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PROPERTIES OF DIFFERENCES OF MEROMORPHIC FUNCTIONS

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Abstract. Let f be a transcendental meromorphic function. We propose a number of results concerning zeros and fixed points of the difference $g(z)=f(z+c)-f(z)$ and the divided difference $g(z)/f(z)$.

Keywords: meromorphic function, difference, divided difference, zero, fixed point

MSC 2010: 30D35, 39A10

1. Introduction and results

Bergweiler and Langley [2] investigated the existence of zeros of the difference $f(z+c)-f(z)$ and the divided difference $(f(z+c)-f(z))/f(z)$. They obtained many profound and significant results. The results may be viewed as difference analogues of the following existing theorem on the zeros of f'.

Theorem A ([3], [8], [15]). *Let f be transcendental and meromorphic in the plane with*

$$\varliminf_{r\to\infty}\frac{T(r,f)}{r}=0. \tag{1.1}$$

Then f' has infinitely many zeros.

Theorem A is sharp, as shown by e^z, $\tan z$ and examples of arbitrary order greater than 1 constructed in [6].

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In this paper we assume the reader is familiar with the basic notions of Nevanlinna's value distribution theory (see e.g. [12], [17], [18]). In addition, we use the notations $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$; $\lambda(f)$ and $\lambda(1/f)$ denote, respectively, the exponents of convergence of zeros and poles of $f(z)$. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of f that is defined as

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log N(r, 1/(f-z))}{\log r}.$$

For f as in the hypotheses of Theorem A it follows from Hurwitz's theorem that if z_1 is a zero of f' then $f(z+c) - f(z)$ has a zero near z_1 for all sufficiently small $c \in \mathbb{C} \setminus \{0\}$. This makes it natural to ask whether $f(z+c) - f(z)$, for such functions f, must always have infinitely many zeros or not. Bergeiler and Langley [2] answered this question, and obtained the following Theorems B–D.

Theorem B. *Let f be a function transcendental and meromorphic of lower order $\mu(f) < 1$ in the plane. Let $c \in \mathbb{C} \setminus \{0\}$ be such that at most finitely many poles z_j, z_k of f satisfy $z_j - z_k = c$.*

Then $g(z) = f(z+c) - f(z)$ has infinitely many zeros.

Theorem C. *Let $\varphi(r)$ be a positive non-decreasing function on $[1, \infty)$ which satisfies $\lim_{r\to\infty} \varphi(r) = \infty$. Then there exists a function f transcendental and meromorphic in the plane with*

$$\overline{\lim_{r\to\infty}} \frac{T(r,f)}{r} < \infty \quad \text{and} \quad \underline{\lim_{r\to\infty}} \frac{T(r,f)}{\varphi(r)\log r} < \infty$$

such that $g(z) = f(z+1) - f(z)$ has only one zero. Moreover, the function g satisfies

$$\overline{\lim_{r\to\infty}} \frac{T(r,g)}{\varphi(r)\log r} < \infty.$$

Theorem D. *Let f be a function transcendental and meromorphic in the plane with*

$$T(r,f) = O(\log r)^2 \quad \text{as} \quad r \to \infty,$$

and set

$$g(z) = f(z+1) - f(z) \quad \text{and} \quad G_1(z) = \frac{g(z)}{f(z)} = \frac{f(z+1) - f(z)}{f(z)}.$$

Then at least one of $g(z)$ and $G_1(z)$ has infinitely many zeros.

Chen and Shon [4] considered zeros and fixed points of the difference and the divided difference of entire functions with order of growth $\sigma(f) = 1$ and obtained the following theorem.

Theorem E. *Let $c \in \mathbb{C} \setminus \{0\}$ and let f be a transcendental entire function of order of growth $\sigma(f) = \sigma = 1$, that has infinitely many zeros with the exponent of convergence of zeros $\lambda(f) = \lambda < 1$. Then $g(z) = \Delta f(z) = f(z+c) - f(z)$ has infinitely many zeros and infinitely many fixed points.*

In particular, if a set $H = \{z_j\}$ consists of all different zeros of $f(z)$ satisfying any one of the following two conditions:

(i) *at most finitely many zeros z_j, z_k satisfy $z_j - z_k = c$;*

(ii) $\underline{\lim}_{j\to\infty} |z_{j+1}/z_j| = l > 1$, *then*

$$G(z) = \frac{\Delta f(z)}{f(z)} = \frac{f(z+c) - f(z)}{f(z)}$$

has infinitely many zeros and infinitely many fixed points.

From Theorem B we see that the condition "at most finitely many poles z_j, z_k of f satisfy $z_j - z_k = c$" guarantees that $g(z)$ has infinitely many zeros.

From Theorem C we see that Theorem B fails without the hypothesis on the value c, even for lower order 0.

Theorem C shows that for any given σ $(0 \leqslant \sigma \leqslant 1)$, there exists a transcendental meromorphic function of order of growth $\sigma(f) = \sigma$, such that $g(z)$ has only one zero.

Theorem D shows that even under the condition "$T(r,f) = O(\log r)^2$ as $r \to \infty$", we cannot prove that $g(z)$ has infinitely many zeros.

Theorem E shows that the fixed points of the difference and the divided difference have the same properties as their zeros.

In this paper, we consider the following three problems:

(i) What conditions will guarantee that the difference $f(z+c) - f(z)$ has infinitely many zeros without the hypothesis on c for a meromorphic function f?

(ii) What is the exponent of convergence of zeros of the difference $f(z+c) - f(z)$ if it has infinitely many zeros?

(iii) What can we say about the zeros of

$$f(z+c) - f(z) - p(z) \quad \text{and} \quad \frac{f(z+c) - f(z)}{f(z)} - p(z),$$

where $p(z)$ is a polynomial?

We prove the following three theorems concerning the above three problems.

Theorem 1. *Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and f a meromorphic function of order of growth $\sigma(f) = \sigma \leqslant 1$. Suppose that f satisfies $\lambda(1/f) < \lambda(f) < 1$ or has infinitely many zeros (with $\lambda(f) = 0$) and finitely many poles. Then*

$$g(z) = f(z + c) - f(z) \tag{1.2}$$

has infinitely many zeros and satisfies $\lambda(g) = \lambda(f)$.

Theorem 2. *Let c and $f(z)$ satisfy the conditions of Theorem 1. Suppose that $p(z)$ is a polynomial. Then $g^*(z) = g(z) - p(z)$ has infinitely many zeros and satisfies $\lambda(g^*) = \sigma(f)$.*

Theorem 3. *Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and f a transcendental meromorphic function of order of growth $\sigma(f) = \sigma < 1$ or of the form $f(z) = h(z)\mathrm{e}^{az}$ where $a \neq 0$ is a constant, $h(z)$ is a transcendental meromorphic function with $\sigma(h) < 1$. Suppose that $p(z)$ is a nonconstant polynomial. Then*

$$G(z) = \frac{f(z + c) - f(z)}{f(z)} - p(z) \tag{1.3}$$

has infinitely many zeros.

From Theorems 2 and 3 we easily obtain the following corollaries on fixed points of differences and divided differences.

Corollary 1. *Let c and $f(z)$ satisfy the conditions of Theorem 2. Then $g(z)$ has infinitely many fixed points and satisfies the exponent of convergence of fixed points $\tau(g) = \sigma(f)$.*

Corollary 2. *Let c and $f(z)$ satisfy the conditions of Theorem 3. Then $G_1(z) = (f(z + c) - f(z))/f(z)$ has infinitely many fixed points.*

Remark 1.1. The following examples show that the condition $\lambda(f) < 1$ of Theorem 1 and Corollary 1 cannot be replaced by $\lambda(f) \leqslant 1$.

For example, the function $f(z) = \mathrm{e}^z + 1$ satisfies $\lambda(f) = 1$, but

$$g(z) = f(z + 1) - f(z) = (e - 1)\mathrm{e}^z$$

has no zero. And for example, the function $f = \mathrm{e}^z + \frac{1}{2}z^2 - \frac{1}{2}z + 1$ satisfies $\lambda(f) = 1$ by Milloux's theorem (see [12], [18]), and $g(z) = f(z + 1) - f(z) = (e - 1)\mathrm{e}^z + z$ has no fixed point, but it has infinitely many zeros.

2. Proof of theorem 1

We need the following lemmas and notion to prove Theorem 1.

ε-set. Following Hayman [13, p. 75–76], we define an ε-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set then the set of $r \geqslant 1$ for which the circle $S(0,r)$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

Lemma 2.1 ([2]). *Let f be a function transcendental and meromorphic in the plane of order < 1. Let $h > 0$. Then there exists an ε-set E such that*

$$f(z+c) - f(z) = cf'(z)(1+o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \leqslant h$.

Lemma 2.2 ([2]). *Let g be a function transcendental and meromorphic in the plane of order < 1. Let $h > 0$. Then there exists an ε-set E such that*

$$\frac{g'(z+c)}{g(z+c)} \to 0, \qquad \frac{g(z+c)}{g(z)} \to 1 \text{ as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \leqslant h$. Further, E may be chosen such that for large z not in E the function g has no zeros or poles in $|\zeta - z| \leqslant h$.

Lemma 2.3 (Rouché's theorem ([7, p. 125]). *Suppose f and g are meromorphic in a neighborhood of $\{z\colon |z-a| \leqslant R\}$ with no zeros or poles on the circle $\gamma = \{z\colon |z-a| = R\}$. If*

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

on γ, then

$$n\Big(R, \frac{1}{f}\Big) - n(R,f) = n\Big(R, \frac{1}{g}\Big) - n(R,g).$$

Proof of Theorem 1. We divide this proof into two cases $\sigma(f) = \sigma < 1$ and $\sigma(f) = \sigma = 1$.

Case I. $\sigma(f) = \sigma < 1$. First, we suppose that f satisfies $\lambda(1/f) < \lambda(f)$. Suppose that $f(z) = u(z)/v(z)$, where $u(z)$ and $v(z)$ are canonical products ($v(z)$ may be a polynomial) formed by zeros and poles of $f(z)$, respectively, and

$$\sigma(u) = \lambda(u) = \lambda(f) > \sigma(v) = \lambda(v) = \lambda\Big(\frac{1}{f}\Big).$$

By Lemma 2.1, there exists an ε-set E such that

$$f(z+c)-f(z)=cf'(z)(1+o(1)) \quad \text{as } z\to\infty \text{ in } \mathbb{C}\setminus E. \tag{2.1}$$

Set

$$H=\{|z|=r\in(1,\infty)\colon\ z\in E \ \text{ or } \ g(z)=0,\ \text{ or } f'(z)=0\}.$$

By $\sigma(f)<1$ and the property of the ε-set, we see that H has finite logarithmic measure. Thus, for large $|z|=r\notin[0,1]\cup H$, $g(z)$ and $f'(z)$ have no zero on the circle $|z|=r$, and by (2.1),

$$|g(z)-cf'(z)|=|cf'(z)o(1)|<|cf'(z)|+|g(z)|. \tag{2.2}$$

Applying Lemma 2.3 (Rouché's theorem) to $g(z)$ and $cf'(z)$, by (2.2) we obtain that

$$n\Big(r,\frac{1}{g}\Big)-n(r,g)=n\Big(r,\frac{1}{f'}\Big)-n(r,f') \quad r\notin[0,1]\cup H. \tag{2.3}$$

Since $f'(z)=(u'(z)v(z)-u(z)v'(z))/v^2(z)$, $\sigma(f)=\sigma(f')$ and $\lambda(1/f)<\lambda(f)=\sigma(f)<1$, we see that

$$\lambda\Big(\frac{1}{f'}\Big)=\lambda\Big(\frac{1}{f}\Big)<\lambda(f)=\sigma(f)=\sigma(f')=\lambda(f'). \tag{2.4}$$

By (1.2) and $\lambda\left(\frac{1}{f}\right)<\lambda(f)=\sigma(f)$, we see that

$$\lambda\Big(\frac{1}{g}\Big)\leqslant\lambda\Big(\frac{1}{f}\Big)<\lambda(f)=\lambda(f'). \tag{2.5}$$

Thus, (2.3)–(2.5) give

$$\lambda(g)=\lambda(f')=\lambda(f).$$

Secondly, we suppose that $f(z)$ has infinitely many zeros (with $\lambda(f)=0$) and only finitely many poles. Using a method similar to the above, we can complete the proof of Case I.

Case II. $\sigma(f)=\sigma=1$. First, we suppose that f satisfies $\sigma(f)=1$ and $\lambda(1/f)<\lambda(f)<1$. Then f can be rewritten as

$$f(z)=h(z)\mathrm{e}^{az}=\frac{u(z)}{v(z)}\,\mathrm{e}^{az}, \tag{2.6}$$

where $a \neq 0$ is a constant, $h(z)$ is a meromorphic function such that $h(z) = u(z)/v(z)$, $u(z)$ and $v(z)$ are canonical products ($v(z)$ may be polynomial) formed by zeros and poles of $f(z)$ respectively. Also,

$$\begin{aligned} 1 &> \sigma(h) = \lambda(h) = \sigma(u) = \lambda(u) = \lambda(f) \\ &> \lambda\Big(\frac{1}{h}\Big) = \sigma(v) = \lambda(v) = \lambda\Big(\frac{1}{f}\Big). \end{aligned} \tag{2.7}$$

Thus,

$$g(z) = [h(z+c)e^{ac} - h(z)]\, e^{az} = g_1(z)e^{az},$$

where

$$g_1(z) = h(z+c)e^{ac} - h(z).$$

Thus,

$$\sigma(g) = 1,\ \sigma(g_1) < 1,\ \lambda(g) = \lambda(g_1) \ \text{ and } \ \lambda\Big(\frac{1}{g}\Big) = \lambda\Big(\frac{1}{g_1}\Big).$$

If $e^{ac} = 1$, then by Case I and (2.7), we see that the assertion holds in Case II.

Next, we suppose that $e^{ac} \neq 1$. By Lemma 2.3, there exists an ε-set E such that

$$h(z+c) = h(z)(1+o(1)) \quad \text{as} \quad z \to \infty \quad \text{in} \quad \mathbb{C} \setminus E. \tag{2.8}$$

Thus (2.8) yields

$$g_1(z) = e^{ac}h(z)(1+o(1)) - h(z) = (e^{ac}-1)h(z)(1+o(1)). \tag{2.9}$$

So, since h is transcendental, we see that g_1 is transcendental. Set

$$H = \{|z| = r \in (1,\infty)\colon\ z \in E \ \text{ or } \ g_1(z) = 0, \ \text{ or } \ h(z) = 0\}.$$

By $\sigma(g_1) < 1$ and the property of the ε-set, we see that H has finite logarithmic measure. Thus, for large $|z| = r \notin [0,1] \cup H$, $g_1(z)$ and $(e^{ac}-1)h(z)$ have no zero on the circle $|z| = r$, and by (2.9),

$$|g_1(z) - (e^{ac}-1)h(z)| = |(e^{ac}-1)h(z)o(1)| < |(e^{ac}-1)h(z)| + |g_1(z)|. \tag{2.10}$$

Using a method similar to the proof of Case I, by (2.10) we get

$$\lambda(g_1) = \lambda(h) = \lambda(u) = \lambda(f).$$

Secondly, we suppose that $f(z)$ has infinitely many zeros (with $\lambda(f) = 0$) and only finitely many poles. Using a method similar to the above, we can complete the proof of Case II.

3. Proof of theorem 2

We need the following lemma to prove Theorem 2.

Lemma 3.1 ([19]). *Let $f_j(z)$ $(j = 1, \ldots, n)$ $(n \geqslant 2)$ be meromorphic functions, $g_j(z)$ $(j = 1, \ldots, n)$ entire functions, and let them satisfy*

(i) $\sum_{j=1}^{n} f_j(z)\mathrm{e}^{g_j(z)} \equiv 0$;

(ii) *when $1 \leqslant j < k \leqslant n$, then $g_j(z) - g_k(z)$ is not a constant;*

(iii) *when $1 \leqslant j \leqslant n$, $1 \leqslant h < k \leqslant n$, then*

$$T(r, f_j) = o\{T(r, \mathrm{e}^{g_h - g_k})\} \quad (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.

Proof of Theorem 2. We divide this proof into two cases $\sigma(f) = \sigma < 1$ and $\sigma(f) = \sigma = 1$.

Case I. $\sigma(f) = \sigma < 1$. We suppose that f satisfies $\lambda(f) > \lambda(1/f)$. From Theorem 1 and its proof of Case I, we see that

$$\sigma(g) = \lambda(g) = \sigma(f) = \lambda(f), \quad \lambda\Big(\frac{1}{g}\Big) \leqslant \lambda\Big(\frac{1}{f}\Big) < \sigma(g).$$

Since $g^*(z) = g(z) - p(z)$ where $p(z)$ is a polynomial, we have

$$1 > \sigma(g^*) = \sigma(g) = \lambda(g) > \lambda\Big(\frac{1}{g}\Big) = \lambda\Big(\frac{1}{g^*}\Big).$$

So, $\lambda(g^*) = \sigma(g^*) = \sigma(g) = \lambda(f) = \sigma(f)$.

For the case that f has infinitely many zeros (with $\lambda(f) = 0$) and only finitely many poles, using a method similar to the above, we can complete the proof of Case I.

Case II. $\sigma = 1$. We suppose that f satisfies $\lambda(1/f) < \lambda(f) < 1$. From Theorem 1 and its proof of Case II, we see that

$$f(z) = h(z)\mathrm{e}^{az} \quad \text{and} \quad g(z) = [h(z + c)\mathrm{e}^{ac} - h(z)]\,\mathrm{e}^{az}$$

where $a \neq 0$ is a constant, $h(z)$ is a meromorphic function such that $\sigma(g) = 1$ and

$$1 > \lambda(h) = \lambda(f) > \lambda\Big(\frac{1}{f}\Big) = \lambda\Big(\frac{1}{h}\Big), \quad \lambda(g) = \lambda(f) > \lambda\Big(\frac{1}{f}\Big) \geqslant \lambda\Big(\frac{1}{g}\Big). \tag{3.1}$$

Suppose that $\lambda(g^*) < 1$. Then by $\sigma(g^*) = \sigma(g-p) = 1$, $g^*(z)$ can be rewritten as

$$g^*(z) = g(z) - p(z) = h^*(z)\mathrm{e}^{dz} \tag{3.2}$$

where $h^*(z)$ is a meromorphic function such that

$$\lambda(h^*) = \lambda(g^*),\ \lambda\Big(\frac{1}{h^*}\Big) = \lambda\Big(\frac{1}{g^*}\Big),\ \sigma(h^*) = \max\Big\{\lambda(h^*),\ \lambda\Big(\frac{1}{h^*}\Big)\Big\} < 1.$$

By (3.1), we see that $h^*(z) \not\equiv 0$ and

$$\lambda\Big(\frac{1}{g^*}\Big) = \lambda\Big(\frac{1}{g}\Big) = \lambda\Big(\frac{1}{h^*}\Big) \leqslant \lambda\Big(\frac{1}{f}\Big). \tag{3.3}$$

Thus (3.2) gives

$$[h(z+c)\mathrm{e}^{ac} - h(z)]\,\mathrm{e}^{az} - h^*(z)\mathrm{e}^{dz} - p(z)\mathrm{e}^{0z} = 0. \tag{3.4}$$

If $a \neq d$, then by Lemma 3.1 we see that

$$h(z+c)\mathrm{e}^{ac} - h(z) \equiv h^*(z) \equiv p(z) \equiv 0.$$

This is a contradiction. So, $a = d$. By (3.4), we see that

$$[h(z+c)\mathrm{e}^{ac} - h(z) - h^*(z)]\,\mathrm{e}^{az} - p(z)\mathrm{e}^{0z} = 0. \tag{3.5}$$

Again applying Lemma 3.1, we obtain that

$$p(z) \equiv 0, \quad h(z+c)\mathrm{e}^{ac} - h(z) - h^*(z) \equiv 0.$$

This is also a contradiction. Hence $\lambda(g-p) = 1$. Case II of Theorem 2 is thus proved.

4. Proof of theorem 3

We need the following lemmas to prove Theorem 3.

Lemma 4.1 ([2]). *Let $c \in \mathbb{C}\setminus\{0\}$ be a constant and f a function transcendental and meromorphic in the plane which satisfies* (1.1). *Then both $f(z+c) - f(z)$ and $(f(z+c) - f(z))/f(z)$ are transcendental.*

Lemma 4.2 ([9]). *Let f be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, let $H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \geqslant 0$ for $i = 1, \ldots, q$. Let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| \notin E \cup [0, 1]$ and for all $(k, j) \in H$, we have*

$$\Bigl|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Bigr| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

The following Lemma 4.3 can be got by using a method similar to the proof of Lemma 4.1 (see [2]).

Lemma 4.3. *Let a and $c \in \mathbb{C} \setminus \{0\}$ be constants and h a function transcendental and meromorphic in the plane which satisfies* (1.1). *Then $(h(z+c)\mathrm{e}^{ac} - h(z))/h(z)$ is transcendental.*

Proof of Theorem 3. We divide this proof into two cases $\sigma(f) = \sigma < 1$, and $f(z)$ is of the form $f(z) = h(z)\mathrm{e}^{az}$ where $a \neq 0$ is a constant and $h(z)$ is a transcendental meromorphic function with $\sigma(h) < 1$.

Case I. $\sigma(f) = \sigma < 1$. By $\sigma(f) < 1$, we see that f satisfies (1.1). By Lemma 4.1, we see that $(f(z+c) - f(z))/f(z)$ is transcendental, and so is $G(z)$.

By Lemma 2.1, there is an ε-set E, such that

$$f(z+c) - f(z) = cf'(z)(1 + o(1)) \quad \text{as} \quad z \to \infty \text{ in } \mathbb{C} \setminus E. \tag{4.1}$$

By Lemma 4.2, for a given $\varepsilon > 0$ there exists a set $H_1 \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| \notin [0, 1] \cup H_1$ we have

$$\Bigl|\frac{f'(z)}{f(z)}\Bigr| \leqslant |z|^{\sigma-1+\varepsilon} \tag{4.2}$$

where $\sigma(f) = \sigma < 1$. Set

$$H_2 = \{|z| = r \in (1, \infty)\colon\ z \in E, \quad \text{or} \quad G(z) = 0, \quad \text{or} \quad p(z) = 0\}.$$

Using the inequality $\sigma(f) < 1$ and the property of an ε-set, we see that H_2 has finite logarithmic measure. Thus for large $|z| = r \notin [0, 1] \cup H_1 \cup H_2$, $G(z)$ and $p(z)$ have no zero on the circle $|z| = r$. By (4.1) and (4.2), we obtain that

$$\begin{aligned} |G(z) + p(z)| &= \Bigl|\frac{cf'(z)}{f(z)}(1 + o(1))\Bigr| \\ &\leqslant |c(1 + o(1))||z|^{\sigma-1+\varepsilon} < |G(z)| + |p(z)|. \end{aligned} \tag{4.3}$$

Applying Lemma 2.3 (Rouché's theorem) to $G(z)$ and $p(z)$, by (4.3) we obtain that

$$n\Big(r, \frac{1}{G}\Big) - n(r, G) = n\Big(r, \frac{1}{p}\Big) - n(r, p) = \deg p, \quad r \notin [0,1] \cup H_1 \cup H_2. \tag{4.4}$$

Since G is transcendental and $\sigma(G) < 1$, we see that at least one of $n(r, 1/G) \to \infty$ and $n(r, G) \to \infty$ $(r \to \infty)$ is true. So, by (4.4), we see that both $n(r, 1/G) \to \infty$ and $n(r, G) \to \infty$ $(r \to \infty)$ hold. Hence $G(z)$ must have infinitely many zeros. Thus, Case I of Theorem 3 is proved.

Case II. $f(z)$ is of the form $f(z) = h(z)\mathrm{e}^{az}$ where $a \neq 0$ is a constant and $h(z)$ is a transcendental meromorphic function with $\sigma(h) < 1$. Substituting $f(z) = h(z)\mathrm{e}^{az}$ into $G(z)$, we get that

$$G(z) = \frac{h(z+c)\mathrm{e}^{ac} - h(z)}{h(z)} - p(z), \tag{4.5}$$

where $h(z)$ is transcendental and $\sigma(h) < 1$.

If $\mathrm{e}^{ac} = 1$, then by Case I and (4.5) we see that $G(z)$ has infinitely many zeros.

Assume henceforth that $\mathrm{e}^{ac} \neq 1$. We use a method similar to the proof of Case I. By Lemmas 2.1 and 4.2, for a given $\varepsilon > 0$ there exist an ε-set E and a set $H_1 \subset (1, \infty)$ having finite logarithmic measure, such that for all z satisfying $z \in \mathbb{C} \setminus E$ and $|z| \notin [0,1] \cup H_1$ we have

$$\begin{aligned} \Big|\frac{h(z+c)\mathrm{e}^{ac} - h(z)}{h(z)}\Big| &= \Big|\frac{ch'(z)}{h(z)}\mathrm{e}^{ac} + (\mathrm{e}^{ac} - 1)\Big| \\ &\leqslant |c\mathrm{e}^{ac}||z|^{\sigma-1+\varepsilon} + |\mathrm{e}^{ac} - 1|, \end{aligned} \tag{4.6}$$

where $\sigma(h) = \sigma < 1$. Set

$$H_2 = \{|z| = r \in (1, \infty)\colon\ z \in E, \quad \text{or} \quad G(z) = 0, \quad \text{or} \quad p(z) = 0\}.$$

So, H_2 has finite logarithmic measure. Thus for large $|z| = r \notin [0,1] \cup H_1 \cup H_2$, $G(z)$ and $p(z)$ have no zero on the circle $|z| = r$. By (4.5) and (4.6), we obtain that

$$|G(z) + p(z)| \leqslant |c\mathrm{e}^{ac}||z|^{\sigma-1+\varepsilon} + |\mathrm{e}^{ac} - 1| < |G(z)| + |p(z)|. \tag{4.7}$$

By Lemma 2.3 (Rouché's theorem) and (4.7), we obtain (4.4). By the same argument as in the proof of Case I and noting that $G(z)$ is transcendental, by Lemma 4.3 we obtain $n(r, 1/G) \to \infty$ $(r \to \infty)$. Case II of Theorem 3 is thus proved.

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