Charles S. Kahane

A note on the convolution theorem for the Fourier transform

*Czechoslovak Mathematical Journal*, Vol. 61 (2011), No. 1, 199–207

Persistent URL: http://dml.cz/dmlcz/141527

**Czechoslovak Mathematical Journal, 61 (136) (2011), 199–207**

# A NOTE ON THE CONVOLUTION THEOREM FOR THE FOURIER TRANSFORM

Charles S. Kahane, Nashville

(Received November 5, 2009)

*Abstract.* In this paper we characterize those bounded linear transformations $Tf$ carrying $L^1(\mathbb{R}^1)$ into the space of bounded continuous functions on $\mathbb{R}^1$, for which the convolution identity $T(f*g) = Tf \cdot Tg$ holds. It is shown that such a transformation is just the Fourier transform combined with an appropriate change of variable.

*Keywords*: convolution, Fourier transform

*MSC 2010*: 42A38, 39B22

## 1. Introduction

The purpose of this note is to answer the following question: To what extent does the convolution theorem characterize the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}x\xi} f(x)\,\mathrm{d}x$$

on $L^1(\mathbb{R}^1)$? More precisely, suppose $Tf$ is a bounded linear transformation sending $L^1(\mathbb{R}^1)$ into the space $C_b(\mathbb{R}^1)$ of uniformly bounded continuous functions on $\mathbb{R}^1$; for which such transformations does the relation

$$T(f*g) = Tf \cdot Tg, \tag{1}$$

where $f*g$ denotes the convolution of $f$ and $g$, hold pointwise for all $f$ and $g \in L^1(\mathbb{R}^1)$?

Restricting ourselves for the moment to those transformations satisfying the additional property that

$$\text{for no } \eta \in \mathbb{R}^1 \text{ is } Tf(\eta) = 0 \text{ for all } f \in L^1(\mathbb{R}^1) \tag{2}$$

(where $Tf(\eta)$ denotes the value of the image function $Tf$ at the point $\eta \in \mathbb{R}^1$), we have the following

**Theorem 1.** *Let $Tf$ be a bounded linear transformation mapping $L^1(\mathbb{R}^1)$ into $C_b(\mathbb{R}^1)$ satisfying condition* (2). *Then the convolution property* (1) *holds if and only if it has the form*

$$Tf(\eta) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}x\beta(\eta)} f(x)\,\mathrm{d}x = \hat{f}(\beta(\eta)), \tag{3}$$

*where $\beta(\eta)$ is a continuous real valued function on $\mathbb{R}^1$. In other words, $Tf$ is nothing more than the Fourier transform combined with a continuous change of variable.*

Proof of Sufficiency. Since the convolution property holds for the Fourier transform: $\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$, it follows immediately, on replacing $\xi$ by $\beta(\eta)$, that it will also hold for all $T$'s of the form (3). It is also clear that such $T$'s are bounded linear transformations carrying $L^1(\mathbb{R}^1)$ into $C_b(\mathbb{R}^1)$, and that they satisfy the restriction (2).

Before entering into the proof of the necessity we want to address the characterization of those transformations satisfying the convolution property, but without the restriction (2) holding. It will be shown that they also have a similar description: $Tf(\eta) = \hat{f}(\beta(\eta))$, the Fourier transform combined with a change of variable; but with a more elaborate function $\beta(\eta)$ whose precise description will be specified after the necessity of Theorem 1 has been proved.

## 2. The necessity proof

The proof depends on two lemmas which follow. Both of them involve some definitions and notation from the set and measure theory which we proceed to describe.

We shall be dealing with sets $E$ in $\mathbb{R}^j$, $j = 1, 2$. Their complements in $\mathbb{R}^j$ will be denoted by $\widetilde{E}$. A set in $\mathbb{R}^j$ will be said to be of *full measure* if its complement in $\mathbb{R}^j$ has $j$-dimensional measure zero. Finally, if $E$ is a set in $\mathbb{R}^2$, we define $E_x$, its "$x$-slice," to be the set in $\mathbb{R}^1$ consisting of all points $y \in \mathbb{R}^1$ such that $(x, y) \in E$. This being understood we now state our first lemma.

**Lemma 1.** *Suppose $S$ is a set of full measure in $\mathbb{R}^2$, then for almost all $x \in \mathbb{R}^1$, the sets $S_x$ are sets of full measure in $\mathbb{R}^1$.*

Proof. As a result of Fubini's theorem [4, Corollary 3.3, p. 82],

$$m(\widetilde{S}) = \int_{\mathbb{R}^1} m(\widetilde{S})_x \,\mathrm{d}x.$$

Since $m(\widetilde{S}) = 0$, we have $m(\widetilde{S})_x = 0$ a.e. in $\mathbb{R}^1$. But $(\widetilde{S})_x = (S_x)^\sim$; and so for almost all $x \in \mathbb{R}^1$, the sets $S_x$ are sets of full measure in $\mathbb{R}^1$, as claimed.

**Lemma 2.** *Let $k(x)$ be a bounded measurable function on $\mathbb{R}^1$satisfying the functional equation*

$$k(x)k(y) = k(x+y) \text{ a.e. in } \mathbb{R}^2. \tag{4}$$

*Then either*

$$k(x) = 0 \quad \text{a.e. in } \mathbb{R}^1,$$

*or*

$$k(x) = \mathrm{e}^{\mathrm{i}\beta x} \quad \text{a.e. in } \mathbb{R}^1, \text{ with } \beta \text{ real.}$$

**Remark.** The fact that $k(x)$ is a bounded measurable function in $\mathbb{R}^1$ implies that both sides of (4) are bounded measurable functions in $\mathbb{R}^2$.

P r o o f. Either $k(x) = 0$ a.e. in $\mathbb{R}^1$ or it is not. In the latter case, if $k(x)$ were everywhere defined and continuous in $\mathbb{R}^1$, and (4) held everywhere in $\mathbb{R}^2$, there would be nothing to prove as the conclusion is well known under these circumstances (see [2, pp. 44–45] or [3, pp. 194–195]). We shall reduce to this situation by showing that there exists an everywhere defined continuous function $k^*(x)$ satisfying (4) in all of $\mathbb{R}^2$ and agreeing with $k(x)$ a.e. in $\mathbb{R}^1$.

To accomplish this we first note that the set $S$ in $\mathbb{R}^2$ for which (4) holds is a set of full measure. Accordingly, by Lemma 1, the set $X$ consisting of those $x$'s for which $S_x$ is a set of full measure in $\mathbb{R}^1$ is itself a set of full measure in $\mathbb{R}^1$. Since the points $(x, y)$ with $x \in X$ and $y \in S_x$ all fall into $S$, it follows that the relation

$$k(x)k(y) = k(x+y) \text{ holds for } x \in X \text{ and } y \in S_x. \tag{5}$$

Next, since $k(x)$ does not vanish a.e. in $\mathbb{R}^1$, there exists a bounded set $E$ of positive measure over which the real part or the imaginary part of $k(x)$ is of one sign. Hence $\int_E k(y)\,\mathrm{d}y \neq 0$.

Integrating both sides of (5) with respect to $y$ over $E \cap S_x$, we have for a fixed $x \in X$

$$k(x) \int_{E\cap S_x} k(y)\,\mathrm{d}y = \int_{E\cap S_x} k(x+y)\,\mathrm{d}y.$$

But as $S_x$ is a set of full measure, we can replace the set $E \cap S_x$ over which we are integrating by $E$; and so in view of the non-vanishing of $\int_E k(y)\,\mathrm{d}y$, we have

$$k(x) = \int_E k(x+y)\,\mathrm{d}y \Big/ \int_E k(y)\,\mathrm{d}y \quad \text{for } x \in X.$$

Setting $k^*(x)$ equal to the quotient on the right, we see that it defines a function in all of $\mathbb{R}^1$ which is continuous there due to the translational continuity of the integral $\int_E k(x+y)\,\mathrm{d}y$. Furthermore, since $k^*(x)$ agrees with $k(x)$ on $X$, it agrees with $k(x)$ almost everywhere. Because of the latter, the satisfaction by $k$ of the functional equation (4) implies the same for $k^*$, i.e.

$$k^*(x)k^*(y) = k^*(x+y) \text{ a.e. in } \mathbb{R}^2. \tag{6}$$

Indeed, since $k^*(x)k^*(y) = k(x)k(y)$ on the set $\{(x,y)\colon\ x \in X,\ y \in X\}$ of full measure in $\mathbb{R}^2$ and $k^*(x+y) = k(x+y)$ on the set $\{(x,y)\colon\ (x+y) \in X\}$ also of full measure in $\mathbb{R}^2$, (6) is then seen to be a consequence of (4) by virtue of the transitivity of the "a.e. equality" relation.

Finally, as sets of full measure in $\mathbb{R}^2$ are dense in $\mathbb{R}^2$, it follows from the continuity of $k^*$, that the validity of the functional equation $k^*(x)k^*(y) = k^*(x+y)$ a.e. in $\mathbb{R}^2$, leads to its validity everywhere in $\mathbb{R}^2$. Thus we conclude that $k^*(x) = \mathrm{e}^{cx}$ for some complex number $c$; and so $k(x) = \mathrm{e}^{cx}$ a.e. in $\mathbb{R}^1$. But as $k(x)$ is a bounded function, $c$ must be pure imaginary, which completes the proof.

We are now in position to give the

Proof of the necessity for Theorem 1. From the assumption that $Tf$ is a bounded linear transformation from $L^1(\mathbb{R}^1)$ into $C_b(\mathbb{R}^1)$, it follows that, for fixed $\eta$, $Tf(\eta)$ is a bounded linear functional from $L^1(\mathbb{R}^1)$ into the complex numbers. Consequently, it must have the form

$$Tf(\eta) = \int_{-\infty}^{+\infty} f(x)k(x,\eta)\,\mathrm{d}x,$$

where as a function of $x$, $k(x,\eta)$ is a bounded measurable function on $\mathbb{R}^1$.

If now the convolution identity (1) is to hold, we must have

$$\begin{aligned}\left(\int_{-\infty}^{+\infty} f(x)k(x,\eta)\,\mathrm{d}x\right)&\left(\int_{-\infty}^{+\infty} g(y)k(y,\eta)\,\mathrm{d}y\right)\\ &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(u-y)g(y)\,\mathrm{d}y\right)k(u,\eta)\,\mathrm{d}u\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)g(y)k(x+y,\eta)\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

which leads to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)g(y)[k(x,\eta)k(y,\eta) - k(x+y,\eta)]\,\mathrm{d}x\,\mathrm{d}y = 0$$

for all $f$ and $g \in L^1(\mathbb{R}^1)$. This implies that

$$k(x,\eta)k(y,\eta) = k(x+y,\eta) \quad \text{a.e. for } (x,y) \in \mathbb{R}^2.$$

From Lemma 2 it then follows that either $k(x,\eta)$ is zero a.e. for $x \in \mathbb{R}^1$, or that $k(x,\eta) = \mathrm{e}^{\mathrm{i}\beta(\eta)x}$ a.e. for $x \in \mathbb{R}^1$, with $\beta(\eta)$ a real number. The first alternative is not possible because of condition (2). Thus $Tf$ is of the form

$$Tf(\eta) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\beta(\eta)x} f(x)\,\mathrm{d}x = \hat{f}(\beta(\eta)).$$

Now in order that $T$ map $L^1(\mathbb{R}^1)$ into $C_b(\mathbb{R}^1)$, the space of uniformly bounded continuous functions on $\mathbb{R}^1$, we claim that $\beta(\eta)$ will have to be a continuous function on $\mathbb{R}^1$. Because of the known continuity of $\hat{f}(\xi)$, this is clearly sufficient to guarantee that $Tf(\eta) = \hat{f}(\beta(\eta))$ will be a continuous function of $\eta$. It is also necessary. To see this suppose $\beta(\eta)$ is discontinuous at $\eta = \gamma$; we will then show that for a suitable $f \in L^1(\mathbb{R}^1)$, $Tf(\eta) = \hat{f}(\beta(\eta))$ will be discontinuous at $\gamma$.

The construction of the appropriate $f$ is carried out as follows: Since $\beta(\eta)$ is discontinuous at $\gamma$, for some $\delta > 0$ we can find a sequence $\{\eta_k\}_{k=1}^{\infty}$ such that $\eta_k \to \gamma$ as $k \to \infty$, but that $|\beta(\eta_k) - \beta(\gamma)| > \delta$ for all $k$.

Now let $g(x)$ be any $L^1(\mathbb{R}^1)$ function for which $\hat{g}(0) = \int_{-\infty}^{+\infty} g(x)\,\mathrm{d}x \neq 0$. Because of the Riemann-Lebesgue lemma, $\hat{g}(\xi) \to 0$ as $|\xi| \to \infty$. It follows that for $|\xi|$ sufficiently large, say

$$\text{for } |\xi| > M, \text{ we have } |\hat{g}(\xi)| < \tfrac{1}{2}|\hat{g}(0)|. \tag{7}$$

Setting $f(x) = \mathrm{e}^{-\mathrm{i}\beta(\gamma)x}\delta M^{-1} g(\delta M^{-1}x)$, so that $\hat{f}(\xi) = \hat{g}(M\delta^{-1}(\xi - \beta(\gamma)))$; and replacing $\xi$ in (7) by $M\delta^{-1}(\xi - \beta(\gamma))$, we find, as $\hat{g}(0) = \hat{f}(\beta(\gamma))$, that

$$|\hat{f}(\xi)| < \tfrac{1}{2}|\hat{f}(\beta(\gamma))| \quad \text{for } |\xi - \beta(\gamma)| > \delta.$$

Hence, since $|\beta(\eta_k) - \beta(\gamma)| > \delta$ for all $k$,

$$|\hat{f}(\beta(\eta_k))| < \tfrac{1}{2}|\hat{f}(\beta(\gamma))| \quad \text{for all } k;$$

and so in view of $\hat{f}(\beta(\gamma)) = \hat{g}(0) \neq 0$, it is impossible for $\{\hat{f}(\beta(\eta_k))\}_{k=1}^{\infty}$ to converge to $\hat{f}(\beta(\gamma))$ as $k \to \infty$. This shows that $\hat{f}(\beta(\eta))$ is discontinuous at $\eta = \gamma$ and completes the proof of Theorem 1.

## 3. The general case

We now turn to the characterization of the transformations $T$ with the desired properties, but without requiring condition (2) to hold. In this situation we are going to show that the transformations in question have a description similar to the one given when condition (2) holds. Namely, $Tf(\eta) = \hat{f}(\beta(\eta))$ where $\beta(\eta)$ will now be a suitably defined extended real valued function.

To accomplish this we note that before invoking condition (2), the argument given in the proof of Theorem 1 allowed us to conclude that

$$Tf(\eta) = \int_{-\infty}^{+\infty} k(x,\eta) f(x)\,\mathrm{d}x$$

where for fixed $\eta$ (leaving aside "almost everywhere" considerations which are immaterial here), either

$$k(x,\eta) \equiv 0 \quad \text{for } x \in \mathbb{R}^1,$$

or

$$k(x,\eta) = \mathrm{e}^{\mathrm{i}\beta(\eta)x} \quad \text{for } x \in \mathbb{R}^1, \text{ with } \beta(\eta) \text{ a real number.}$$

The effect of invoking condition (2) was to preclude the first alternative. Thus, if condition (2) no longer holds we have to allow $k(x,\eta)$ to be $\equiv 0$ as a function of $x$. In order to accommodate this possibility we adopt the following conventions: We shall allow $\beta(\eta)$ to assume the value $\infty$. If $\beta(\eta) = \infty$, then we interpret $\mathrm{e}^{\mathrm{i}\beta(\eta)x} = \mathrm{e}^{\mathrm{i}\infty x}$ to be zero for all $x \in \mathbb{R}^1$. This permits us to express the function $k(x,\eta)$ in the form

$$k(x,\eta) = \mathrm{e}^{\mathrm{i}\beta(\eta)x} \quad \text{for } x \in \mathbb{R}^1 \tag{8}$$

in all cases, even when for some $\eta$, $k(x,\eta)$ is identically zero as a function of $x$, in which case we take $\beta(\eta) = \infty$ in (8). Finally, we also adopt the convention $\hat{f}(\infty) = 0$, so that if $\beta(\eta) = \infty$, then $\hat{f}(\beta(\eta)) = 0$.

Thus, as a consequence of the argument given in the proof of necessity of Theorem 1 together with the conventions just adopted, it follows, in the general case, that the transformations $Tf$ mapping $L^1(\mathbb{R}^1)$ into $C_b(\mathbb{R}^1)$ for which the convolution identity (1) holds must be of the form

$$Tf(\eta) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\beta(\eta)x} f(x)\,\mathrm{d}x = \hat{f}(\beta(\eta)),$$

where for any $\eta \in \mathbb{R}^1, \beta(\eta)$ is either a real number or $\infty$.

Not only must $T$ have this form if the convolution identity (1) is to hold; but it is easily checked that if it does have this form, that identity does hold. In fact, at any point $\eta$ where $\beta(\eta)$ is finite, this is an immediate consequence, as noted earlier, of the convolution identity holding for the Fourier transform. At a point $\eta$ where $\beta(\eta) = \infty$ so that $\mathrm{e}^{\mathrm{i}\beta(\eta)x} \equiv 0$ for $x \in \mathbb{R}^1$, and, as a result, $Tf(\eta) = 0$ for all $f \in L^1(\mathbb{R}^1)$, the convolution identity holds trivially.

We also note that when $T$ has the above form, the image functions $Tf(\eta)$ are bounded on $\mathbb{R}^1$ by the $L^1(\mathbb{R}^1)$ norm of $f$.

Next we address the question of characterizing those extended real valued functions $\beta(\eta)$ for which the image functions $Tf(\eta)$ will be continuous. To this end we adopt the following definition of continuity for such functions.

At a point $\gamma$ where $\beta(\gamma)$ is finite, continuity of $\beta$ means that there exists a neighborhood of $\gamma$ inside of which $\beta(\eta)$ is finite and within which $\beta(\eta)$ is continuous at $\gamma$ in the conventional sense.

At a point $\gamma$ where $\beta(\gamma) = \infty$, continuity means either that $\gamma$ belongs to a neighborhood throughout which $\beta(\eta) = \infty$; or, if not, then for any infinite sequence of points $\{\eta_k\}$ converging to $\gamma$ as $k \to \infty$, and at which the $\beta(\eta_k)$'s are all finite valued, we should have

$$|\beta(\eta_k)| \to \infty \quad \text{as } k \to \infty.$$

From these definitions it should be clear that $Tf(\eta) = \hat{f}(\beta(\eta))$ defines a continuous function of $\eta$. Namely, at a point $\gamma$ where $\beta(\gamma)$ is finite, if we restrict ourselves to a suitable neighborhood of $\gamma$ where $\beta(\eta)$ is finite valued, then $\beta(\eta)$ is continuous at $\gamma$ in the conventional sense and the continuity of $\hat{f}(\beta(\eta))$ at $\gamma$ is then a consequence of the continuity of $\hat{f}(\xi)$.

If, on the other hand, $\beta(\gamma) = \infty$, then the continuity of $\beta$ at $\gamma$ entails one of two possibilities. The first of these is that there is a neighborhood of $\gamma$ throughout which $\beta(\eta) = \infty$. In that case $\hat{f}(\beta(\eta)) = \hat{f}(\infty) = 0$ for $\eta$ in that neighborhood; and so $\hat{f}(\beta(\eta))$ is then clearly continuous at $\gamma$.

In the other possible case that $\beta(\eta)$ is continuous at $\gamma$, we argue as follows: Consider any infinite sequence $\{\eta_k\}$ converging to $\gamma$ as $k \to \infty$, and divide this sequence into two sequences $\{\eta'_k\}$ and $\{\eta''_k\}$ on which, respectively, $\beta(\eta'_k)$ is finite and $\beta(\eta''_k) = \infty$. Now

$$\hat{f}(\beta(\eta''_k)) = \hat{f}(\infty) = 0, \tag{9}$$

and if $\{\eta'_k\}$ is an infinite sequence, the continuity of $\beta(\eta)$ at $\gamma$ requires that $|\beta(\eta'_k)| \to \infty$ as $k \to \infty$, so that

$$\hat{f}(\beta(\eta'_k)) \to 0 \quad \text{as } k \to \infty, \tag{10}$$

due to the Riemann-Lebesgue lemma. Since $\hat{f}(\beta(\gamma)) = \hat{f}(\infty) = 0$, it follows from (9) and (10) that

$$\hat{f}(\beta(\eta_k)) \to 0 = \hat{f}(\beta(\gamma)) \quad \text{as } k \to \infty,$$

thereby establishing the continuity of $\hat{f}(\beta(\eta))$ at $\eta = \gamma$.

Next, to demonstrate the necessity for the continuity of $\beta(\eta)$, we need to show that if such an extended real valued $\beta(\eta)$ is discontinuous at $\eta = \gamma$, there is an $f \in L^1(\mathbb{R}^1)$ such that $\hat{f}(\beta(\eta))$ is discontinuous at $\eta = \gamma$.

Suppose first that $\beta(\gamma)$ is finite valued, then $\beta$ may be discontinuous at $\eta = \gamma$ either (a) because there is no neighborhood of $\gamma$ which is free of points where $\beta$ assumes $\infty$; or (b) there is such a neighborhood but $\beta(\eta)$ restricted to this neighborhood fails to be continuous at $\eta = \gamma$ in the usual sense. In the latter case, the construction of an $f \in L^1(\mathbb{R}^1)$ such that $\hat{f}(\beta(\eta))$ is discontinuous at $\eta = \gamma$ has already been carried out in the course of proving the necessity of Theorem 1.

Now in the case the discontinuity at $\eta = \gamma$ is due to the impossibility of finding a neighborhood of $\gamma$ free of points where $\beta(\eta)$ assumes the value $\infty$, there exists an infinite sequence of points $\{\eta_k\}$ converging to $\gamma$ as $k \to \infty$, at which $\beta(\eta_k) = \infty$ for all $k$. Consequently, $\hat{f}(\beta(\eta_k)) = \hat{f}(\infty) = 0$ for all $k$; and so $\lim_{k\to\infty}\hat{f}(\beta(\eta_k)) = 0$. Thus, to obtain a function with the desired discontinuity it suffices to produce an $f \in L^1(\mathbb{R}^1)$ such that $\hat{f}(\beta(\gamma)) \neq 0$. But this is easily done: Take $f(x) = \mathrm{e}^{-\mathrm{i}\beta(\gamma)x} g(x)$, where $g(x)$ is an $L^1(\mathbb{R}^1)$ function with $\hat{g}(0) = \int_{-\infty}^{+\infty} g(x)\,\mathrm{d}x \neq 0$; then $\hat{f}(\xi) = \hat{g}(\xi - \beta(\gamma))$, so that $\hat{f}(\beta(\gamma)) = \hat{g}(0) \neq 0$.

Finally, consider the case where $\beta(\eta)$ is discontinuous at a point $\eta = \gamma$ where $\beta(\gamma) = \infty$. In this situation the discontinuity occurs because we can find an infinite sequence of points $\{\eta_k\}$ converging to $\gamma$ as $k \to \infty$, at which the $\beta(\eta_k)$'s are finite valued and remain bounded as $k \to \infty$. Working with a subsequence if necessary, we may assume, without loss of generality, that $\{\beta(\eta_k)\}$ converges to some finite value $\lambda$ as $k \to \infty$: $\beta(\eta_k) \to \lambda$ as $k \to \infty$. Consequently, on account of the continuity of $\hat{f}(\xi)$,

$$\hat{f}(\beta(\eta_k)) \to \hat{f}(\lambda) \;\; \text{as } k \to \infty.$$

But $\hat{f}(\beta(\gamma)) = \hat{f}(\infty) = 0$, and so $\hat{f}(\beta(\eta))$ will not be continuous at $\eta = \gamma$ for any $f \in L^1(\mathbb{R}^1)$ for which $\hat{f}(\lambda) \neq 0$. Such $f$'s exist; as in the construction in the previous paragraph, we need only to take $f(x) = \mathrm{e}^{-\mathrm{i}\lambda x} g(x)$ with $g(x)$ an $L^1(\mathbb{R}^1)$ function such that $\hat{g}(0) \neq 0$, so that, as $\hat{f}(\xi) = \hat{g}(\xi - \lambda)$, $\hat{f}(\lambda) = \hat{g}(0) \neq 0$. This completes the argument characterizing the desired transformations.

In summary we have established the following

**Theorem 2.** *Let $Tf$ be a bounded linear transformation mapping $L^1(\mathbb{R}^1)$ into $C_b(\mathbb{R}^1)$. Then in order that the convolution identity* (1) *hold for $Tf$ it is necessary and sufficient that it be of the form*

$$Tf(\eta) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\beta(\eta)x} f(x)\,\mathrm{d}x = \hat{f}(\beta(\eta)) \tag{11}$$

*where $\beta(\eta)$ is an extended real valued function which is continuous in the sense defined above; and where the conventions $\mathrm{e}^{\mathrm{i}\infty x} = 0$ for $x \in \mathbb{R}^1$ and $\hat{f}(\infty) = 0$ are used when $\beta(\eta) = \infty$ in* (11).

**Note.** It has come to author's attention that Lemma 2 is a special case of a general result [1] due to A. Járai.

*References*

[1] *A. Járai*: Measurable solutions of functional equations satisfied almost everywhere. Math. Pannon. *10* (1999), 103–110.

[2] *S. Krantz*: A Panorama of Harmonic Analysis, The Carus Mathematical Monographs. Number 27, The Mathematical Association of America, Washington D.C., 1999.

[3] *W. Rudin*: Real and Complex Analysis. McGraw-Hill, New York, 1st edition, 1966.

[4] *E. M. Stein and R. Shakarchi*: Real Analysis. Princeton University Press, Princeton, New Jersey, 2005.

*Author's address*: Charles S. Kahane, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA, e-mail: c.kahane@att.net.