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LIOUVILLE THEOREMS, A PRIORI ESTIMATES, AND BLOW-UP RATES FOR SOLUTIONS OF INDEFINITE SUPERLINEAR PARABOLIC PROBLEMS

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Abstract. In this paper we establish new nonlinear Liouville theorems for parabolic problems on half spaces. Based on the Liouville theorems, we derive estimates for the blow-up of positive solutions of indefinite parabolic problems and investigate the complete blow-up of these solutions. We also discuss a priori estimates for indefinite elliptic problems.

Keywords: a priori estimates, Liouville theorems, blow-up rate, positive solution, indefinite parabolic problem

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1. Introduction

In this paper we consider the problem

$$\begin{aligned} u_t &= \Delta u + a(x)|u|^{p-1}u, && (x,t) \in \Omega \times (0,T), \\ u &= 0, && (x,t) \in \partial\Omega \times (0,T), \end{aligned} \tag{1.1}$$

which, if needed, is completed with an initial condition

$$u(\cdot,0) = u_0(\cdot) \in L^\infty(\Omega). \tag{1.2}$$

We assume that Ω is a smooth domain in $\mathbb{R}^N$ and $p > 1$. Furthermore, we suppose that $a\colon \bar{\Omega} \to \mathbb{R}$ belongs to $C^2(\bar{\Omega})$ and

$$\text{if} \quad \lim_{k\to\infty} a(x_k) = 0, \quad \text{then} \quad \limsup_{k\to\infty} |\nabla a(x_k)| > 0. \tag{1.3}$$

Here $C^k(D)$ denotes the space of k-times differentiable, bounded functions on $D \subset \mathbb{R}^N$, with bounded, continuous derivatives up to the kth order.

If Ω is bounded and if we denote

$$\Gamma := \{x \in \bar{\Omega}\colon\ a(x) = 0\}, \tag{1.4}$$

$$\Omega^+ := \{x \in \Omega\colon\ a(x) > 0\}, \tag{1.5}$$

$$\Omega^- := \{x \in \Omega\colon\ a(x) < 0\}, \tag{1.6}$$

then (1.3) is equivalent to

$$|\nabla a(x)| \neq 0 \qquad (x \in \Gamma), \tag{1.7}$$

that is, a has nondegenerate zeros in $\bar{\Omega}$. Since u_0 and a are bounded, standard results [21] yield the unique, strong solution of the problem (1.1), (1.2), with the maximal existence time $T_{\max} \in (0,\infty]$. Moreover, by regularity results, if $T_{\max} < \infty$, then $\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty$ as $t \to T_{\max}$. We do not indicate the dependence of $T_{\max}$ on u_0 if no confusion seems possible. Here and in the rest of the paper we assume $T \in (0, T_{\max}]$.

As the main result of this paper, we derive an upper bound for the blow-up rate of nonnegative solutions of (1.1). The blow-up rates and related a priori estimates were studied under various assumptions on a, Ω and u in [1], [10], [11], [17], [13], [14], [15], [22], [26], [27], [28], [36], [34], [35], see also references therein. We just briefly describe the results directly connected to our results. First, Friedman and McLeod [11] studied blowing up solutions ($T_{\max} < \infty$) of the problem

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u, && (x,t) \in \Omega \times (0,T),\\ u &= 0, && (x,t) \in \partial\Omega \times (0,T), \end{aligned} \tag{1.8}$$

with $T = T_{\max}$, and the initial condition (1.2). They proved

$$|u(x,t)| \leqslant C\bigl(1 + (T_{\max} - t)^{-1/(p-1)}\bigr) \qquad (x \in \Omega), \tag{1.9}$$

where Ω is a bounded convex domain, $p > 1$, and u is a positive, increasing (in time) solution of (1.8). These results were generalized by Giga and Kohn [13] and later by Giga et al. [14], [15]. With help of localized energy estimates and iterative arguments, they proved that (1.9) holds true if Ω is a bounded convex domain or $\Omega = \mathbb{R}^N$, u is, a not necessarily positive, solution of (1.8), (1.2), and $1 < p < p_S$, where

$$p_S = p_S(N) := \begin{cases} \infty, & N \leqslant 2,\\ \dfrac{N+2}{N-2}, & N \geqslant 3. \end{cases}$$

In [9] Fila and Souplet employed scaling and Fujita type results to remove the assumption on convexity of Ω and established (1.9) for all positive solutions of (1.8), (1.2), and $1 < p \leqslant 1 + 2/(N+1)$.

Finally, Poláčik et al. [26] investigated positive solutions of (1.8) with a sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ and $1 < p < p_B$, where

$$p_B = p_B(N) := \begin{cases} \infty, & N \leqslant 1, \\ \dfrac{N(N+2)}{(N-1)^2}, & N \geqslant 2. \end{cases} \tag{1.10}$$

Using scaling, doubling lemma and Liouville theorems they obtained

$$u(x,t) \leqslant C(1 + t^{-1/(p-1)} + (T-t)^{-1/(p-1)}) \qquad ((x,t) \in \Omega \times (0,T)), \tag{1.11}$$

where C is a universal constant depending only on p, N and Ω. We remark that the estimates for the initial blow-up rate had been previously established by Bidaut-Véron [5] (see also [3]) for $1 < p < p_B$ and $\Omega = \mathbb{R}^N$. Some estimates on the initial blow-up rates for bounded Ω were proved by Quittner et al. [29].

The first a priori estimates for positive solutions of (1.1), (1.2) with sign-changing a were derived in the form (see [27] and references therein)

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} &\leqslant C(\|u_0\|_{L^\infty(\Omega)}, \delta, N, p, \Omega, a) \\ &\qquad (t \in [0, T_{\max} - \delta],\ \delta > 0,\ T_{\max} < \infty). \end{aligned} \tag{1.12}$$

Later, Xing [36] obtained an upper estimate for the blow-up rate of positive solutions of (1.1), (1.2)

$$u(x,t) \leqslant C\big(1 + (T_{\max} - t)^{-3/(2(p-1))}\big) \qquad ((x,t) \in \Omega \times (0, T_{\max}),\ T_{\max} < \infty)$$

when Ω is bounded, $1 < p < p_B$ and $\Gamma \subset \Omega$, that is, when a does not vanish on $\partial\Omega$. Here C depends on $\|u_0\|_{L^\infty(\Omega)}$, N, p, Ω, a.

The next theorem refines the results in [36] in various directions. It includes unbounded domains and it allows for a very general behavior of a on $\partial\Omega$. In addition, it also yields an estimate for the initial blow-up rate. Denote by $\nu_\Omega(x)$ the unit outward normal vector to $\partial\Omega$ at x.

Theorem 1.1. *Let Ω be a uniformly regular domain of class C^2 in $\mathbb{R}^N$ (cf. [2]) and let $1 < p < p_B$. Suppose that $a \in C^2(\bar{\Omega})$ satisfies* (1.3) *and*

$$\left| \frac{\nabla a(x_0)}{|\nabla a(x_0)|} - \nu_\Omega(x_0) \right| \geqslant \tilde{c} > 0 \qquad (x_0 \in \Gamma \cap \partial\Omega). \tag{1.13}$$

Then every nonnegative solution u *of* (1.1) *satisfies*

$$u(x,t) \leqslant C\big(1 + t^{-3/(2(p-1))} + (T-t)^{-3/(2(p-1))}\big) \qquad ((x,t) \in \Omega \times (0,T)), \tag{1.14}$$

where C *depends on* N, p, Ω *and* a.

Remark 1.2. (a) The nonlinearity $|u|^{p-1}u$ in (1.1) can be replaced by $f(u)$ with

$$\lim_{v\to\infty} \frac{f(v)}{v^p} = l > 0.$$

Then (1.14) holds with C depending on N, f, Ω and a. Also, we can add lower order terms to the right hand side, that is, we can add a function $g\colon \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{u\to\infty} \sup_{(x,t)\in\Omega\times(0,T)} \frac{g(x,t,u)}{u^p} = 0.$$

Then (1.14) holds with C depending on N, p, Ω, a and g.

(b) For the blowing-up solutions ($T_{\max} < \infty$) of (1.8) one has (cf. [28, Proposition 23.1]) $\sup_{x\in\mathbb{R}^N} u(x,t) \geqslant C(T_{\max} - t)^{-1/(p-1)}$. This shows the optimality of the final blow up estimate in (1.11) for $a \equiv 1$. However, it is not known whether or not the weaker estimate (1.14) is optimal for sign changing a. Below, we show that under additional assumptions the stronger estimate (1.11) holds true even if a changes sign.

(c) If a also depends on t and $p > (N+2)/N$, the initial blow-up estimate in (1.14) does not hold even if $0 \leqslant a \leqslant 1$ (see e.g. [32], [33]). If Ω is bounded, then (1.13) is equivalent to $\nabla a(x_0)/|\nabla a(x_0)| \neq \nu_\Omega(x_0)$ for any $x_0 \in \Gamma \cap \partial\Omega$. It is not known if this assumption is technical or not.

(d) Universal estimates of the form (1.11) or (1.14) are not true for $p \geqslant p_S$, $N \geqslant 3$, $\Omega = \mathbb{R}^N$, due to the existence of arbitrarily large stationary radial solutions of (1.1). We require $p < p_B < p_S$ mainly because the Liouville theorem for the problem

$$u_t = \Delta u + u^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.15}$$

with $p_B \leqslant p < p_S$ is not known. If one proved such a Liouville theorem for some $p \in [p_B, p_S)$, then the conclusion of Theorem 1.1 would hold for the same p as well.

(e) If we restrict ourselves to the class of radial solutions (of course now Ω and a are radially symmetric), then similarly to [26], one can prove Theorem 1.1 for each $1 < p < p_S$. This is possible, since the Liouville theorem is known for nonnegative radial solutions of (1.15) for any $1 < p < p_S$ (see [24]).

(f) If a nonnegative solution u of (1.1) is global ($T_{\max} = \infty$), then after letting $T \to \infty$ in (1.14) we obtain

$$u(x,t) \leqslant C\big(1 + t^{-3/(2(p-1))}\big) \qquad ((x,t) \in \Omega \times (0,\infty)). \tag{1.16}$$

In particular, u is bounded on $\Omega \times (1,\infty)$. For previous results, see [5], [26].

Remark 1.3. Observe that (1.14) is equivalent to

$$M(x,t) \leqslant C(1+d^{-1}(t)) \qquad ((x,t)\in\Omega\times(0,T)), \tag{1.17}$$

where

$$M := u^{(p-1)/3} \quad \text{and} \quad d(t) := \min\{t, T-t\}^{1/2}.$$

Also, for each $x\in\Omega$, one has $d(t) = d_P[(x,t),\Theta]$, where $\Theta := \Omega\times\{0,T\}$ and d_P denotes the parabolic distance:

$$d_P[(x,t),(y,s)] = |x-y| + |t-s|^{\frac{1}{2}} \qquad ((x,t),(y,s)\in\Omega\times(0,T)). \tag{1.18}$$

In this notation we obtain yet another form of (1.14):

$$u(x,t) \leqslant C\big(1 + d_P^{-3/(p-1)}[(x,t),\Theta]\big) \qquad ((x,t)\in\Omega\times(0,T)).$$

If u is a stationary solution of (1.1), that is, if u solves

$$\begin{aligned} 0 &= \Delta u + a(x)|u|^{p-1}u, && x\in\Omega,\\ u &= 0, && x\in\partial\Omega, \end{aligned} \tag{1.19}$$

we obtain the following corollary.

Corollary 1.4. *Let $\Omega\subset\mathbb{R}^N$ be a uniformly regular domain of class C^2 (cf. [2]), $1<p<p_S$, and let $a\in C^2(\bar\Omega)$ satisfy* (1.3) *and* (1.13). *If u is a nonnegative solution of* (1.19), *then $u\leqslant C(p,N,\Omega,a)$.*

This corollary extends the results of Du and Li [7] (see also references therein), as it allows a to vanish on $\partial\Omega$. If $1<p<p_B(N)$, then since $T_{\max}=\infty$, Corollary 1.4 follows from (1.16). If we merely assume $1<p<p_S(N)$, then one has to reprove Theorem 1.1 for solutions of (1.19). The only difference is the application of elliptic Liouville theorems [12], instead of parabolic ones, whenever $p<p_B$ is required.

The next propositions shows that final blow-up rates in Theorem 1.1 (and the main results in [36]) can be improved if $a>0$ and Ω is a convex bounded set. Notice that a is allowed to vanish on $\partial\Omega$. In this case, the universal bounds (1.12) were already obtained in [27].

Proposition 1.5. *Let $\Omega\subset\mathbb{R}^N$ be a bounded, smooth, convex set and let $1<p<p_B$. Assume $a\in C^2(\bar\Omega)$ satisfies* (1.7) *and $a(x)>0$ for $x\in\Omega$. Then a nonnegative solution u of* (1.1), (1.2) *satisfies*

$$u(x,t) \leqslant C\big(1+(T-t)^{-1/(p-1)}\big) \qquad ((x,t)\in\Omega\times(0,T)), \tag{1.20}$$

where C depends on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.

If a changes sign in Ω, we formulate sufficient conditions for (1.20) only in the one-dimensional case. However, one can generalize the following propositions to the higher dimensional case if Ω is convex and certain monotonicity of a and u_0 near $\partial\Omega$ is assumed.

Proposition 1.6. *Let $N = 1$ and $\Omega = (0, 1)$. Suppose that $a \in C^2([0, 1])$ and has exactly one nondegenerate zero $\mu \in [0, 1]$, that is, $a(\mu) = 0$ and $a'(\mu) \neq 0$. If*

$$\operatorname{sign}[a(x)](u_0(2\mu - x) - u_0(x)) \leqslant 0 \qquad (x \in (\max\{0, 2\mu - 1\}, \mu)),$$

then a nonnegative classical solution u of (1.1), (1.2) *satisfies* (1.20) *with C depending on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.*

Proposition 1.7. *Let $N = 1$ and $\Omega = (0, 1)$. Suppose that $a \in C^2([0, 1])$ and has exactly two nondegenerate zero $\mu_1 < \mu_2$ in $[0, 1]$, that is, $a(\mu_i) = 0$ and $a'(\mu_i) \neq 0$ for $i = 1, 2$. If $\max\{\mu_1, 1 - \mu_2\} < \mu_2 - \mu_1$ and*

$$\begin{aligned} a(x) < 0, \quad u_0(2\mu_1 - x) &\geqslant u_0(x) \qquad (x \in (0, \mu_1)), \\ u_0(2\mu_2 - x) &\geqslant u_0(x) \qquad (x \in (\mu_2, 1)), \end{aligned}$$

then a nonnegative classical solution u of (1.1), (1.2) *satisfies* (1.20) *with C depending on N, p, Ω, a, T and $\|u_0\|_{L^\infty(\Omega)}$.*

One can also employ Liouville theorems and universal estimates in the investigation of the complete blow-up and the continuity of the blow-up time. Let us recall these notions and explain the results.

Let u be a nonnegative solution of (1.1), (1.2) with $T_{\max} < \infty$. Let u_k ($k \in \mathbb{N}$) be the solution of the approximation problem

$$\begin{aligned} (u_k)_t - \Delta u_k &= f_k(x, u_k), & (x, t) &\in \Omega \times (0, \infty), \\ u_k &= 0, & (x, t) &\in \partial\Omega \times (0, \infty), \\ u_k(x, 0) &= u_0(x) \geqslant 0, & x &\in \Omega, \end{aligned}$$

where

$$f_k(x, v) := \begin{cases} a(x)\min\{v^p, k\} & \text{if } a(x) \geqslant 0,\ v \in \mathbb{R}, \\ a(x)v^p & \text{if } a(x) < 0,\ v \in \mathbb{R}. \end{cases}$$

Since f_k is bounded from above, the nonnegative solution u_k exists globally (for all positive times). Since $f_k \leqslant f_{k+1}$, the maximum principle implies $u_{k+1}(x, t) \geqslant u_k(x, t)$ for any $(x, t) \in \Omega \times (0, \infty)$. Thus

$$\bar{u}(x, t) := \lim_{k\to\infty} u_k(x, t) \in [0, \infty] \qquad ((x, t) \in \Omega \times [0, \infty))$$

is well defined. Moreover, $\bar{u}(x,t) = u(x,t)$ for any $(x,t) \in \bar{\Omega} \times [0, T_{\max})$. We say that u *blows-up completely* in $D \subset \Omega$ at T, if $\bar{u}(x,t) = \infty$ for any $x \in D$ and $t > T$.

Theorem 1.8. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $1 < p < p_B$. Suppose that $a \in C^2(\bar{\Omega})$ satisfies* (1.7) *and* (1.13)*. If $T_{\max} < \infty$ for a nonnegative solution u of* (1.1), (1.2), *then u blows-up completely in Ω^+ at $T_{\max}$. In addition, the function*

$$T\colon \{u_0 \in L^\infty(\Omega)\colon u_0 \geqslant 0\} \to (0, \infty], \qquad T\colon u_0 \mapsto T_{\max}(u_0)$$

is continuous.

If $a \equiv 1$, Baras and Cohen [4] proved complete blow-up of nonnegative solutions of (1.8), (1.2) in Ω at $T_{\max} < \infty$ for each $1 < p < p_S$ (see also [28]). However, for $p > p_S$, $N \leqslant 10$, and Ω being a ball, there exist radial solutions of (1.8) that do not blow-up completely in Ω at $T_{\max}$. For further discussion see [28] and references therein.

If a changes sign, then one cannot expect the complete blow-up in the whole Ω, since $\bar{u}$ stays bounded in Ω^- for any $t > 0$ (see [20]). Quittner and Simondon [27] proved the complete blow-up of u in Ω^+ at $T_{\max} < \infty$ for $1 < p \leqslant 1 + 3/(N+1)$ and $\Gamma \subset \Omega$. Later Poláčik and Quittner [23] replaced the former assumption by $1 < p < p_B$ and proved Theorem 1.8 under an additional assumption $\Gamma \subset \Omega$.

The rest of the paper is organized as follows. In Section 2 we state and prove parabolic Liouville theorems. In Section 3 we formulate the doubling lemma and prove our main results.

2. Liouville theorems

Since some results in this section can be of independent interest, we formulate them in a more general setting than that required for the proofs of the main results. Let us define

$$\mathbb{R}^N_\lambda := \{x = (x_1, x') \in \mathbb{R}^N\colon\ x_1 > \lambda\} \qquad (\lambda \in \mathbb{R}), \tag{2.1}$$

$$H_\lambda := \partial\mathbb{R}^N_\lambda = \{x = (x_1, x') \in \mathbb{R}^N\colon\ x_1 = \lambda\} \qquad (\lambda \in \mathbb{R}). \tag{2.2}$$

The following two lemmas were proved in [36] for increasing functions f. Here we propose simpler proofs that remove this unnecessary assumption. The elliptic counterparts can be found in [8], [30], [31], see also references therein.

Lemma 2.1. *Let f be a continuous function with $f(v) > 0$ for any $v > 0$. If $u\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonnegative bounded solution of*

$$u_t - \Delta u = -f(u), \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

then $u \equiv 0$.

P r o o f. We proceed by way of contradiction, that is, we assume $u \not\equiv 0$. Fix $(x^*, t^*) \in \mathbb{R}^N \times \mathbb{R}$ such that

$$u(x^*, t^*) \geqslant C^* := \frac{1}{2} \sup_{(x,t)\in\mathbb{R}^N\times\mathbb{R}} u(x,t) > 0.$$

For each $\varepsilon > 0$ denote

$$v_\varepsilon(x,t) := u(x,t) - \varepsilon|x - x^*|^2 - \varepsilon\bigl(\sqrt{(t-t^*)^2+1} - 1\bigr) \qquad ((x,t) \in \mathbb{R}^N \times \mathbb{R}).$$

Since $v_\varepsilon(x,t) \to -\infty$ whenever $|t| \to \infty$ or $|x| \to \infty$, there exists $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^N \times \mathbb{R}$ with

$$v_\varepsilon(x_\varepsilon, t_\varepsilon) = \sup_{(x,t)\in\mathbb{R}^N\times\mathbb{R}} v_\varepsilon(x,t).$$

Then for each $\varepsilon > 0$

$$2C^* \geqslant u(x_\varepsilon, t_\varepsilon) \geqslant v_\varepsilon(x_\varepsilon, t_\varepsilon) \geqslant v_\varepsilon(x^*, t^*) = u(x^*, t^*) \geqslant C^* > 0,$$

and

$$(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) = 0, \qquad \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \leqslant 0.$$

Consequently,

$$\begin{aligned}
0 &\leqslant (v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) - \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon)\\
&= u_t(x_\varepsilon, t_\varepsilon) - \Delta u(x_\varepsilon, t_\varepsilon) - \varepsilon\frac{t_\varepsilon - t^*}{\sqrt{(t_\varepsilon - t^*)^2 + 1}} + 2\varepsilon N\\
&= -f(u(x_\varepsilon, t_\varepsilon)) - \varepsilon\frac{t_\varepsilon - t^*}{\sqrt{(t_\varepsilon - t^*)^2 + 1}} + 2\varepsilon N\\
&\leqslant -\inf_{2C^*\geqslant v\geqslant C^*} f(v) + \varepsilon + 2\varepsilon N \qquad (\varepsilon > 0).
\end{aligned}$$

Since the first term on the right hand side is negative and independent of ε, we obtain a contradiction for sufficiently small $\varepsilon > 0$. □

Lemma 2.2. *Suppose $f \in C^1$ satisfies $f(0) = 0$ and $f(v) > 0$ for any $v > 0$. Let h be a continuous function with $h(x_1) < 0$ for each $x_1 > 0$, and let $\limsup_{x_1 \to \infty} h(x_1) < 0$. If u is a nonnegative bounded solution of the problem*

$$\begin{aligned} u_t - \Delta u &= h(x_1) f(u), \qquad && (x,t) \in \mathbb{R}^N_0 \times \mathbb{R}, \\ u &= 0, && (x,t) \in H_0 \times \mathbb{R}, \end{aligned}$$

then $u \equiv 0$.

Proof. The proof is similar to that of Lemma 2.1. We again proceed by a contradiction, that is, we assume $u \not\equiv 0$. Fix $(x^*, t^*) \in \mathbb{R}^N_0 \times \mathbb{R}$ such that

$$u(x^*, t^*) \geqslant C^* := \frac{1}{2} \sup_{(x,t) \in \mathbb{R}^N_0 \times \mathbb{R}} u(x,t) > 0.$$

It is easy to see that there exists a function $\varphi \in C^2(\mathbb{R}^N \times \mathbb{R})$ with

$$\begin{gathered} \varphi(x,t) \geqslant 0, \quad |\nabla \varphi(x,t)| \leqslant 1, \quad |\varphi_t - \Delta \varphi| \leqslant 1 \qquad ((x,t) \in \mathbb{R}^N \times \mathbb{R}), \\ \varphi(0,0) = 0, \quad \varphi(x,t) \to \infty \quad \text{if} \quad |x| \to \infty \quad \text{or} \quad t \to \pm\infty. \end{gathered}$$

For each $\varepsilon \in (0,1)$ denote

$$v_\varepsilon(x,t) := u(x,t) - \varepsilon \varphi(x - x^*, t - t^*) \qquad ((x,t) \in \mathbb{R}^N_0 \times \mathbb{R}).$$

Since u is bounded, $v_\varepsilon(x,t) \to -\infty$ whenever $|t| \to \infty$ or $|x| \to \infty$. Moreover, $v_\varepsilon(x,t) \leqslant 0 < v_\varepsilon(x^*, t^*)$ for any $(x,t) \in H_0 \times \mathbb{R}$, and therefore there exists $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^N_0 \times \mathbb{R}$ such that

$$v_\varepsilon(x_\varepsilon, t_\varepsilon) = \sup_{(x,t) \in \mathbb{R}^N_0 \times \mathbb{R}} v_\varepsilon(x,t).$$

Consequently,

$$2C^* \geqslant u(x_\varepsilon, t_\varepsilon) \geqslant v_\varepsilon(x_\varepsilon, t_\varepsilon) \geqslant v_\varepsilon(x^*, t^*) = u(x^*, t^*) \geqslant C^* > 0,$$

and

$$(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) = 0, \qquad (\Delta v_\varepsilon)(x_\varepsilon, t_\varepsilon) \leqslant 0.$$

Observe that u satisfies

$$u_t = \Delta u + h(x_1) \frac{f(u)}{u} u = \Delta u + c(x,t) u.$$

Since $f \in C^1$, $f(0) = 0$, and u is bounded, c is a bounded function in $\{(x,t) \in \mathbb{R}_0^N \times \mathbb{R} : x_1 < 2\}$. Hence, standard parabolic regularity (see for example [19, Theorem 1.15]) implies

$$|\nabla u(x,t)| \leqslant C \qquad ((x,t) \in \bar{\mathbb{R}}_0^N \times \mathbb{R},\ x_1 < 1),$$

and consequently,

$$|\nabla v_\varepsilon(x,t)| \leqslant C + 1 \qquad ((x,t) \in \bar{\mathbb{R}}_0^N \times \mathbb{R},\ x_1 < 1),$$

where C is independent of $\varepsilon \in (0,1)$. Furthermore, $v_\varepsilon(x_\varepsilon, t_\varepsilon) \geqslant C^* > 0$ and $v_\varepsilon(x,t) \leqslant 0$ for all $(x,t) \in H_0 \times \mathbb{R}$ yield $\operatorname{dist}(x_\varepsilon, H_0) = (x_\varepsilon)_1 \geqslant c_0$, where c_0 is a constant independent of ε. Finally,

$$\begin{aligned}
0 &\leqslant (v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) - \Delta v_\varepsilon(x_\varepsilon, t_\varepsilon) \\
&= u_t(x_\varepsilon, t_\varepsilon) - \Delta u(x_\varepsilon, t_\varepsilon) - \varepsilon[\varphi_t(x_\varepsilon, t_\varepsilon) - \Delta\varphi(x_\varepsilon, t_\varepsilon)] \\
&\leqslant h((x_\varepsilon)_1) f(u(x_\varepsilon, t_\varepsilon)) + \varepsilon \\
&\leqslant \sup_{y \geqslant c_0} h(y) \inf_{2C^* \geqslant v \geqslant C^*} f(v) + \varepsilon.
\end{aligned}$$

Since the first term on the right hand side is negative and independent of ε, we obtain a contradiction for sufficiently small $\varepsilon > 0$. □

Next, consider the problem

$$\begin{aligned}
u_t - \Delta u &= h(x \cdot v) f(u), && (x,t) \in \Omega \times \mathbb{R}, \\
u &= 0, && (x,t) \in \partial\Omega \times \mathbb{R},
\end{aligned} \tag{2.3}$$

where

(v1) $v = (v_1, v_2, \ldots, v_N) \in \mathbb{R}^N$ is a unit vector with $v_1 > 0$ and $v_i = 0$ for $i \geqslant 3$.

About Ω, we assume that

(d1) Ω is a subset of $\mathbb{R}^N$, convex and unbounded in x_1, that is, $x + \xi e_1 \in \Omega$ for any $x \in \Omega$ and $\xi > 0$;
(d2) there is a constant d^* such that $x_2 v_2 \leqslant d^*$ for any $x = (x_1, x_2, \ldots, x_N) \in \Omega$.

Next, the function $h\colon \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis.

(h1) h is continuous, nondecreasing, and strictly increasing on $(0, \infty)$;
(h2) $h(0) = 0$ and $\lim_{y \to \infty} h(y) = \infty$.

About f we assume

(f1) $f \in C^1([0, \infty))$, with $f(0) = f'(0) = 0$, and $f(v) > 0$, $f'(v) \geqslant 0$ for each $v > 0$.

The following theorem is a generalization of elliptic [7] and parabolic [23] results proved for $v = e_1$ and $\Omega = \mathbb{R}^N$. The general framework of the proof is similar to one used in [7], [23].

Theorem 2.3. *If* (v1), (d1), (d2), (h1), (h2), *and* (f1) *hold true, then the only nonnegative, bounded solution* u *of* (2.3) *is* $u \equiv 0$.

As a corollary we obtain the Liouville theorem for indefinite problems on half spaces.

Corollary 2.4. *Given unit vectors* $b, v \in \mathbb{R}^N$ *and a constant* c^*, *let* $\Omega := \{x \in \mathbb{R}^N : x \cdot b > c^*\}$. *Consider functions* h *and* f *that satisfy* (h1), (h2), *and* (f1), *respectively. Let* u *be a nonnegative, bounded solution of* (2.3). *If* $v \neq -b$, *then* $u \equiv 0$.

Remark 2.5. The statement of Corollary 2.4 still holds true if $v = -b$, $c^* \geqslant 0$, and h in addition to (h1), (h2) satisfies $h(y) < 0$ for $y < 0$. This follows after suitable rotation and translation from Lemma 2.2. However, if $v = -b$ and $c^* < 0$, there are nontrivial, nonnegative solutions of (2.3). This result will be published elsewhere.

Proof of Corollary 2.4. We rotate the coordinates so that $b = e_2$, $v_1 \geqslant 0$, and $v_i = 0$ for $i \geqslant 3$. Then $\Omega = \{x \in \mathbb{R}^N : x_2 > c^*\}$ and (d1) holds true. Notice that (2.3), (h1), (h2), and (f1) are invariant under rotations.

If $v_1 > 0$ and $v_2 \leqslant 0$, then (v1) and (d2) are satisfied with $d^* = c^* v_2$, and the corollary follows from Theorem 2.3.

If $v_2 > 0$, consider another rotation that maps v to e_1 and fixes the space spanned by $\{e_3, \ldots, e_N\}$. Then (v1) and (d2) are clearly satisfied with $d^* = 0$. Also, Ω is transformed to $\Omega' := \{x \in \mathbb{R}^N : x \cdot b' > c^*\}$, where $b' = (v_2, v_1, 0, \ldots, 0)$. In particular, $b'_1 > 0$ and (d1) holds. Now, the corollary follows from Theorem 2.3.

If $v_1 = 0$ and $v_2 \leqslant 0$, then $v = -e_2 = -b$, a contradiction to our assumptions. □

Before we proceed, define $Lu := u_t - \Delta u$ and $M := \sup\limits_{\Omega} u$. Furthermore, given $\lambda \in \mathbb{R}$ set

$$
\begin{aligned}
\Sigma_\lambda &:= \{x \in \Omega : x_1 < \lambda\}, \\
x^\lambda &:= (2\lambda - x_1, x_2, \ldots, x_N) \qquad (x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N), \\
w_\lambda(x,t) &:= u(x^\lambda, t) - u(x,t) \qquad ((x,t) \in \bar{\Sigma}_\lambda \times \mathbb{R}), \\
\lambda(t) &:= \sup\{\mu : w_\lambda(x,t) \geqslant 0 \ \text{ for all } \ x \in \Sigma_\lambda \ \text{ and } \ \lambda < \mu\}, \\
\lambda^* &:= \inf\{\lambda(t) : t \in \mathbb{R}\}.
\end{aligned} \tag{2.4}
$$

Observe that (d1) implies $x^\lambda \in \bar{\Omega}$ for any $x \in \bar{\Sigma}_\lambda$, and therefore w_λ is well defined. Moreover, since u is nonnegative in Ω and vanishes on $\partial\Omega$,

$$w_\lambda(x,t) = u(x^\lambda,t) - u(x,t) = u(x^\lambda,t) \geqslant 0 \qquad ((x,t) \in (\partial\Omega \cap \bar{\Sigma}_\lambda) \times \mathbb{R}).$$

Clearly $w_\lambda(x,t) = 0$ if $(x,t) \in (\Omega \cap \partial\Sigma_\lambda) \times \mathbb{R}$, and therefore

$$w_\lambda(x,t) \geqslant 0 \qquad ((x,t) \in \partial\Sigma_\lambda \times \mathbb{R}). \tag{2.5}$$

We divide the proof of Theorem 2.3 into several lemmas, in which we implicitly suppose the assumptions of the theorem.

First, notice that $v_1 > 0$ implies

$$x^\lambda \cdot v - x \cdot v = 2(\lambda - x_1)v_1 \geqslant 0 \qquad (x \in \Sigma_\lambda), \tag{2.6}$$

and consequently by (h1)

$$h(x \cdot v) \leqslant h(x^\lambda \cdot v) \qquad (x \in \Sigma_\lambda). \tag{2.7}$$

Lemma 2.6. *If there are $\lambda \in \mathbb{R}$, $\tilde{x} \in \Sigma_\lambda$ and $\tilde{t} \in \mathbb{R}$ with $h(\tilde{x} \cdot v) \leqslant 0$ and $w_\lambda(\tilde{x},\tilde{t}) \leqslant 0$, then $Lw_\lambda(\tilde{x},\tilde{t}) \geqslant 0$. Moreover, if $\tilde{x}_1 \leqslant -d^*/v_1$, then $w_\lambda(\tilde{x},\tilde{t}) \leqslant 0$ implies $Lw_\lambda(\tilde{x},\tilde{t}) \geqslant 0$.*

Proof. The positivity and monotonicity of f, together with (2.7) yields

$$\begin{aligned} Lw_\lambda(\tilde{x},\tilde{t}) &= h(\tilde{x}^\lambda \cdot v)f(u(\tilde{x}^\lambda,\tilde{t})) - h(\tilde{x} \cdot v)f(u(\tilde{x},\tilde{t})) \\ &\geqslant h(\tilde{x} \cdot v)[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x},\tilde{t}))] \geqslant 0, \end{aligned}$$

and the first statement follows. Next, assume $\tilde{x}_1 \leqslant -d^*/v_1$. Then $v_1 > 0$ and (d2) imply

$$\tilde{x} \cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \leqslant \tilde{x}_1 v_1 + d^* \leqslant 0,$$

and by (h1) and (h2) one has $h(\tilde{x} \cdot v) \leqslant 0$. Now, the second statement follows from the first one. □

Lemma 2.7. *$\lambda(t) \geqslant -d^*/v_1$ for all $t \in \mathbb{R}$.*

Proof. We proceed by a contradiction, that is, we assume the existence of $\lambda < -d^*/v_1$ and $(\tilde{x},\tilde{t}) \in \Sigma_\lambda \times \mathbb{R}$ with $w_\lambda(\tilde{x},\tilde{t}) < 0$. Then $Lw_\lambda(\tilde{x},\tilde{t}) \geqslant 0$ by the second statement of Lemma 2.6. One can easily verify that for any sufficiently smooth function $g\colon (-\infty,\lambda] \to (0,\infty)$

$$g(x_1)L\overline{w}_\lambda(x,t) = Lw_\lambda(x,t) + 2(\partial_{x_1}\overline{w}_\lambda(x,t))g'(x_1) + \overline{w}_\lambda(x,t)g''(x_1) \qquad ((x,t) \in \Sigma_\lambda \times (0,\infty)), \tag{2.8}$$

where $\overline{w}_\lambda(x,t) := w_\lambda(x,t)/g(x_1)$. If we set

$$g(y) := \ln(\lambda+1-y)+1 \qquad (y \in (-\infty,\lambda]),$$

then $g>0$ and for already fixed $\tilde{x}$ and $\tilde{t}$ we have

$$L\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant 2(\partial_{x_1}\overline{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)} + \overline{w}_\lambda(\tilde{x},\tilde{t})\frac{g''(\tilde{x}_1)}{g(\tilde{x}_1)}. \tag{2.9}$$

Consider the solution of the problem

$$\begin{aligned} z_t - z_{yy} &= F(y,z,z_y), && (y,t) \in \mathbb{R}\times(0,\infty), \\ z(y,0) &= -M, && y \in \mathbb{R}, \end{aligned} \tag{2.10}$$

where

$$F(y,z,z_y) = \begin{cases} 2z_y g'/g & y < \lambda-1, \\ 2z_y g'/g - az & y \in [\lambda-1,\lambda], \\ 0 & y > \lambda, \end{cases}$$

and $a := -g''(\lambda-1)/g(\lambda-1) > 0$. Then the maximum principle implies $z(y,t) < 0$ for all $(y,t) \in \mathbb{R}\times(0,\infty)$, and since $F(y,-M,0) \geqslant 0$, z is increasing in t. Also, for any $T \geqslant 0$ the function $Z\colon (x,t) \mapsto z(x_1, t+T)$ satisfies

$$L[Z] \leqslant 2\frac{g'(x_1)}{g(x_1)}\partial_{x_1}Z + \frac{g''(x_1)}{g(x_1)}Z \qquad ((x,t) \in \mathbb{R}^N\times(0,\infty),\ x_1<\lambda).$$

Then the maximum principle on the set where $\overline{w}_\lambda \leqslant 0$ yields $\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant Z(\tilde{x},\tilde{t}) = z(\tilde{x}_1, \tilde{t}+T)$ for any $T>0$.

Since z is increasing in t, $\tilde{z}(y) := \lim_{t\to\infty} z(y,t)$ is well defined for each $y \in \mathbb{R}$ and

$$-\tilde{z}_{yy} = F(y,\tilde{z},\tilde{z}_y), \qquad y \in \mathbb{R}.$$

An analysis of this problem (for details see [23, Claim 2]) implies $\tilde{z} \equiv 0$. Thus, $\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant z(\tilde{x}_1,\tilde{t}+T) \to 0$ as $T\to\infty$, a contradiction. □

Lemma 2.8. *The mapping $t \mapsto \lambda(t)$ is nondecreasing. If $\lambda(t_1)=\infty$, this means that $\lambda(t_2)=\infty$ for all $t_2 \geqslant t_1$.*

P r o o f. Fix $t_0 \in \mathbb{R}$ and $\lambda < \lambda(t_0)$. Then

$$w_\lambda(x,t_0) \geqslant 0 \qquad (x \in \Sigma_\lambda),$$

and by (2.5)

$$w_\lambda(x,t) \geqslant 0 \qquad ((x,t) \in \partial\Sigma_\lambda \times [t_0,\infty)).$$

Next, (2.7) and the mean value theorem imply

$$\begin{aligned} Lw_\lambda(x,t) &= h(x^\lambda\cdot v)f(u(x^\lambda,t)) - h(x\cdot v)f(u(x,t)) \\ &\geqslant h(x\cdot v)[f(u(x^\lambda,t)) - f(u(x,t))] \\ &= h(x\cdot v)f'(\theta(x,t))w_\lambda(x,t), \qquad (x,t)\in\Sigma_\lambda\times[t_0,\infty), \end{aligned}$$

where $\theta(x,t)$ is a number between $u(x,t)$ and $u(x^\lambda,t)$. In particular, $\theta\colon (x,t)\mapsto [0,\infty)$ is a bounded function. Since by (d2)

$$x\cdot v = x_1v_1 + x_2v_2 \leqslant x_1v_1 + d^* \leqslant \lambda + d^* \qquad (x\in\Sigma_\lambda),$$

one has $h(x\cdot v)\leqslant h(\lambda+d^*)$ for each $x\in\Sigma_\lambda$. Now, the maximum principle, with the coefficient $c(x,t) := h(x\cdot v)f'(\theta(x,t))$ being possibly unbounded from below (see [6], [18]), gives $w_\lambda(x,t)\geqslant 0$ for all $(x,t)\in\Sigma_\lambda\times[t_0,\infty)$. Since $\lambda<\lambda(t_0)$ was chosen arbitrary, $\lambda(t)\geqslant\lambda(t_0)$ for each $t\geqslant t_0$. □

Lemma 2.9. $\lambda^*=\infty$, *or equivalently,* u *is nondecreasing in* x_1.

Proof. We proceed by contradiction, that is, we suppose $\lambda^*<\infty$. Lemma 2.7 guarantees $\lambda^*\geqslant -d^*/v_1$. By the definition of λ^* and by Lemma 2.8, there exist $\lambda_k\searrow\lambda^*$ and $t_k\searrow-\infty$ with

$$\inf_{x\in\Sigma_{\lambda_k}} w_{\lambda_k}(x,t_k) < 0.$$

Since u is bounded there is $M>0$ with $u\leqslant M$. Consequently, by (f1), there exists C_f such that $f'\leqslant C_f$ on $[0,M]$. Set $b_2 := h(\lambda^*v_1+d^*+1)C_f>0$ and choose $1>\delta>0$ with

$$2\delta^{-2}\geqslant 3^3(2b_2+1). \tag{2.11}$$

Since $f'(0)=0$, we can fix $\eta>0$ with

$$f'(z)\leqslant\frac{\delta}{h(\lambda^*+d^*+1)(\lambda^*+1+d^*/v_1)^3} \qquad (z\in[0,\eta]). \tag{2.12}$$

Let ε with $0<\varepsilon<\delta$ be sufficiently small (as specified below), and fix k such that $\lambda_k<\lambda^*+\varepsilon$. To simplify the notation set $\lambda:=\lambda_k$ and denote

$$\begin{aligned} g(y) &:= 2-\frac{\delta}{\delta+\lambda-y} \qquad (y\in(-\infty,\lambda]), \\ \overline{w}_\lambda(x,t) &:= \frac{w_\lambda(x,t)}{g(x_1)} \qquad ((x,t)\in\Sigma_\lambda\times\mathbb{R}). \end{aligned}$$

Observe that $g''(y) \leqslant 0$ and $g(y) > 0$ for any $y \leqslant \lambda$. For λ already fixed, define

$$S := \{(x,t) \in \Sigma_\lambda \times \mathbb{R}\colon\ w_\lambda(x,t) \leqslant 0\}.$$

Case 1. If $(\tilde{x},\tilde{t}) \in S$ with $\tilde{x}_1 < \lambda^* - \delta$ and $Lw_\lambda(\tilde{x},\tilde{t}) \geqslant 0$, then (2.8) and the concavity of g yield

$$L\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant 2(\partial_{x_1}\overline{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)}.$$

Case 2. If $(\tilde{x},\tilde{t}) \in S$ with $\tilde{x}_1 < \lambda^* - \delta$ and $Lw_\lambda(\tilde{x},\tilde{t}) < 0$, then Lemma 2.6 yields $h(\tilde{x}\cdot v) > 0$. Consequently, (h1) and (d2) yield

$$0 \leqslant \tilde{x}\cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \leqslant \tilde{x}_1 v_1 + d^* \leqslant \lambda^* + d^* + 1. \tag{2.13}$$

Also, Lemma 2.6 implies $\tilde{x}_1 > -d^*/v_1$, and therefore

$$\tilde{x}^\lambda \cdot v = (2\lambda - \tilde{x}_1)v_1 + \tilde{x}_2 v_2 \leqslant 2\lambda v_1 + 2d^* \leqslant 2\lambda^* + 2d^* + 1. \tag{2.14}$$

Now, (2.7) implies $h(\tilde{x}^\lambda \cdot v) \geqslant h(\tilde{x}\cdot v) > 0$ and (h1), (2.13), (2.14) yield

$$h(-1) \leqslant h(x\cdot v) \leqslant h(2(\lambda^* + d^*) + 2) \qquad ((x,t) \in \mathbb{R}^{N+1}, d_P[(x,t), S^*] < 1),$$

where d_P was defined in (1.18) and S^* is the convex hull of S and the set $\{(x^\lambda,t)\colon (x,t) \in S\}$. Next, the boundedness of u and standard local parabolic estimates give

$$|\nabla u(x,t)| \leqslant C_\lambda \qquad ((x,t) \in S^*).$$

Furthermore,

$$u(\tilde{x}^{\lambda^*},\tilde{t}) \geqslant u(\tilde{x},\tilde{t}) \geqslant u(\tilde{x}^\lambda,\tilde{t}) \tag{2.15}$$

and

$$|\tilde{x}^{\lambda^*} - \tilde{x}^\lambda| = |\tilde{x}_1^{\lambda^*} - \tilde{x}_1^\lambda| = 2(\lambda - \lambda^*) \leqslant 2\varepsilon.$$

Also, by (f1) and $h(\tilde{x}\cdot v) \geqslant 0$

$$\begin{aligned} 0 > Lw_\lambda(\tilde{x},\tilde{t}) &= h(\tilde{x}^\lambda\cdot v)f(u(\tilde{x}^\lambda,\tilde{t})) - h(\tilde{x}\cdot v)f(u(\tilde{x},\tilde{t})) \\ &\geqslant h(\tilde{x}^\lambda\cdot v)f(u(\tilde{x}^\lambda,\tilde{t})) - h(\tilde{x}\cdot v)f(u(\tilde{x}^{\lambda^*},\tilde{t})) \\ &= h(\tilde{x}^\lambda\cdot v)[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x}^{\lambda^*},\tilde{t}))] + [h(\tilde{x}^\lambda\cdot v) - h(\tilde{x}\cdot v)]f(u(\tilde{x}^{\lambda^*},\tilde{t})). \end{aligned} \tag{2.16}$$

Let us estimate each term separately. Since the segment connecting $\tilde{x}$ and $\tilde{x}^{\lambda^*}$ belongs to S^*, one has by (2.14), (2.15) and the definition of C_f and C_λ

$$\begin{aligned}(2.17)\quad h(\tilde{x}^\lambda\cdot v)[f(u(\tilde{x}^\lambda,\tilde{t}))-f(u(\tilde{x}^{\lambda^*},\tilde{t}))]\\ &\geqslant h(2(\lambda^*+d^*)+1)C_f(u(\tilde{x}^\lambda,\tilde{t})-u(\tilde{x}^{\lambda^*},\tilde{t}))\\ &\geqslant -2h(2(\lambda^*+d^*)+1)C_fC_\lambda\varepsilon.\end{aligned}$$

To estimate the second term, notice that $\tilde{x}_1\leqslant\lambda^*-\delta$ implies

$$\tilde{x}^\lambda\cdot v-\tilde{x}\cdot v=2(\lambda-\tilde{x}_1)v_1\geqslant 2(\lambda-\lambda^*+\delta)v_1\geqslant 2\delta v_1.$$

Thus by the monotonicity of h and (2.13) we have

$$(2.18)\qquad h(\tilde{x}^\lambda\cdot v)-h(\tilde{x}\cdot v)\geqslant\inf_{y\in[0,\lambda^*+d^*+1]}(h(y+2\delta v_1)-h(y))>0\,.$$

A substitution of (2.17) and (2.18) into (2.16) yields

$$0>-2h(2(\lambda^*+d^*)+1)C_fC_\lambda\varepsilon+\Big[\inf_{y\in[0,\lambda^*+d^*+1]}\big(h(y+2\delta v_1)-h(y)\big)\Big]f(u(\tilde{x}^{\lambda^*},\tilde{t})),$$

or equivalently,

$$f(u(\tilde{x}^{\lambda^*},\tilde{t}))<\frac{2h(2(\lambda^*+d^*)+1)C_fC_\lambda}{\inf\limits_{y\in[0,\lambda^*+d^*+1]}(h(y+2\delta v_1)-h(y))}\,\varepsilon.$$

Hence, by (f1) it follows that for sufficiently small $\varepsilon>0$ one has $u(\tilde{x}^{\lambda^*},\tilde{t})\leqslant\eta$, and for such ε, (2.12) holds true for any $z\in[0,u(\tilde{x}^{\lambda^*},\tilde{t})]$. Then (2.12), (2.13) and (2.15) imply

$$\begin{aligned}Lw_\lambda(\tilde{x},\tilde{t})&\geqslant h(\tilde{x}\cdot v)[f(u(\tilde{x}^\lambda,\tilde{t}))-f(u(\tilde{x},\tilde{t}))]\\ &\geqslant h(\lambda^*+d^*+1)\frac{\delta}{h(\lambda^*+d^*+1)(\lambda^*+1+d^*/v_1)^3}\,w_\lambda(\tilde{x},\tilde{t})\\ &=\frac{\delta}{(\lambda^*+1+d^*/v_1)^3}\,w_\lambda(\tilde{x},\tilde{t}).\end{aligned}$$

Easy calculations show that

$$\frac{\delta}{(\lambda^*+1+d^*/v_1)^3}\leqslant\frac{\delta}{(\delta+\lambda-y)^3}=-\frac{g''(y)}{2}\leqslant-\frac{g''(y)}{g(y)}\qquad\Big(y\in\Big[\frac{-d^*}{v_1},\lambda^*\Big]\Big),$$

and since $\tilde{x}_1\geqslant -d^*/v_1$,

$$Lw_\lambda(\tilde{x},\tilde{t})\geqslant\frac{\delta}{(\lambda^*+1+d^*/v_1)^3}\,w_\lambda(\tilde{x},\tilde{t})\geqslant-\frac{g''(\tilde{x}_1)}{g(\tilde{x}_1)}\,w_\lambda(\tilde{x},\tilde{t})=-g''(\tilde{x}_1)\overline{w}_\lambda(\tilde{x},\tilde{t}).$$

Consequently, (2.8) implies

$$L\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant 2(\partial_{x_1}\overline{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)}.$$

Case 3. Consider $(\tilde{x},\tilde{t}) \in S$ with $\tilde{x}_1 \in [\lambda^* - \delta, \lambda]$. Then by (d2)

$$\tilde{x}\cdot v = \tilde{x}_1 v_1 + \tilde{x}_2 v_2 \leqslant \lambda v_1 + d^* \leqslant \lambda^* v_1 + d^* + 1,$$

and therefore for b_2 and C_f already fixed we have

$$\begin{aligned} Lw_\lambda(\tilde{x},\tilde{t}) &\geqslant h(\tilde{x}\cdot v)[f(u(\tilde{x}^\lambda,\tilde{t})) - f(u(\tilde{x},\tilde{t}))] \geqslant h(\lambda^* v_1 + d^* + 1)C_f w_\lambda(\tilde{x},\tilde{t}) \\ &= b_2 w_\lambda(\tilde{x},\tilde{t}). \end{aligned}$$

Moreover, (2.11) implies

$$-g''(y) = \frac{2\delta}{(\delta+\lambda-y)^3} \geqslant 2b_2 + 1 \geqslant g(y)b_2 + 1 \qquad (y \in [\lambda^* - \delta, \lambda]).$$

After a substitution into the previous estimate and then into (2.8), we obtain

$$L\overline{w}_\lambda(\tilde{x},\tilde{t}) \geqslant 2(\partial_{x_1}\overline{w}_\lambda(\tilde{x},\tilde{t}))\frac{g'(\tilde{x}_1)}{g(\tilde{x}_1)} - \frac{\overline{w}_\lambda(\tilde{x},\tilde{t})}{g(\tilde{x}_1)}.$$

The rest of the proof uses the comparison principle similarly to Lemma 2.7, for more details see [23, Proof of Claim 4]. □

P r o o f of Theorem 2.3. We proceed by a contradiction, that is, we assume $M := \|u\|_{L^\infty(\Omega\times\mathbb{R})} > 0$. Then by the continuity of u, there are $t_0 \in \mathbb{R}$ and a smooth bounded domain $K_0 \subset \Omega$ with $|K_0| \leqslant 1$ (here $|K_0|$ denotes the Lebesgue measure of K_0) such that $u(x,t_0) > 0$ for all $x \in K_0$. Define

$$K_\sigma := \{x + \sigma e_1 \colon x \in K_0\} \qquad (\sigma \geqslant 0).$$

Since Ω is convex and unbounded in x_1, one has $K_\sigma \subset \Omega$ for all $\sigma \geqslant 0$. Let $\mu > 0$ be the first eigenvalue of the problem

$$\begin{aligned} -\Delta\varphi_0 &= \mu\varphi_0, & x &\in K_0, \\ \varphi_0 &= 0, & x &\in \partial K_0, \end{aligned}$$

where the eigenfunction φ_0 is normalized so that $\max\limits_{K_0}\varphi_0 = 1$. Set

$$\varphi_\sigma(x) := \varphi_0(x_1 - \sigma, x') \qquad (x = (x_1, x') \in K_\sigma)$$

and

$$\psi_\sigma(t) := \int_{K_\sigma} u(x,t)\varphi_\sigma(x)\,\mathrm{d}x \qquad (t \in \mathbb{R}).$$

Since by Lemma 2.9 u is nondecreasing in x_1 and $u > 0$ in $K_0 \times \{t_0\}$,

$$\psi_\sigma(t_0) \geqslant \psi_0(t_0) =: c_0 > 0 \qquad (\sigma \geqslant 0).$$

Denote

$$K_\sigma^*(t) := \{x \in K_\sigma \colon\ u(x,t)\varphi_\sigma(x) \geqslant c_0/2\} \qquad (t \geqslant t_0).$$

If $\psi_\sigma(t^*) \geqslant c_0$ for some $t^* \geqslant t_0$, then (using $|K_\sigma| \leqslant 1$)

$$c_0 \leqslant \int_{K_\sigma} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \leqslant |K_\sigma^*(t^*)| \cdot M + \frac{c_0}{2}|K_\sigma| \leqslant |K_\sigma^*(t^*)| \cdot M + \frac{c_0}{2}.$$

Consequently, $|K_\sigma^*(t^*)| \geqslant \xi := c_0/(2M) > 0$. Next,

$$\begin{aligned}\int_{K_\sigma^*(t^*)} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \geqslant \xi\frac{c_0}{2} &\geqslant \xi \int_{K_\sigma \setminus K_\sigma^*(t^*)} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \\ &= \xi \int_{K_\sigma} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x - \xi \int_{K_\sigma^*(t^*)} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x.\end{aligned}$$

It follows that

$$\int_{K_\sigma^*(t^*)} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \geqslant \frac{\xi}{1+\xi}\int_{K_\sigma} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x = \frac{c_0}{2M+c_0}\psi_\sigma(t^*).$$

Since K is bounded, we can choose R such that K is a subset of the ball of radius R centered at the origin. Then for sufficiently large $\sigma \geqslant 0$

$$\begin{aligned} x \cdot v = x_1 v_1 + x_2 v_2 &\geqslant -|x_1 - \sigma| v_1 + v_1\sigma - R|v_2| \\ &\geqslant R(-v_1 - |v_2|) + v_1\sigma \geqslant \frac{1}{2}v_1\sigma \qquad (x \in K_\sigma).\end{aligned}$$

Hence, for sufficiently large $\sigma \geqslant 0$, using (h2) one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\psi_\sigma(t^*) &= \int_{K_\sigma} \Delta u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x + \int_{K_\sigma} h(x\cdot v) f(u(x,t^*))\varphi_\sigma(x)\,\mathrm{d}x \\ &\geqslant \int_{K_\sigma} u(x,t^*)\Delta\varphi_\sigma(x)\,\mathrm{d}x + h\left(\frac{1}{2}v_1\sigma\right)\int_{K_\sigma} f(u(x,t^*))\varphi_\sigma(x)\,\mathrm{d}x \\ &\geqslant \int_{K_\sigma} u(x,t^*)\Delta\varphi_\sigma(x)\,\mathrm{d}x + h\left(\frac{1}{2}v_1\sigma\right)\int_{K_\sigma^*(t^*)} \frac{f(u(x,t^*))}{M} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \\ &\geqslant -\mu\psi_\sigma(t^*) + h\left(\frac{1}{2}v_1\sigma\right) f\left(\frac{c_0}{2}\right)\frac{1}{M}\int_{K_\sigma^*(t^*)} u(x,t^*)\varphi_\sigma(x)\,\mathrm{d}x \\ &\geqslant \psi_\sigma(t^*)\left[-\mu + h\left(\frac{1}{2}v_1\sigma\right) f\left(\frac{c_0}{2}\right)\frac{1}{M}\frac{c_0}{2M+c_0}\right] \\ &\geqslant \psi_\sigma(t^*).\end{aligned}$$

Thus, if $\psi_\sigma(t^*) \geqslant c_0$, then $\psi'_\sigma(t^*) \geqslant 0$, and consequently $\psi'_\sigma(t) \geqslant \psi_\sigma(t) \geqslant c_0$ for each $t \geqslant t^*$. Since $\psi_\sigma(t_0) \geqslant c_0$, one has $\psi'_\sigma(t) \geqslant c_0 > 0$ for each $t > t_0$. Therefore $\psi_\sigma(t) \to \infty$ as $t \to \infty$, a contradiction to the boundedness of u. □

3. Proofs of main results

In this section we use the notation introduced in the previous sections. Especially, recall the definitions of $\mathbb{R}^N_\lambda$ (see (2.1)), H_λ (see (2.2)), x^λ (see (2.4)), and d_p (see (1.18)).

Our main technical tools are the following doubling lemmas.

Lemma 3.1. *Let (X,d) be a compact metric space and let $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Set $\Theta := \Sigma \setminus D$. Also, let $M\colon D \to (0,\infty)$ be a bounded function on compact subsets of D, and fix a real $k > 0$. If $y \in D$ is such that*

$$M(y)d(y,\Theta) > 2k,$$

then there exists $x \in D$ such that

$$M(x)d(x,\Theta) > 2k, \quad M(x) \geqslant M(y),$$

and

$$M(z) \leqslant 2M(x) \qquad (z \in D \cap B^*(x, kM^{-1}(x))), \tag{3.1}$$

where $B^*(y,R) := \{x \in X\colon d^*(x,y) \leqslant R\}$ and $d^*(x,y) = |d(x,\Theta) - d(y,\Theta)|$.

Lemma 3.2. *The statement of Lemma 3.1 holds true if (X,d) is a complete metric space and $B^*(x,kM^{-1}(x))$ in (3.1) is replaced by $B(x,kM^{-1}(x))$, where $B(x,R) := \{x \in X\colon d(x,y) \leqslant R\}$.*

Lemma 3.2 was proved in [25, Lemma 5.1]. The proof of Lemma 3.1 is analogous to the proof of [25, Lemma 5.1]. One only replaces every d by d^* and uses compactness of X when passing to the limit.

Proof of Theorem 1.1. This proof is partly inspired by the proofs of the corresponding results in [7], [26], [36]. We use the equivalent formulation introduced in Remark 1.3. If (1.17) fails, then there exist $(T_k)_{k\in\mathbb{N}} \subset (0,\infty)$, a sequence $(u_k)_{k\in\mathbb{N}}$ of nonnegative solutions of (1.1) with T replaced by T_k, and $(y_k,s_k)_{k\in\mathbb{N}} \subset \Omega \times (0,T_k)$ such that

$$M_k(y_k,s_k) := u_k^{(p-1)/3}(y_k,s_k) > 2k(1 + d_k^{-1}(s_k)) \qquad (k \in \mathbb{N}),$$

where $d_k(s) := \min\{s, T_k - s\}^{1/2}$. Now, for each $k \in \mathbb{N}$, Lemma 3.2 with $X_k = \Sigma_k = \bar{\Omega} \times [0, T_k]$, $d = d_P$, $D_k = \bar{\Omega} \times (0, T_k)$ and $\Theta_k = \Omega \times \{0, T_k\}$ implies the existence of $(x_k, t_k) \in \bar{\Omega} \times (0, T_k)$ with

$$\begin{aligned} M_k(x_k, t_k) &\geqslant M_k(y_k, s_k) > 2k d_k^{-1}(t_k), \\ M_k(x_k, t_k) &\geqslant M_k(y_k, s_k) > 2k, \\ 2M_k(x_k, t_k) &\geqslant M_k(x, t) \qquad ((x,t) \in G_k), \end{aligned} \tag{3.2}$$

where

$$G_k := \{(x,t) \in \Omega \times (0, T_k)\colon\ d_P((x,t),(x_k,t_k)) < k\lambda_k\},$$

and

$$\lambda_k := M_k^{-1}(x_k, t_k) \to 0 \quad \text{as } k \to \infty.$$

Here we have used that $d_P((x,t), \Theta_k) = d_k(t)$ for each $(x,t) \in \Sigma_k$. By (3.2)

$$|t - t_k| < k^2 \lambda_k^2 < \frac{d_k^2(t_k)}{4} = \frac{1}{4} \min\{t_k, T_k - t_k\} \qquad ((x,t) \in G_k),$$

and therefore

$$\left\{ x \in \Omega\colon\ |x - x_k| < \frac{k\lambda_k}{2} \right\} \times \left(t_k - \frac{k^2\lambda_k^2}{4}, t_k + \frac{k^2\lambda_k^2}{4} \right) \subset G_k.$$

Since the function a is bounded, we can, after passing to a subsequence, assume that $\mathcal{A} := \lim_{k\to\infty} a(x_k)$ exists.

Case (1). First assume $\mathcal{A} \neq 0$. We define a sequence $(v_k)_{k\in\mathbb{N}}$ of rescaled copies of u as

$$v_k(x,t) := \lambda_k^{3/(p-1)} u\big(x_k + \lambda_k^{3/2} x, t_k + \lambda_k^3 t\big) \qquad ((x,t) \in D_k),$$

where

$$D_k := \left\{ x \in \lambda_k^{-3/2}(\Omega - x_k)\colon\ |x| < \frac{k}{2\lambda_k^{1/2}} \right\} \times \left(-\frac{k^2}{4\lambda_k}, \frac{k^2}{4\lambda_k} \right). \tag{3.3}$$

Then $v_k(0,0) = 1$ and, by (3.2), $0 \leqslant v_k(x,t) \leqslant 2$ for each $(x,t) \in D_k$. Moreover, v_k satisfies

$$(v_k)_t = \Delta v_k + a\big(x_k + \lambda_k^{\frac{3}{2}} x\big) v_k^p, \quad (x,t) \in D_k, \tag{3.4}$$

$$v_k = 0, \quad (x,t) \in \left\{ y \in \lambda_k^{-3/2}(\partial\Omega - x_k)\colon\ |y| < \frac{k}{2\lambda_k^{\frac{1}{2}}} \right\} \times \left(-\frac{k^2}{4\lambda_k}, \frac{k^2}{4\lambda_k} \right). \tag{3.5}$$

By passing to a suitable subsequence we may assume either

$$\text{(i)}\quad \frac{\operatorname{dist}(x_k,\partial\Omega)}{\lambda_k^{\frac{3}{2}}} \to \infty \qquad \text{or} \qquad \text{(ii)}\quad \frac{\operatorname{dist}(x_k,\partial\Omega)}{\lambda_k^{\frac{3}{2}}} \to c^* \geqslant 0.$$

If (i) holds, then (3.4), the L^p estimates, and Schauder's estimates yield a subsequence of $(v_k)_{k\in\mathbb{N}}$ converging in $C^{2+\sigma,1+\sigma/2}_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$, $\sigma \in (0,1)$ to a function v_∞ satisfying

$$(v_\infty)_t = \Delta v_\infty + \mathcal{A} v_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Moreover, $v_\infty(0,0) = 1$ and $v_\infty \leqslant 2$. However, if $\mathcal{A} > 0$ and $p < p_B(N)$ (for the definition of $p_B(N)$ see (1.10)) this contradicts [5, Remark 2.6]. If $\mathcal{A} < 0$ and $p > 1$ we have a contradiction to Lemma 2.1.

If (ii) holds, then after an application of a suitable orthogonal change of coordinates, the L^p estimates and Schauder's estimates yield a subsequence of $(v_k)_{k\in\mathbb{N}}$ converging in $C^{2+\sigma,1+\sigma/2}_{\mathrm{loc}}(\mathbb{R}^N_{c^*} \times \mathbb{R})$ to a function v_∞ satisfying

$$\begin{aligned} (v_\infty)_t &= \Delta v_\infty + \mathcal{A} v_\infty^p, && (x,t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}, \\ v_\infty &= 0, && (x,t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}, \end{aligned}$$

with $v_\infty(0,0) = 1$ and $v_\infty \leqslant 2$. However, if $\mathcal{A} > 0$ and $p < p_S(N) \leqslant p_B(N-1)$, then this contradicts [26, Theorem 2.1]. If $\mathcal{A} < 0$ and $p > 1$, we have a contradiction to Lemma 2.2.

Case (2). Assume $\mathcal{A} = 0$. Since a is bounded in $C^2(\bar{\Omega})$, we can assume, after passing to a subsequence, that there exists a vector $\mathcal{B} := \lim_{k\to\infty} \nabla a(x_k) \in \mathbb{R}^N$. Then (1.3) implies $\mathcal{B} \neq 0$.

If $(x_k)_{k\in\mathbb{N}}$ has a convergent subsequence, we can, after appropriate restriction, assume the existence of $x_\infty := \lim_{k\to\infty} x_k$. Then $\mathcal{A} = a(x_\infty) = 0$. Set $\tilde{z}_k := x_\infty$ and $V_k := \mathcal{V} := \Omega$ for each $k \in \mathbb{N}$

If $(x_k)_{k\in\mathbb{N}}$ has no convergent subsequence, we can assume $|x_k - x_l| \geqslant 3$ for each $k \neq l$. Let V_k be the connected component of $B_1(x_k) \cap \Omega$ containing x_k, where $B_1(y)$ is the unit ball centered at y. By [16, Lemma 6.37], there exists an extension of $a \in C^2(\overline{V}_k)$ to $C^2(\overline{B}_1(x_k))$, which we denote again by a. Since $V_k \cap V_l = \emptyset$ for $k \neq l$, the function a is well defined on $\mathcal{V} := \bigcup_{k\in\mathbb{N}} \overline{B}_1(x_k)$.

Denote $\tilde{\Gamma} := \{x \in \overline{\mathcal{V}} \colon a(x) = 0\}$. Since $a \in C^2(\mathcal{V})$, $\mathcal{A} = 0$, and $\mathcal{B} \neq 0$, there is $(\tilde{z}_k)_{k\in\mathbb{N}} \subset \tilde{\Gamma}$ with $|x_k - \tilde{z}_k| \to 0$ as $k \to \infty$. Define δ_k and $(z_k)_{k\in\mathbb{N}} \subset \tilde{\Gamma}$ such that

$$\delta_k := |z_k - x_k| = \operatorname{dist}(x_k, \tilde{\Gamma}) \leqslant |x_k - \tilde{z}_k| \to 0.$$

Then $a \in C^2(\mathcal{V})$ yields $\lim_{k\to\infty} \nabla a(z_k) = \lim_{k\to\infty} \nabla a(x_k) \neq 0$. Thus we may assume $|\nabla a(z_k)| \neq 0$, and therefore

$$\delta_k = \frac{|\nabla a(z_k)(x_k - z_k)|}{|\nabla a(z_k)|} \qquad (k \in \mathbb{N}).$$

Using that $z_k \in \tilde{\Gamma}$, that is, $a(z_k) = 0$, we obtain

$$a(x_k + \lambda_k x) = \nabla a(z_k)(x_k + \lambda_k x - z_k) + O(|\delta_k|^2 + \lambda_k^2 |x|^2). \tag{3.6}$$

We define a sequence $(w_k)_{k\in\mathbb{N}}$ of rescaled copies of u as

$$w_k(x,t) := \lambda_k^{3/(p-1)} u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \qquad ((x,t) \in \tilde{D}_k),$$

where

$$\tilde{D}_k := \left\{ x \in \lambda_k^{-1}(V_k - x_k) \colon \; |x| < \frac{k}{2} \right\} \times \left(-\frac{k^2}{4}, \frac{k^2}{4} \right).$$

Then $w_k(0,0) = 1$ and $0 \leqslant w_k(x,t) \leqslant 2$ for each $(x,t) \in \tilde{D}_k$, and w_k satisfies

$$(w_k)_t = \Delta w_k + \frac{1}{\lambda_k} a(x_k + \lambda_k x) w_k^p, \quad (x,t) \in \tilde{D}_k, \tag{3.7}$$

$$w_k = 0, \quad (x,t) \in \left\{ y \in \lambda_k^{-1}(\partial\Omega - x_k) \colon \; |y| < \frac{k}{2} \right\} \times \left(-\frac{k^2}{4}, \frac{k^2}{4} \right). \tag{3.8}$$

Hence, by (3.6),

$$(w_k)_t = \Delta w_k + \frac{1}{\lambda_k}\big[\nabla a(z_k)(x_k + \lambda_k x - z_k) + O(|\delta_k|^2 + \lambda_k^2 |x|^2)\big] w_k^p, \qquad (x,t) \in \tilde{D}_k. \tag{3.9}$$

Case (2a). Assume that there is a suitable subsequence of $(x_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{\nabla a(z_k)(x_k - z_k)}{\lambda_k} = \pm|\mathcal{B}| \lim_{k\to\infty} \frac{\delta_k}{\lambda_k} =: d^* \in \mathbb{R}.$$

By passing to a yet another subsequence we may assume that either

$$\text{(i)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k} \to \infty \qquad \textit{or} \qquad \text{(ii)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k} \to c^* \geqslant 0.$$

If (i) holds, then (3.9), L^p estimates, and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N \times \mathbb{R})$ that is a weak solution of the problem

$$(w_\infty)_t = \Delta w_\infty + (d^* + \mathcal{B} \cdot x) w_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

satisfying $w_\infty(0,0) = 1$, $0 \leqslant w_\infty \leqslant 2$. Standard regularity theory implies that w_∞ is in fact a classical solution. After a suitable orthogonal transformation and translation, we obtain a nontrivial nonnegative bounded solution of the problem

$$(w_\infty)_t = \Delta w_\infty \pm |\mathcal{B}| x_n w_\infty^p, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

a contradiction to [23, Theorem 1.1] for any $p > 1$.

If (ii) holds, then $\mathrm{dist}(x_k, \partial\Omega) \to 0$ as $k \to \infty$. After a suitable rotation we have $\nu_\Omega(x_k) \to -e_1$ as $k \to \infty$. Then (3.9), L^p estimates, and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N_{c^*} \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N_{c^*} \times \mathbb{R})$ that is a weak solution of the problem

$$\begin{aligned} (w_\infty)_t &= \Delta w_\infty + (d^* + \mathcal{B}\cdot x) w_\infty^p, \qquad && (x,t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}, \\ w_\infty &= 0, && (x,t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}, \end{aligned}$$

with $w_\infty(0,0) = 1$ and $0 \leqslant w_\infty \leqslant 2$. Standard regularity theory yields that w_∞ is in fact a classical solution. Also, $a \in C^2(\bar{\Omega})$, $\mathrm{dist}(x_k, \partial\Omega) \to 0$ and (1.13) imply

$$0 < \frac{\tilde{c}}{2} \leqslant \liminf_{k\to\infty} \left| \frac{\nabla a(x_k)}{|\nabla a(x_k)|} + e_1 \right| = \left| \frac{\mathcal{B}}{|\mathcal{B}|} + e_1 \right|.$$

Thus, $\mathcal{B}$ is not a multiple of $-e_1$. Now, after a suitable translation, we obtain a contradiction to Corollary 2.4 for any $p > 1$.

Case (2b). After passing to a subsequence, we may assume that

$$\lim_{k\to\infty} \frac{\nabla a(z_k)(x_k - z_k)}{\lambda_k} = \pm|\mathcal{B}| \lim_{k\to\infty} \frac{\delta_k}{\lambda_k} = \pm\infty.$$

Setting

$$y = \frac{x}{\alpha_k}, \quad s = \frac{t}{\alpha_k^2},$$

where

$$\alpha_k := \left(\frac{\lambda_k}{\delta_k |\nabla a(z_k)|} \right)^{\frac{1}{2}} = \left(\frac{\lambda_k}{|\nabla a(z_k)(x_k - z_k)|} \right)^{\frac{1}{2}} \to 0$$

we transform (3.9) to

$$\begin{aligned} (w_k)_s &= \Delta_y w_k + \frac{\alpha_k^2}{\lambda_k} a(x_k + \lambda_k \alpha_k y) w_k^p \\ &= \Delta_y w_k + \frac{\nabla a(z_k)(x_k - z_k + \lambda_k x) + O(\delta_k^2 + \lambda_k^2 |x|^2)}{|\nabla a(z_k)(x_k - z_k)|} w_k^p \\ &= \Delta_y w_k + [\pm 1 + \alpha_k^3 \nabla a(z_k) y + O(\delta_k + \alpha_k^4 \lambda_k |y|^2)] w_k^p, \qquad (y,s) \in \hat{D}_k, \end{aligned}$$

where

$$\hat{D}_k := \left\{ y \in (\lambda_k \alpha_k)^{-1}(\Omega - x_k)\colon\ |y| < \frac{k}{2\alpha_k} \right\} \times \left(-\frac{k^2}{4\alpha_k^2}, \frac{k^2}{4\alpha_k^2} \right).$$

Moreover, by (3.8)

$$w_k = 0, \quad (y, s) \in \left\{ y \in (\lambda_k \alpha_k)^{-1}(\partial\Omega - x_k)\colon\ |y| < \frac{k}{2\alpha_k} \right\} \times \left(-\frac{k^2}{4\alpha_k^2}, \frac{k^2}{4\alpha_k^2} \right).$$

By passing to a yet another subsequence, we may assume either

$$\text{(i)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k \alpha_k} \to \infty \qquad \text{or} \qquad \text{(ii)} \quad \frac{\operatorname{dist}(x_k, \partial\Omega)}{\lambda_k \alpha_k} \to c^* \geqslant 0.$$

If (i) holds, the L^p estimates and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N \times \mathbb{R})$ that is a weak solution of the problem

$$(w_\infty)_t = \Delta w_\infty \pm w_\infty^p, \qquad (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

and $w_\infty(0,0) = 1$, $0 \leqslant w_\infty \leqslant 2$. Standard regularity theory implies that w_∞ is a classical solution. However, this contradicts [5] (with "+" sign) for any $1 < p < p_B(N)$ and Lemma 2.1 (with "−" sign) for any $p > 1$.

If (ii) holds, then after a suitable orthogonal change of coordinates and a translation, the L^p estimates and standard imbeddings yield a subsequence of $(w_k)_{k\in\mathbb{N}}$ converging in $C_{\mathrm{loc}}(\mathbb{R}^N_{c^*} \times \mathbb{R})$ to a function $w_\infty \in C(\mathbb{R}^N_{c^*} \times \mathbb{R})$ that is a weak solution of the problem

$$\begin{aligned} (w_\infty)_t &= \Delta w_\infty \pm w_\infty^p, && (x, t) \in \mathbb{R}^N_{c^*} \times \mathbb{R}, \\ w_\infty &= 0, && (x, t) \in \partial\mathbb{R}^N_{c^*} \times \mathbb{R}, \end{aligned}$$

and $w_\infty(0,0) = 1$, $0 \leqslant w_\infty \leqslant 2$. Standard regularity theory implies that w_∞ is a classical solution. However, this contradicts [26, Theorem 2.1] (with "+" sign) for any $1 < p < p_S(N) \leqslant p_B(N-1)$ and Lemma 2.2 (with "−" sign) for any $p > 1$. □

Let us formulate a sufficient condition that guarantees (1.20).

Lemma 3.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $1 < p < p_B(N)$, and assume that $a \in C^2(\bar{\Omega})$. For a nonnegative classical solution u of* (1.1), (1.2) *define $x^*\colon (0, T) \to \Omega$ such that*

$$u(x^*(t), t) = \sup_{x\in\Omega} u(x, t) \qquad (t \in (0, T)).$$

If there exist $\varepsilon^* > 0$ and $t_0 \in [0, T]$ such that $\operatorname{dist}(x^*(t), \Gamma) \geqslant \varepsilon^*$ for each $t \in [t_0, T]$, then (1.20) *holds with C depending on N, p, Ω, a, $\|u_0\|_{L^\infty(\Omega)}$, ε^* and t_0.*

P r o o f. As in the proof of Theorem 1.1, we use the equivalent formulation introduced in Remark 1.3. Assume that (1.20) fails. Then there exist $(T_k)_{k\in\mathbb{N}} \subset (0,\infty)$, a sequence $(u_k)_{k\in\mathbb{N}}$ of nonnegative solutions of (1.1), and a sequence $(y_k, s_k)_{k\in\mathbb{N}} \subset \Omega \times (0, T_k)$ such that

$$\tilde{M}_k(y_k, s_k) > 2k(1 + d_k^{-1}(s_k)),$$

where

$$\tilde{M}_k := u_k^{(p-1)/2}, \qquad d_k(t) = \min\{t, T_k - t\}^{1/2}.$$

Now, Lemma 3.1 with compact $X_k = \Sigma_k = \bar{\Omega} \times [0, T_k]$, $D_k = \bar{\Omega} \times (0, T_k)$ and $\Theta_k = \bar{\Omega} \times \{0, T_k\}$ implies the existence of a sequence $(x'_k, t_k) \in \Omega \times (0, T_k)$ with

$$\begin{aligned} \tilde{M}_k(x'_k, t_k) &\geqslant \tilde{M}_k(y_k, s_k) > 2k d_k^{-1}(t_k), \\ \tilde{M}_k(x'_k, t_k) &\geqslant \tilde{M}_k(y_k, s_k) > 2k, \\ 2\tilde{M}_k(x'_k, t_k) &\geqslant \tilde{M}_k(x, t) \qquad ((x,t) \in G'_k), \end{aligned} \tag{3.10}$$

where

$$G'_k := \{(x,t) \in \Omega \times (0,T)\colon\ d_k^*((x,t), (x'_k, t_k)) < k\lambda'_k\},$$
$$d_k^*((x,t),(y,s)) := |d_k(t) - d_k(s)| \qquad ((x,t),(y,s) \in X_k),$$

and

$$\lambda'_k := \tilde{M}^{-1}(x'_k, t_k) \to 0 \quad \text{as } k \to \infty.$$

Observe that d_k^* does not depend on x, and therefore (3.10) remains true if we replace x'_k by $x_k := x^*(t_k)$ and G'_k by

$$G_k := \{(x,t) \in \Omega \times (0,T)\colon\ d_k^*((x,t),(x_k,t_k)) < k\lambda_k\} \subset G'_k,$$

where

$$\lambda_k := \tilde{M}^{-1}(x_k, t_k) \to 0.$$

By our assumptions $\lim\limits_{k\to\infty} a(x_k) \neq 0$. The rest of the proof is now the same as Case (1) in the proof of Theorem 1.1 (see also [26, Theorem 4.1]) with v_k replaced by

$$v_k(x,t) := \lambda^{2/(p-1)} u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \qquad ((x,t) \in D_k),$$

and D_k by

$$D_k := \left\{(x,t) \in \lambda_k^{-1}(\Omega - x_k)\colon\ |x| < \frac{k}{2}\right\} \times \left(-\frac{k^2}{2}, \frac{k^2}{2}\right).$$

□

P r o o f of Proposition 1.5. In the proof we implicitly assume that all constants depend on N, p, Ω, a, $\|u_0\|_{L^\infty(\Omega)}$ and T. Fix any $\xi \in \partial\Omega$ with $a(\xi) = 0$. Since Ω is convex, we can, after a suitable rotation, assume

$$\xi_1 = \sup_{x\in\Omega} x_1, \quad \text{and therefore} \quad \nu_\Omega(\xi) = e_1.$$

Since ξ is a local minimizer of a in $\bar{\Omega}$, all tangential derivatives of a vanish at ξ. Then (1.7) implies $\partial_{x_1} a(\xi) < 0$. Denote

$$\Omega_\lambda := \{x \in \Omega\colon\ x_1 > \lambda\}.$$

Assume $u \not\equiv 0$, otherwise the statement is trivial. Observe that u satisfies

$$u_t = \Delta u + \alpha(x,t)u, \qquad (x,t) \in \Omega \times (0,T),$$

where $\alpha(x,t) = a(x)u^{p-1}$. By Theorem 1.1, α is bounded on $\Omega \times (0, T/2)$ and the bound depends only on the constants implicitly assumed. Next, the Hopf boundary lemma (see [19, Lemma 2.6]) implies $\partial_{e_1} u(\xi, T/2) < 0$. By the convexity of Ω, we can choose $\lambda < \xi_1$, sufficiently close to ξ_1 such that

$$w_\lambda(x,t) := u(x^\lambda, t) - u(x,t) \qquad ((x,t) \in \Omega_\lambda \times (0,T))$$

is well defined (for the definition of x^λ and Ω_λ see (2.4)). Since $\partial_{x_1} u(\xi, T/2) < 0$ and $\partial_{x_1} a(\xi) < 0$, we can increase $\lambda < \xi_1$ such that

$$w_\lambda(x, T/2) > 0, \quad \text{and} \quad a(x^\lambda) > a(x) \qquad (x \in \Omega_\lambda).$$

Observe that $\xi_1 - \lambda \geqslant c_1 > 0$, where c_1 is independent of ξ. Since $a(x^\lambda) > a(x)$ for $x \in \Omega_\lambda$, w_λ satisfies

$$(w_\lambda)_t \geqslant \Delta w_\lambda + \alpha^*(x,t) w_\lambda \qquad (x,t) \in \Omega_\lambda \times (0,T),$$

where

$$\alpha^*(x,t) := a(x)\frac{u^p(x^\lambda,t) - u^p(x,t)}{u(x^\lambda,t) - u(x,t)} \qquad ((x,t) \in \Omega_\lambda \times (0,T))$$

is bounded on compact subintervals of $(0,T)$. Similarly to (2.5)

$$w_\lambda(x,t) \geqslant 0 \qquad ((x,t) \in \partial\Omega_\lambda \times (0,T)).$$

Now, the maximum principle implies $w_\lambda > 0$ in $\Omega_\lambda \times (T/2, T)$. Therefore $|x^*(t) - \xi| \geqslant c_0$ for each $t \in (T/2, T)$. Since c_0 is independent of ξ and $\Gamma \subset \partial\Omega$, one has

$$\operatorname{dist}(x^*(t), \Gamma) \geqslant \operatorname{dist}(x^*(t), \partial\Omega) \geqslant c_0 > 0 \qquad (t \in (T/2, T)),$$

and the statement of the proposition follows from Lemma 3.3. □

Lemma 3.4. *Let $N = 1$, $\Omega = (0,1)$ and fix $\mu \in [0, \frac{1}{2})$. Assume $a \in C^2([0,1])$ has exactly one nondegenerate zero $\mu \in [0, 2\mu]$. Also assume $a(x) < 0$ for $x \in [0,\mu)$ and*

$$u_0(x) \leqslant u_0(x^\mu) \qquad (x \in (0,\mu)). \tag{3.11}$$

If $u \not\equiv 0$ is a nonnegative solution of the problem (1.1), (1.2), *then $|x^*(t) - \mu| \geqslant c_0 > 0$ and c_0 depends on N, p, a, $\|u_0\|_{L^\infty((0,1))}$, T.*

Proof. For each $\lambda \in (0, \frac{1}{2})$, define $w_\lambda\colon (0,\lambda) \times (0,\infty) \to \mathbb{R}$ as $w_\lambda(x,t) := u(x^\lambda, t) - u(x,t)$. Since $a(x^\mu) \geqslant 0 \geqslant a(x)$ for each $x \in [0,\mu]$,

$$a(x^\mu)u^p(x^\mu,t) - a(x)u^p(x,t) \geqslant 0 \qquad ((x,t) \in [0,\mu] \times (0,T)).$$

Thus,

$$(w_\mu)_t - (w_\mu)_{xx} \geqslant 0 \qquad ((x,t) \in (0,\mu) \times (0,T)).$$

By (3.11)

$$w_\mu(x,0) = u_0(x^\mu) - u_0(x) \geqslant 0 \qquad (x \in (0,\mu)).$$

Since $u \not\equiv 0$, the maximum principle implies $u > 0$ in $(0,1) \times (0,T)$. Then similarly to (2.5)

$$w_\mu(0,t) > 0 \quad \text{and} \quad w_\mu(\mu,t) = 0 \qquad (t \in (0,T)).$$

Then by the maximum principle $w_\mu > 0$ in $(0,\mu) \times (0,T)$ and $\partial_x w_\mu(\mu,t) < 0$ for $t \in (0,T)$. Hence, for sufficiently small $\varepsilon_0 > 0$ we obtain

$$w_\lambda(x, T/2) \geqslant 0 \qquad (x \in (0,\lambda), \lambda \in [\mu, \mu + \varepsilon_0)).$$

As above one can show

$$w_\lambda(0,t) > 0 \quad \text{and} \quad w_\lambda(\lambda,t) = 0 \qquad (t \in (T/2, T)).$$

Since $a'(\mu) > 0$, we can decrease $\varepsilon_0 > 0$ to obtain $a(x^\lambda) \geqslant a(x)$ for each $x \in (0,\lambda)$ and each $\lambda \in [\mu, \mu + \varepsilon_0)$. Then

$$(w_\lambda)_t - \Delta w_\lambda \geqslant a(x)[u^p(x^\lambda,t) - u^p(x,t)] = c(x,t)w_\lambda \quad ((x,t) \in (0,\lambda) \times (t_0,T)),$$

where $c(x,t)$ is a continuous function on $[0,\lambda] \times [t_0,T)$ (possibly unbounded as $t \to T$). The maximum principle implies $w_\lambda(x,t) > 0$ for each $(x,t) \in (0,\lambda) \times (t_0,T)$. In particular, $x^*(t) \geqslant \lambda > \mu$ and therefore $|x^*(t) - \mu| \geqslant c_0 > 0$ for each $t \in (t_0,T)$. □

Proof of Proposition 1.7. Lemma 3.4 with $\mu = \mu_1$ implies $|x^*(t) - \mu_1| > \varepsilon^* > 0$. If we replace x by $1 - x$ and use Lemma 3.4 with $\mu = 1 - \mu_2$ again, we obtain $|x^*(t) - \mu_2| > \varepsilon^* > 0$. Now, the proposition follows from Lemma 3.3. □

P r o o f of Proposition 1.6. Without loss of generality assume $a(0) \leqslant 0$, otherwise replace x by $1-x$. If $\mu < \frac{1}{2}$, then the proposition follows from Lemma 3.4 and Lemma 3.3. Assume $\mu \in [\frac{1}{2}, 1]$. Similarly to the proof of Lemma 3.4, we can show that $w_\mu(x,t) := u(x^\mu, t) - u(x,t)$ is well defined on $[\mu, 1]$ and satisfies

$$w_\mu(x,t) < 0 \qquad ((x,t) \in (\mu,1) \times (0,T)) \quad \text{and} \quad w'_\mu(\mu,t) < 0 \qquad (t \in (0,T)).$$

Hence, for $\lambda > \mu$ sufficiently close to μ we have $w_\lambda(x, T/2) < 0$ for any $x \in (\lambda, 1)$. Similarly to Lemma 3.4 (using the maximum principle), we prove $w_\lambda(x,t) < 0$ for any $(x,t) \in (\lambda,1) \times (T/2, T)$. Consequently, $|x^*(t) - \mu| > \lambda - \mu > 0$ for all $t \in (T/2, T)$ and the proposition follows from Lemma 3.3. □

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