t < \infty$, $0 \leqslant i \leqslant j \leqslant n-1$. In [2] there were obtained upper and lower estimates for the functions $u_i(t) = t^{\alpha_0} K_{1,i}(t, 1, -\overline{\alpha})$, $i = 0, 1, \dots, n-1$. Below we give three statements about estimates for the functions $K_{i+1,j}(t,1)$ and $K_{i+1,j}(1,t)$, which follow from these results. Moreover, for convenience we use the following equalities:

$$\begin{aligned}
&\min_{i \leqslant s \leqslant j} \left(\alpha_0 + \sum_{k=i+1}^{s} (1 - \alpha_k) \right) \\
&\quad = \min_{i \leqslant s \leqslant j} \left[\alpha_0 + j - i + 1 - \sum_{k=i+1}^{j+1} \alpha_k - \left(j - s + 1 - \sum_{k=s+1}^{j+1} \alpha_k \right) \right] \\
&\quad = \alpha_0 + j - i + 1 - \sum_{k=i+1}^{j+1} \alpha_k - M_{i,j}.
\end{aligned}$$

Lemma 2.1. *Let* $0 \leqslant i \leqslant j \leqslant n-1$. *Then*

$$K_{i+1,j}(t,1) \ll t^{\,j-i+1-\sum\limits_{k=i+1}^{j+1} \alpha_k - M_{i,j}} |\ln t|^{l_{i,j}}, \quad t \in (0,1],$$

where $l_{i,j}$ *is the number of* k, $k_{i,j} + 1 \leqslant k \leqslant j$, *such that* $\sum\limits_{s=k_{i,j}+1}^{k} (\alpha_s - 1) = 0$, *if* $k_{i,j} < j$, *and* $l_{i,j} = 0$, *if* $k_{i,j} = j$.

Lemma 2.2. *Let* $0 \leqslant i \leqslant n-1$. *Then there exists* δ, $0 < \delta < 1$, *such that for any* $t \in (0, \delta]$ *the following estimate*

$$K_{i+1,n-1}(t,1) \gg t^{\,n-i-\sum\limits_{k=i+1}^{n} \alpha_k - M_i}$$

holds.

Lemma 2.3. *Let $0 \leqslant i \leqslant n-1$. Then*

$$t^{-\alpha_n} K_{i+1,n-1}(1,t) \ll t^{M_i-1} |\ln t|^{l_i}, \quad t \geqslant 1,$$

where l_i is the number of k, $i+1 \leqslant k \leqslant k_i - 1$, such that $\sum_{s=k}^{k_i-1} (\alpha_s - 1) = 0$ when $k_i > i+1$, and $l_i = 0$ when $k_i = i+1$.

We also recall the following Lemma by T. Andô [3]:

Lemma 2.4. *Every linear integral operator, acting from L_p to L_q, where $1 \leqslant q < p < \infty$, is compact.*

Consider the following integral operators:

$$K_i D_{\overline{\alpha}}^n f(t) = t^{\mu_{m,i}} \int_t^1 x^{-\alpha_n} K_{i+1,n-1}(t,x) D_{\overline{\alpha}}^n f(x)\, \mathrm{d}x, \quad i = i_0, i_0+1, \dots, m, \tag{2.12}$$

acting from $L_p(0,1)$ to $L_q(0,1)$.

From the results in [7] we have the following:

Lemma 2.5. *Let $1 \leqslant q < p < \infty$. The integral operators* (2.12) *are bounded from $L_p(0,1)$ to $L_q(0,1)$ if and only if*

$$B_n = \max_{i_0 \leqslant i \leqslant m} \max_{i \leqslant j \leqslant n-1} B_{i,j}^n < \infty,$$

where

$$\begin{aligned} B_{i,j}^n = \Bigg\{ \int_0^1 \bigg(\int_t^1 |x^{-\alpha_n} K_{j+1,n-1}(t,x)|^{p'}\, \mathrm{d}x \bigg)^{q(p-1)/(p-q)} \\ \times \bigg(\int_0^t |s^{\mu_{m,i}} K_{i+1,j}(s,t)|^q\, \mathrm{d}s \bigg)^{q/(p-q)} \\ \times \mathrm{d}\bigg(\int_0^t |s^{\mu_{m,i}} K_{i+1,j}(s,t)|^q\, \mathrm{d}s \bigg) \Bigg\}^{(p-q)/pq}. \end{aligned} \tag{2.13}$$

3. Embedding theorems for the space $W^n_{p,\overline{\alpha}}(0,1)$

Denote $i_0 = \min\{i\colon\ 0 \leqslant i \leqslant m,\ c_{m,i} \neq 0\}$, where $c_{m,i}$, $i = 0,1,\ldots,m$, are defined as in (2.1).

Our main result in this paper reads:

Theorem 3.1. *Let $I = (0,1)$, $1 \leqslant q < p < \infty$ and $0 \leqslant m < n$. Then the following conditions are equivalent:*

i) *the embedding* (1.1) *is bounded;*

ii) *the embedding* (1.1) *is compact;*

iii)

$$|\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} > \max\Big\{\frac{1}{p}, M_{i_0}\Big\}. \tag{3.1}$$

P r o o f. First we prove that i) $\Rightarrow$ ii).

Assume that i) holds, i.e., for all $f \in W^n_{p,\overline{\alpha}}$ the following estimate

$$\|f\|_{W^m_{q,\overline{\beta}}} \leqslant c\|f\|_{W^n_{p,\overline{\alpha}}}$$

holds. Then, by the definition of the norm in the space $W^m_{q,\overline{\beta}}$, the following estimate

$$\|D^m_{\overline{\beta}} f\|_q \leqslant c\|f\|_{W^n_{p,\overline{\alpha}}} \tag{3.2}$$

holds, where $c > 0$ does not depend on $f \in W^n_{p,\overline{\alpha}}$.

Now we take a set L of functions from $W^n_{p,\overline{\alpha}}$ such that for all $f \in L$:

$$D^j_{\overline{\alpha}} f(1) = 0, \quad j = 0,1,\ldots,n-1. \tag{3.3}$$

It is obvious that L is a subset of the space $W^n_{p,\overline{\alpha}}$. For any $F \in L_p(0,1)$ there exists a unique function $f \in L$ as a solution of the equation $D^n_{\overline{\alpha}} f(t) = F(t)$ with initial condition (3.3). Therefore, due to the fact that $\|f\|_{W^n_{p,\overline{\alpha}}} = \|F\|_p$, the operator $D^n_{\overline{\alpha}}$ establishes an isometry between the subspace $L \subset W^n_{p,\overline{\alpha}}$ and the space $L_p(0,1)$.

Let

$$\sum_{i=i_0}^{m} c_{m,i} x^{-\alpha_n} t^{\mu_{m,i}} K_{i+1,n-1}(t,x) = \overline{K}(t,x).$$

Then, for all $f \in L$, the expression (2.5) has the following form:

$$D^m_{\overline{\beta}} f(t) = \int_t^1 \overline{K}(t,x) D^n_{\overline{\alpha}} f(x)\,\mathrm{d}x = \overline{K} D^n_{\overline{\alpha}} f(t).$$

Using this expression in (3.2), for all $f \in L$ we have that

$$\|\overline{K} D_{\overline{\alpha}}^{n} f\|_q \leqslant c \|D_{\overline{\alpha}}^{n} f\|_p,$$

or

$$\|\overline{K} F\|_q \leqslant c \|F\|_p,$$

which means that the operator $\overline{K}$ is bounded from L_p to L_q. In our case $1 \leqslant q < p < \infty$, and, thus, by Lemma 2.4, the integral operator $\overline{K}$ is compact from L_p to L_q. Since the first sum in (2.5) is finite-dimensional, the expression (2.5), as an operator, is compact from $W_{p,\overline{\alpha}}^{n}$ to L_q. Hence, the embedding (1.1) is compact, i.e. ii) holds.

Next we prove that iii) $\Rightarrow$ i). Let iii) hold. According to (2.1) for $f \in W_{p,\overline{\alpha}}^{n}$ when $t = 1$ we have that

$$\sum_{k=0}^{m-1} |D_{\overline{\beta}}^{k} f(1)| \ll \sum_{k=i_0}^{n-1} |D_{\overline{\alpha}}^{k} f(1)|. \tag{3.4}$$

From (2.5) and (3.4) it follows that the embedding (1.1) is bounded whenever

$$\int_0^1 |t^{\mu_{m,i}} K_{i+1,j}(t,1)|^q \,\mathrm{d}t < \infty, \quad i = i_0, i_0+1, \ldots, m; \;\; j = i, i+1, \ldots, n-1, \tag{3.5}$$

and the integral operators (2.12) are bounded from $L_p(0,1)$ to $L_q(0,1)$.

By using Lemma 2.1 for $0 \leqslant i \leqslant j \leqslant n-1$ we find that

$$\int_0^1 |t^{\mu_{m,i}} K_{i+1,j}(t,1)|^q \,\mathrm{d}t \ll \int_0^1 t^{q\left[\mu_{m,i} - \max\limits_{i \leqslant s \leqslant j}\left(\sum\limits_{k=i+1}^{s} \alpha_k + i - s\right)\right]} |\ln t|^{q l_{i,j}} \,\mathrm{d}t.$$

The last integral converges, if, for $i_0 \leqslant i \leqslant j \leqslant m \leqslant n-1$, the following conditions hold:

$$\mu_{m,i} - \max_{i \leqslant s \leqslant j}\left(\sum_{k=i+1}^{s} \alpha_k + i - s\right) + \frac{1}{q} > 0,$$

i.e.

$$\begin{aligned} |\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} &> \max_{i \leqslant s \leqslant j}\left(\sum_{k=i+1}^{s} \alpha_k - s\right) - \sum_{k=i+1}^{n} \alpha_k + n \\ &= \max_{i \leqslant s \leqslant j}\left(n - s - \sum_{k=s+1}^{n} \alpha_k\right). \end{aligned} \tag{3.6}$$

Since $M_{i_0} \geqslant \max\limits_{i \leqslant s \leqslant j}\left(n - s - \sum\limits_{k=s+1}^{n} \alpha_k\right)$ for $i_0 \leqslant i \leqslant j \leqslant n-1$, due to (3.1) the conditions (3.6) hold for all $i = 0, 1, \ldots, m$, $j = i, i+1, \ldots, n-1$, and we conclude that (3.5) holds.

To prove boundedness of the integral operators (2.12) due to Lemma 2.5 we estimate each integral in $B_{i,j}$. By using the properties (2.3) of homogeneity of the functions $K_{i+1,j}$, we find that

$$\begin{aligned}
(3.7)\quad \int_0^t |s^{\mu_{m,i}} K_{i+1,j}(s,t)|^q \,\mathrm{d}s &= [s = tz,\ \mathrm{d}s = t\,\mathrm{d}z] \\
&= t^{\mu_{m,i}q+1}\left(\int_0^1 |z^{\mu_{m,i}} K_{i+1,j}(tz,t)|^q \,\mathrm{d}z\right) \\
&= t^{\mu_{m,i}q+1+q\sum\limits_{k=i+1}^{j}(1-\alpha_k)}\left(\int_0^1 |z^{\mu_{m,i}} K_{i+1,j}(z,1)|^q \,\mathrm{d}z\right).
\end{aligned}$$

Moreover, due to (3.5), we know that the last integral converges. By using now the assumptions of our theorem, we find that

$$\begin{aligned}
|\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} &= \sum_{k=0}^{m} \beta_k - \sum_{k=0}^{i} \alpha_k + i - m + n - i - \sum_{k=i+1}^{n} \alpha_k + \frac{1}{q} \\
&> M_{i_0} \geqslant n - j - \sum_{k=j+1}^{n} \alpha_k.
\end{aligned}$$

Thus

$$\mu_{m,i} + j - i - \sum_{k=i+1}^{j} \alpha_k + \frac{1}{q} > 0$$

or

$$1 + q\mu_{m,i} + q \sum_{k=i+1}^{j} (1-\alpha_k) > 0,$$

and, consequently,

$$\begin{aligned}
(3.8)\qquad \mathrm{d}\left(\int_0^t |s^{\mu_{m,i}} K_{i+1,j}(s,t)|^q \,\mathrm{d}s\right) &= c \cdot \mathrm{d}\left(t^{1+q\mu_{m,i}+q\sum\limits_{k=i+1}^{j}(1-\alpha_k)}\right) \\
&= c_1 \cdot t^{q\left(\mu_{m,i}+\sum\limits_{k=i+1}^{j}(1-\alpha_k)\right)}\,\mathrm{d}t,
\end{aligned}$$

where

$$c = \int_0^1 |s^{\mu_{m,i}} K_{i+1,j}(s,1)|^q \,\mathrm{d}s, \quad c_1 = c \cdot \left(1 + q\mu_{m,i} + q \sum_{k=i+1}^{j} (1-\alpha_k)\right),$$

$$i = i_0, i_0+1, \dots, m,\ \ j = i, i+1, \dots, n-1.$$

Putting (3.7) and (3.8) into (2.13), we find that

$$B^n_{i,j} \ll \Bigg\{ \int_0^1 t^{\left(q\left(\mu_{m,i}+\sum\limits_{k=i+1}^{j}(1-\alpha_k)\right)+1\right)q(p-q)+q\left(\mu_{m,i}+\sum\limits_{k=i+1}^{j}(1-\alpha_k)\right)}$$
$$\times \left(\int_t^1 |x^{-\alpha_n} K_{j+1,n-1}(t,x)|^{p'}\,\mathrm{d}x\right)^{q(p-1)/(p-q)} \mathrm{d}t \Bigg\}^{(p-q)/pq}$$
$$= \Bigg\{ \int_0^1 t^{\left(\mu_{m,i}+\sum\limits_{k=i+1}^{j}(1-\alpha_k)+1/p\right)pq/(p-q)}$$
$$\times \left(\int_t^1 |x^{-\alpha_n} K_{j+1,n-1}(t,x)|^{p'}\,\mathrm{d}x\right)^{q(p-1)/(p-q)} \mathrm{d}t \Bigg\}^{(p-q)/pq}.$$

Since $(p-1)/p = 1/p'$ we conclude that

$$B^n_{i,j} \ll \Bigg\{ \int_0^1 \Bigg(t^{\mu_{m,i}+\sum\limits_{k=i+1}^{j}(1-\alpha_k)+1/p}$$
$$\times \left(\int_t^1 |x^{-\alpha_n} K_{j+1,n-1}(t,x)|^{p'}\,\mathrm{d}x\right)^{1/p'} \Bigg)^{pq/(p-q)} \mathrm{d}t \Bigg\}^{(p-q)/pq}. \tag{3.9}$$

Using again the properties (2.3) of homogeneity of the functions $K_{i+1,j}$ and Lemma 2.3, we obtain that

$$\left(\int_t^1 |x^{-\alpha_n} K_{j+1,n-1}(t,x)|^{p'}\,\mathrm{d}x\right)^{1/p'} \tag{3.10}$$
$$= t^{-\alpha_n+1/p'} \left(\int_1^{1/t} |x^{-\alpha_n} K_{j+1,n-1}(t,tx)|^{p'}\,\mathrm{d}x\right)^{1/p'}$$
$$= t^{-\alpha_n+1/p'+\sum\limits_{k=j+1}^{n-1}(1-\alpha_k)} \left(\int_1^{1/t} |x^{-\alpha_n} K_{j+1,n-1}(1,x)|^{p'}\,\mathrm{d}x\right)^{1/p'}$$
$$\ll t^{-1/p+\sum\limits_{k=j+1}^{n}(1-\alpha_k)} \left(\int_1^{1/t} |x^{p'(M_j-1)}|\ln x|^{p' l_j}\,\mathrm{d}x\right)^{1/p'},$$
$$j = i_0, i_0+1, \ldots, n-1.$$

Since

$$\int_1^\infty x^{p'(M_j-1)}|\ln x|^{p' l_j}\,\mathrm{d}x < \infty \text{ when } M_j < \frac{1}{p},\ \ j = i_0, i_0+1, \ldots, n-1,$$

from (3.10) for small enough $t > 0$ we have that

$$(3.11) \qquad \left(\int_t^1 |x^{-\alpha_n} K_{j+1,n-1}(t,x)|^{p'}\,\mathrm{d}x\right)^{1/p'} \ll \begin{cases} t^{\sum\limits_{k=j+1}^{n}(1-\alpha_k)-M_j}|\ln t|^{l_j} & \text{if } M_j > \dfrac{1}{p}, \\ t^{\sum\limits_{k=j+1}^{n}(1-\alpha_k)-1/p} & \text{if } M_j < \dfrac{1}{p}, \\ t^{\sum\limits_{k=j+1}^{n}(1-\alpha_k)-1/p}|\ln t|^{l_j+1/p'} & \text{if } M_j = \dfrac{1}{p}. \end{cases}$$

From (3.9) and (3.11) we get that

$$(3.12) \quad B_{i,j}^n \ll \begin{cases} \left(\displaystyle\int_0^1 t^{\left(\mu_{m,i}+\sum\limits_{k=i+1}^{j}(1-\alpha_k)+1/p-M_j\right)pq/(p-q)}|\ln t|^{l_j\cdot pq/(p-q)}\,\mathrm{d}t\right)^{(p-q)/pq} \\ \quad \text{if } M_j > 1/p, \\ \left(\displaystyle\int_0^1 t^{\left(\mu_{m,i}+\sum\limits_{k=i+1}^{j}(1-\alpha_k)\right)pq/(p-q)}\,\mathrm{d}t\right)^{(p-q)/pq} \\ \quad \text{if } M_j < 1/p, \\ \left(\displaystyle\int_0^1 t^{\left(\mu_{m,i}+\sum\limits_{k=i+1}^{j}(1-\alpha_k)\right)pq/(p-q)}|\ln t|^{(l_j+1/p')pq/(p-q)}\,\mathrm{d}t\right)^{(p-q)/pq} \\ \quad \text{if } M_j = 1/p. \end{cases}$$

From (3.12) it follows that $B_{i,j}^n$, $i_0 \leqslant i \leqslant m$, $i \leqslant j \leqslant n-1$, will be finite if

$$\mu_{m,i} + \sum_{k=i+1}^{n}(1-\alpha_k) + \frac{1}{p} - M_j > \frac{q-p}{pq},$$

or

$$(3.13) \qquad |\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} > M_j \quad \text{when } M_j > \frac{1}{p},$$

and

$$\mu_{m,i} + \sum_{k=i+1}^{n}(1-\alpha_k) > \frac{q-p}{pq},$$

or

$$(3.14) \qquad |\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} > \frac{1}{p} \quad \text{when } M_j \leqslant \frac{1}{p}.$$

Since the left-hand sides of (3.13) and (3.14) are the same and do not depend on i, j, and the quantities M_i do not increase with the index $i = i_0, i_0+1, \ldots, n-1$, the quantity $B_n = \max\limits_{i_0\leqslant i\leqslant m} \max\limits_{i\leqslant j\leqslant n-1} B^n_{i,j}$ will be finite, if (3.1) holds. Consequently, iii) implies i).

To complete the proof it is sufficient to prove that ii) $\Rightarrow$ iii), so we assume that ii) holds. Then the embedding (1.1) is bounded, and (3.2) holds for every $f \in W^n_{p,\overline{\alpha}}$.

Let us put $f_0(t) = t^{-\alpha_0} K_{1,n-1}(t,1)$. Then $D^n_{\overline{\alpha}} f_0(t) = 0$ when $t \in (0,1)$ and $D^i_{\overline{\alpha}} f_0(1) = 0$, $i = 0,1,\ldots,n-2$, $|D^{n-1}_{\overline{\alpha}} f_0(1)| = 1$. Consequently, $f_0 \in W^n_{p,\overline{\alpha}}$ and $\|f_0\|_{W^n_{p,\overline{\alpha}}} = 1$. Hence, (3.2) implies that

$$\|D^m_{\overline{\beta}} f_0\|_q \leqslant c.$$

Due to (2.6) this yields that $\|D^m_{\overline{\beta}} f_0\|_q > 0$. By using (2.1), we have that

$$\int_0^1 \left| \sum_{i=i_0}^{m} (-1)^i c_{m,i} t^{\mu_{m,i}} K_{i+1,n-1}(t,1)\right|^q \mathrm{d}t \leqslant c^q. \tag{3.15}$$

Since, due to Lemma 2.2, $K_{i+1,n-1}(t,1) \gg t^{n-i-\sum\limits_{k=i+1}^{n}\alpha_k - M_i}$, $0 \leqslant i \leqslant n-1$, for small enough $t > 0$, then

$$t^{\mu_{m,i}} K_{i+1,n-1}(t,1) \gg t^{|\overline{\beta}|-|\overline{\alpha}|+n-m-M_i}, \quad i = i_0, i_0+1, \ldots, m,$$

for small enough $t > 0$. By our condition $c_{m,i_0} \neq 0$ and $M_{i_0} \geqslant M_i$, $i_0 \leqslant i \leqslant m$, this yields that when $M_{i_0} > 1/p$ the order of the integrand in (3.15) is not less than $t^{|\overline{\beta}|-|\overline{\alpha}|+n-m-M_{i_0}}$. Therefore, the function $t^{(|\overline{\beta}|-|\overline{\alpha}|+n-m-M_{i_0})q}$ is integrable in a neighbourhood of $t = 0$ and this is equivalent to the following condition

$$|\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} > M_{i_0}. \tag{3.16}$$

Now let us take the function $f_1(t) = t^{n-|\overline{\alpha}|-\varepsilon/p}$, where $0 < \varepsilon < 1$. Then

$$D^n_{\overline{\alpha}} f_1(t) = \prod_{j=0}^{n-1}\left(n - j - \sum_{k=j+1}^{n} \alpha_k - \frac{\varepsilon}{p}\right) t^{-\varepsilon/p}.$$

Consequently, $f_1 \in W^n_{p,\overline{\alpha}}$. By making some calculations we find that

$$D^m_{\overline{\beta}} f_1(t) = \prod_{i=0}^{m-1}\left(\sum_{k=0}^{i} \beta_k - |\overline{\alpha}| + n - i - \frac{\varepsilon}{p}\right) t^{|\overline{\beta}|-|\alpha|+n-m-\varepsilon/p}.$$

Since we have finite many factors in the product, there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (\varepsilon_0, 1)$,

$$\prod_{i=0}^{m-1} \left(\sum_{k=0}^{i} \beta_k - |\overline{\alpha}| + n - i - \frac{\varepsilon}{p} \right) \neq 0.$$

Due to the continuous embedding (1.1) it must hold that $D^m_{\overline{\beta}} f_1 \in L_q(0,1)$, but this is possible if and only if

$$|\overline{\beta}| - |\overline{\alpha}| + n - m - \frac{\varepsilon}{p} + \frac{1}{q} > 0 \quad \text{for all } \varepsilon \in (\varepsilon_0, 1).$$

Hence, by letting $\varepsilon \to 1$, we have that

$$|\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} \geqslant \frac{1}{p}. \tag{3.17}$$

Let $M_{i_0} < 1/p$. We suppose that

$$|\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} - \frac{1}{p} = 0. \tag{3.18}$$

We consider the following set of the functions:

$$f_\varepsilon(t) = c_\varepsilon t^{-\alpha_0} \int_t^1 K_{1,n-1}(t,x) x^{-\alpha_n} \chi_{0,\varepsilon}(x) x^{-\varepsilon/p} \, \mathrm{d}x, \quad \varepsilon_0 < \varepsilon < 1,$$

where c_ε is a constant and $\chi_{0,\varepsilon}(\cdot)$ denotes the characteristic function of the interval $(0,\varepsilon)$.

Since $D^n_{\overline{\alpha}} f_\varepsilon(t) = c_\varepsilon (-1)^n \chi_{(0,\varepsilon)}(t) t^{-\varepsilon/p}$, we have $f_\varepsilon \in W^n_{p,\overline{\alpha}}$ for all $\varepsilon \in (0,1)$.

We choose a constant c_ε such that $\|f_\varepsilon\|_{W^n_{p,\overline{\alpha}}} = \|D^n_{\overline{\alpha}} f_\varepsilon\|_p = 1$. Then

$$c_\varepsilon = (1-\varepsilon)^{1/p} \varepsilon^{(\varepsilon-1)/p}.$$

We now prove that the set of functions f_ε, $0 < \varepsilon < 1$, converges weakly to zero when $\varepsilon \to 0$. By definition of the space $W^n_{p,\overline{\alpha}}$ it follows that it is isometric to the space $L_p(I) \times \mathbb{R}^n$. Therefore, $(W^n_{p,\overline{\alpha}})^* = (L_p(I) \times \mathbb{R}^n)^* = L_{p'}(I) \times \mathbb{R}^n$. Since $D^i_{\overline{\alpha}} f_\varepsilon(1) = 0$, $i = 0, 1, \ldots, n-1$, we have, according to Hölder's inequality, for each $G = (g, a) \in L_{p'}(I) \times \mathbb{R}^n$:

$$\begin{aligned} |\langle f_\varepsilon, G \rangle| &= \left| \int_0^1 D^n_{\overline{\alpha}} f_\varepsilon(t) g(t) \, \mathrm{d}t \right| = c_\varepsilon \left| \int_0^\varepsilon t^{-\varepsilon/p} g(t) \, \mathrm{d}t \right| \\ &\leqslant c_\varepsilon \left(\int_0^\varepsilon t^{-\varepsilon} \, \mathrm{d}t \right)^{1/p} \left(\int_0^\varepsilon |g(t)|^{p'} \, \mathrm{d}t \right)^{1/p'} \\ &= \left(\int_0^\varepsilon |g(t)|^{p'} \, \mathrm{d}t \right)^{1/p'}. \end{aligned}$$

Hence, it follows that $\langle f_\varepsilon, G\rangle \to 0$ when $\varepsilon \to 0$ for all $G \in (W^n_{p,\overline{\alpha}})^*$. Therefore, due to the compactness of the embedding (1.1), the set of functions f_ε, $0<\varepsilon<1$, when $\varepsilon \to 0$ converges strongly to zero in $W^m_{q,\overline{\beta}}$. Moreover, by using (2.1), (2.4) and (2.5), we have that

$$\begin{aligned} D^m_{\overline{\beta}} f_\varepsilon(t) &= \sum_{i=i_0}^{m} c_{m,i} t^{\mu_{m,i}} D^i_{\overline{\alpha}} f_\varepsilon(t) \\ &= \sum_{i=i_0}^{m} (-1)^i c_{m,i} t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n} \chi_{0,\varepsilon}(x) x^{-\varepsilon/p}\,\mathrm{d}x. \end{aligned} \tag{3.19}$$

Now we prove that for $i = i_0, i_0+1, \dots, m$ and for all $\varepsilon \in (0,1)$, the estimate

$$\int_0^1 \left| t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n} \chi_{0,\varepsilon}(x) x^{-\varepsilon/p}\,\mathrm{d}x \right|^q \mathrm{d}t < \infty, \tag{3.20}$$

holds.

By changing variables, due to Lemma 2.3 we get that

$$\begin{aligned} &\int_0^1 \left| t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n - \varepsilon/p}\,\mathrm{d}x \right|^q \mathrm{d}t \\ &\quad \ll \int_0^1 \left| t^{\mu_{m,i} - \alpha_n - \varepsilon/p + 1 + \sum\limits_{k=i+1}^{n-1}(1-\alpha_k)} \int_1^{1/t} z^{M_i - 1 - \varepsilon/p} |\ln z|^{l_i}\,\mathrm{d}z \right|^q \mathrm{d}t. \end{aligned} \tag{3.21}$$

Since $M_{i_0} < 1/p$ and $M_i \leqslant M_{i_0}$, $i = i_0, i_0+1, \dots, m$, for all $\varepsilon \in (0,1)$ we have that $M_i - 1 - \varepsilon/p < 0$, $i = 0, 1, \dots, m$. Therefore,

$$\int_1^{1/t} z^{M_i - 1 - \varepsilon/p} |\ln z|^{l_i}\,\mathrm{d}z \leqslant \int_1^{1/t} |\ln z|^{l_i}\,\mathrm{d}z \leqslant \frac{1}{t} |\ln t|^{l_i},$$

and, hence, from (3.21) it follows that

$$\begin{aligned} &\int_0^1 \left| t^{\mu_{m,i}} \int_t^1 K_{i+1,n-1}(t,x) x^{-\alpha_n - \varepsilon/p}\,\mathrm{d}x \right|^q \mathrm{d}t \\ &\quad \ll \int_0^1 t^{\left(\mu_{m,i} - \alpha_n - \varepsilon/p + \sum\limits_{k=i+1}^{n-1}(1-\alpha_k)\right) q} |\ln t|^{q l_i}\,\mathrm{d}t. \end{aligned} \tag{3.22}$$

Moreover, according to (3.18) we have that

$$\mu_{m,i} - \alpha_n - \frac{\varepsilon}{p} + \sum_{k=i+1}^{n-1}(1-\alpha_k) > -\frac{1}{q}, \quad \forall\, \varepsilon \in (0,1).$$

Consequently, the last integral in (3.22) converges and this fact yields the estimate (3.20).

Further, by taking the norm in (3.19) we get that

$$
\begin{aligned}
(3.23)\quad & \|D^{m}_{\overline{\beta}} f_\varepsilon\|_q \\
&= c_\varepsilon\left(\int_0^1\left|\sum_{i=i_0}^{m}(-1)^i c_{m,i} t^{\mu_{m,i}}\int_t^1 K_{i+1,n-1}(t,x)x^{-\alpha_n-\varepsilon/p}\chi_{0,\varepsilon}(x)\,\mathrm{d}x\right|^q \mathrm{d}t\right)^{1/q} \\
&= c_\varepsilon\left(\int_0^\varepsilon\left|\sum_{i=i_0}^{m}(-1)^i c_{m,i} t^{\mu_{m,i}}\int_t^\varepsilon K_{i+1,n-1}(t,x)x^{-\alpha_n-\varepsilon/p}\,\mathrm{d}x\right|^q \mathrm{d}t\right)^{1/q}.
\end{aligned}
$$

In (3.23) first we change variables $t \to \varepsilon t$ in the outer integral, next we change variables $x \to \varepsilon x$ in the inter integral, and taking into account the relation (3.18), we find that

$$
\|D^{m}_{\overline{\beta}} f_\varepsilon\|_q = \varepsilon^{|\overline{\beta}|-|\overline{\alpha}|+n-m+1/q-1/p} T_\varepsilon = T_\varepsilon,
$$

where

$$
T_\varepsilon = (1-\varepsilon)^{1/p}\left(\int_0^1\left|\sum_{i=i_0}^{m}(-1)^i c_{m,i} t^{\mu_{m,i}}\int_t^1 K_{i+1,n-1}(t,x)x^{-\alpha_n-\varepsilon/p}\,\mathrm{d}x\right|^q \mathrm{d}t\right)^{1/q}.
$$

Due to (3.20) this yields that $T_\varepsilon < \infty$ for all $\varepsilon \in (0,1)$. Moreover,

$$
\begin{aligned}
T_0 &= \lim_{\varepsilon\to 0} T_\varepsilon \\
&= \lim_{\varepsilon\to 0}(1-\varepsilon)^{1/p}\left(\int_0^1\left|\sum_{i=i_0}^{m}(-1)^i c_{m,i} t^{\mu_{m,i}}\int_t^1 K_{i+1,n-1}(t,x)x^{-\alpha_n-\varepsilon/p}\,\mathrm{d}x\right|^q \mathrm{d}t\right)^{1/q} \\
&= \left(\int_0^1\left|\sum_{i=i_0}^{m}(-1)^i c_{m,i} t^{\mu_{m,i}}\int_t^1 K_{i+1,n-1}(t,x)x^{-\alpha_n}\,\mathrm{d}x\right|^q \mathrm{d}t\right)^{1/q} \\
&= \left(\int_0^1 |D^{m}_{\overline{\beta}}(t^{-\alpha_0}K_{1,n}(t,1))|^q\,\mathrm{d}t\right)^{1/q} \neq 0,
\end{aligned}
$$

since, according to (2.6), $D^{m}_{\overline{\beta}}(t^{-\alpha_0}K_{1,n}(t,1)) \neq 0$ for almost every $t \in (0,1]$. Consequently, $\|D^{m}_{\overline{\beta}} f_\varepsilon\|_q \not\to 0$ when $\varepsilon \to 0$, that is, f_ε does not converge to zero in $W^{m}_{q,\overline{\beta}}$ when $\varepsilon \to 0$. The contradiction obtained shows that strict inequality occurs in (3.17) when $M_{i_0} < 1/p$, that is,

$$
|\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} > \frac{1}{p},
$$

which together with (3.16) gives (3.1).

The proof is complete. □

Now on the interval $I = (0,1)$ when $\alpha_k = 0$, $k = 0,1,\ldots,n-1$, $\alpha_n = \gamma$, $\beta_i = 0$, $i = 0,1,\ldots,m-1$, and $\beta_m = \upsilon$ we consider the Kudryavtsev spaces $L^n_{p,\gamma}$ and $L^m_{q,\upsilon}$, respectively. Then $M_{i_0} = \max\limits_{i_0 \leqslant s \leqslant n-1}(n-s-\gamma) = n-\gamma-i_0$. Hence, Theorem 3.1 implies the following new information about the embedding between these spaces and the spaces with multiweighted derivatives:

Corollary 3.1. *Let $0 \leqslant m < n$ and $1 \leqslant q < p < \infty$. Then the following conditions are equivalent:*

i) *the embedding $L^n_{p,\gamma} \hookrightarrow W^m_{q,\overline{\beta}}$ is bounded;*
ii) *the embedding $L^n_{p,\gamma} \hookrightarrow W^m_{q,\overline{\beta}}$ is compact;*
iii) $|\overline{\beta}| - \gamma + n - m + 1/q > \max\{n-\gamma-i_0, 1/p\}$.

Corollary 3.2. *Let $0 \leqslant m < n$ and $1 \leqslant q < p < \infty$. Then the following conditions are equivalent:*

i) *the embedding $W^n_{p,\overline{\alpha}} \hookrightarrow L^m_{q,\upsilon}$ is bounded;*
ii) *the embedding $W^n_{p,\overline{\alpha}} \hookrightarrow L^m_{q,\upsilon}$ is compact;*
iii) $\upsilon - |\overline{\alpha}| + n - m + 1/q > \max\{M_{i_0}, 1/p\}$.

4. Embedding theorems for the space $W^n_{p,\overline{\alpha}}(1,\infty)$

The connection between the spaces $W^n_{p,\overline{\alpha}}(0,1)$ and $W^n_{p,\overline{\alpha}}(1,\infty)$ can be seen by making the change of variable $x = 1/t$. In this way every function $f \in W^n_{p,\overline{\alpha}}(1,\infty)$ can be transformed into a function $\tilde{f}(x) = f(1/x)$ from the space $W^n_{p,\overline{\tilde{\alpha}}}(0,1)$, where $\overline{\tilde{\alpha}} = (\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$, $\tilde{\alpha}_n = -\alpha_n + 2 - 2/p$, $\tilde{\alpha}_i = -\alpha_i + 2$, $i = 1,2,\ldots,n-1$, $\tilde{\alpha}_0 = -\alpha_0$. Moreover,

$$
\begin{aligned}
&\|D^n_{\overline{\alpha}} f\|_{p,(1,+\infty)} \\
&\quad = \left(\int_1^{+\infty} |D^n_{\overline{\alpha}} f(t)|^p \,\mathrm{d}t\right)^{1/p} = \left(\int_1^{+\infty} \left|t^{\alpha_n} \frac{\mathrm{d}}{\mathrm{d}t} t^{\alpha_{n-1}} \frac{\mathrm{d}}{\mathrm{d}t} \ldots t^{\alpha_1} \frac{\mathrm{d}}{\mathrm{d}t} t^{\alpha_0} f(t)\right|^p \mathrm{d}t\right)^{1/p} \\
&\quad = \left(\int_0^1 \left|x^{-\alpha_n} \frac{\mathrm{d}}{x^{-2}\mathrm{d}x} x^{-\alpha_{n-1}} \frac{\mathrm{d}}{x^{-2}\mathrm{d}x} \ldots x^{-\alpha_1} \frac{\mathrm{d}}{x^{-2}\mathrm{d}x} x^{-\alpha_0} f\left(\frac{1}{x}\right)\right|^p \frac{\mathrm{d}x}{x^2}\right)^{1/p} \\
&\quad = \left(\int_0^1 \left|x^{-\alpha_n+2-2/p} \frac{\mathrm{d}}{\mathrm{d}x} x^{-\alpha_{n-1}+2} \frac{\mathrm{d}}{\mathrm{d}x} \ldots x^{-\alpha_1+2} \frac{\mathrm{d}}{\mathrm{d}x} x^{-\alpha_0} f\left(\frac{1}{x}\right)\right|^p \mathrm{d}x\right)^{1/p} \\
&\quad = \left(\int_0^1 \left|x^{\tilde{\alpha}_n} \frac{\mathrm{d}}{\mathrm{d}x} x^{\tilde{\alpha}_{n-1}} \frac{\mathrm{d}}{\mathrm{d}x} \ldots x^{\tilde{\alpha}_1} \frac{\mathrm{d}}{\mathrm{d}x} x^{\tilde{\alpha}_0} \tilde{f}(x)\right|^p \mathrm{d}x\right)^{1/p} = \|D^n_{\overline{\tilde{\alpha}}} \tilde{f}\|_{p,(0,1)},
\end{aligned}
$$

and $D^i_{\overline{\tilde{\alpha}}} f(1) = D^i_{\overline{\alpha}} f(1)$, $i = 0,1,\ldots,n-1$.

Analogously, from the space $W^m_{q,\overline{\beta}}(1,+\infty)$ we can pass to the space $W^m_{q,\overline{\tilde{\beta}}}(0,1)$. Then the embedding (1.1) is equivalent to the embedding:

$$W^n_{p,\overline{\tilde{\alpha}}}(0,1) \hookrightarrow W^m_{q,\overline{\tilde{\beta}}}(0,1),$$

and all notions and statements for the space $W^n_{p,\overline{\tilde{\alpha}}}(0,1)$ can be rewritten for the space $W^n_{p,\overline{\alpha}}(1,+\infty)$.

Therefore,

$$\begin{aligned}\tilde{M}_i &= \max_{i\leqslant s\leqslant n-1}\left(n-s-\sum_{k=s+1}^{n}\tilde{\alpha}_k\right)\\ &= \max_{i\leqslant s\leqslant n-1}\left(n-s-\sum_{k=s+1}^{n-1}(-\alpha_k+2)+\alpha_n-2+\frac{2}{p}\right)\\ &= \max_{i\leqslant s\leqslant n-1}\left(-\left(n-s-\sum_{k=s+1}^{n}\alpha_k\right)+\frac{2}{p}\right) = -\mathcal{M}_i+\frac{2}{p},\end{aligned}$$

where $\mathcal{M}_i = \min\limits_{i\leqslant s\leqslant n-1}\Big(n-s-\sum\limits_{k=s+1}^{n}\alpha_k\Big)$, $i=0,1,\dots,n-1$.

Since $|\overline{\tilde{\beta}}| = \sum\limits_{i=1}^{m-1}(-\beta_i+2)-\beta_0-\beta_m+2-2/q = -|\overline{\beta}|+2m-2/q$ and $|\overline{\tilde{\alpha}}| = -|\overline{\alpha}|+2n-2/p$, from the condition (3.1) we have that

$$\begin{aligned}(4.1)\qquad |\overline{\tilde{\beta}}|-|\overline{\tilde{\alpha}}|+n-m+1/q &= |\overline{\alpha}|-|\overline{\beta}|+2m-2n+n-m+\frac{1}{q}-\frac{2}{q}+\frac{2}{p}\\ &= |\overline{\alpha}|-|\overline{\beta}|+m-n-\frac{1}{q}+\frac{2}{p} > \max\Big\{\frac{1}{p},\tilde{M}_{i_0}\Big\}.\end{aligned}$$

In the case $\tilde{M}_{i_0} = -\mathcal{M}_{i_0}+2/p > 1/p$, this is equivalent to $\mathcal{M}_{i_0} < 1/p$ and from (4.1) it follows that

$$|\overline{\alpha}|-|\overline{\beta}|+m-n-\frac{1}{q}+\frac{2}{p} > -\mathcal{M}_{i_0}+\frac{2}{p},$$

i.e.

$$|\overline{\beta}|-|\overline{\alpha}|+n-m+\frac{1}{q} < \mathcal{M}_{i_0} \quad \text{when } \mathcal{M}_{i_0} < \frac{1}{p}.$$

In the case $\tilde{M}_{i_0} \leqslant 1/p$, that is $\mathcal{M}_{i_0} \geqslant 1/p$, from (4.1) we get that

$$|\overline{\alpha}|-|\overline{\beta}|+m-n-\frac{1}{q}+\frac{2}{p} > \frac{1}{p},$$

i.e.

$$|\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} < \frac{1}{p} \quad \text{when } \mathcal{M}_{i_0} \geqslant \frac{1}{p}.$$

Hence, the condition

$$|\tilde{\overline{\beta}}| - |\tilde{\overline{\alpha}}| + n - m + \frac{1}{q} > \max\Big\{\frac{1}{p}, \tilde{M}_{i_0}\Big\}$$

will be changed into the condition

$$|\overline{\beta}| - |\overline{\alpha}| + n - m + \frac{1}{q} < \min\Big\{\frac{1}{p}, \mathcal{M}_{i_0}\Big\}.$$

Thus, from Theorem 3.1 and Corollary 3.1, Corollary 3.2, respectively, we obtain the following results:

Theorem 4.1. *Let* $I = (1, +\infty)$, $1 \leqslant q < p < \infty$ *and* $0 \leqslant m < n$. *Then the following conditions are equivalent:*

i) *the embedding* (1.1) *is bounded;*
ii) *the embedding* (1.1) *is compact;*
iii) $|\overline{\beta}| - |\overline{\alpha}| + n - m + 1/q < \min\{\mathcal{M}_{i_0}, 1/p\}$.

In the space $L^n_{p,\gamma}(1, +\infty)$ we have that $M_{i_0} = 1 - \gamma$. Therefore, we get the following results:

Corollary 4.1. *Let* $I = (1, +\infty)$, $0 \leqslant m < n$ *and* $1 \leqslant q < p < \infty$. *Then the following conditions are equivalent:*

i) *the embedding* $L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\overline{\beta}}(I)$ *is bounded;*
ii) *the embedding* $L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\overline{\beta}}(I)$ *is compact;*
iii) $|\overline{\beta}| - \gamma + n - m + 1/q < \min\{1 - \gamma, 1/p\}$.

Corollary 4.2. *Let* $I = (1, +\infty)$, $0 \leqslant m < n$ *and* $1 \leqslant q < p < \infty$. *Then the following conditions are equivalent:*

i) *the embedding* $W^n_{p,\overline{\alpha}}(I) \hookrightarrow L^m_{q,\upsilon}(I)$ *is bounded;*
ii) *the embedding* $W^n_{p,\overline{\alpha}}(I) \hookrightarrow L^m_{q,\upsilon}(I)$ *is compact;*
iii) $\upsilon - |\overline{\alpha}| + n - m + 1/q < \min\{\mathcal{M}_{i_0}, 1/p\}$.

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References

[1] *Z. T. Abdikalikova, A. Baiarystanov, R. Oinarov*: Compactness of embedding between spaces with multiweighted derivatives – the case $p \leqslant q$. Math. Inequal. Appl. Submitted.

[2] *Z. T. Abdikalikova, A. A. Kalybay*: Summability of a Tchebysheff system of functions. J. Funct. Spaces Appl. *8* (2010), 87–102.

[3] *T. Andô*: On compactness of integral operators. Nederl. Akad. Wet., Proc., Ser. A 65 *24* (1962), 235–239.

[4] *A. A. Kalybay*: Interrelation of spaces with multiweighted derivatives. Vestnik Karaganda State University (1999), 13–22. (In Russian.)

[5] *L. D. Kudryavtsev*: Equivalent norms in weighted spaces. Proc. Steklov Inst. Math. *170* (1987), 185–218.

[6] *S. M. Nikol'skiĭ*: Approximation of Functions of Several Variables and Imbedding Theorems, 2nd ed., rev. and suppl. Nauka, Moskva, 1977. (In Russian.)

[7] *R. Oinarov*: Boundedness and compactness of superposition of fractional integration operators and their applications. In: Function Spaces, Differential Operators and Nonlinear Analysis 2004. Math. Institute, Acad. Sci., Czech Republic, 2005, pp. 213–235 (www.math.cas.cz/fsdona2004/oinarov.pdf).

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