Czechoslovak Mathematical Journal

Tiziana Cardinali; Francesco Portigiani; Paola Rubbioni

Nonlocal Cauchy problems and their controllability for semilinear differential inclusions with lower Scorza-Dragoni nonlinearities

Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 1, 225–245

Persistent URL: `http://dml.cz/dmlcz/141530`

Czechoslovak Mathematical Journal, 61 (136) (2011), 225–245

NONLOCAL CAUCHY PROBLEMS AND THEIR CONTROLLABILITY FOR SEMILINEAR DIFFERENTIAL INCLUSIONS WITH LOWER SCORZA-DRAGONI NONLINEARITIES

Tiziana Cardinali, Perugia, Francesco Portigiani, Modena,
Paola Rubbioni, Perugia

(Received November 20, 2009)

Abstract. In this paper we prove the existence of mild solutions and the controllability for semilinear differential inclusions with nonlocal conditions. Our results extend some recent theorems.

Keywords: nonlocal conditions, semilinear differential inclusions, selection theorem, mild solutions, lower Scorza-Dragoni property, controllability

MSC 2010: 34G25, 34H05

1. Introduction

The study of nonlocal Cauchy problems in Banach spaces begins in 1991 with the work of Byszewski [6]. In that paper the author emphasizes the importance of nonlocal conditions, which are more general than the initial ones and make it possible to describe physical problems which cannot be studied by means of classical Cauchy problems. As an example, the nonlocal results can be applied to kinematics to determine the evolution $t \mapsto y(t)$ of the location of a physical object for which the positions $y(0), y(t_1), \ldots, y(t_p)$ are not known, but we know the following nonlocal condition:

$$y(0) + \sum_{k=1}^{p} c_k y(t_k) = x_0, \quad c_k \in \mathbb{R} \setminus \{0\}.$$

Later on, several authors have studied nonlocal Cauchy problems governed by ordinary differential equations or inclusions either with autonomous or non-autonomous linear parts. We refer for instance to the recent papers [2], [10], [23].

Nowadays, some practical situations gave rise to the necessity to study the controllability of this kind of problems. For example, they are a useful tool for obtaining the controllability of the size structured population equation (see [4]).

This fact explains the growing interest by several authors (see e.g. [11], [17]) in the investigation of controllability for nonlocal problems.

In the present paper we consider a Cauchy problem with a nonlocal condition governed by a nonautonomous semilinear differential inclusion.

In Section 3, we estabilish the existence of mild solutions for our problem by requiring the nonlinearity to possess a Scorza-Dragoni property in the sense of lower semicontinuity and the linear part to satisfy the usual conditions. The main tool used in this section is a selection theorem presented in [8].

Then, in Section 4, we study the controllability of a nonlocal Cauchy problem. In the main result of this section we make use again of the selection theorem cited above.

Our existence and controllability results extend in a broad sense some recent theorems existing in literature (see Remarks 3.1 and 4.3).

2. Preliminaries

Let X,Y be Hausdorff topological spaces and, if necessary, let Y be also linear. In the sequel we make use of the following notation: $\mathscr{P}(Y) = \{H \subset Y\colon\ H \neq \emptyset\}$; $\mathscr{P}_c(Y) = \{H \in \mathscr{P}(Y)\colon\ H \text{ convex}\}$; $\mathscr{P}_f(Y) = \{H \in \mathscr{P}(Y)\colon\ H \text{ closed}\}$; $\mathscr{P}_k(Y) = \{H \in \mathscr{P}(Y)\colon\ H \text{ compact}\}$; $\mathscr{P}_{fc}(Y) = \mathscr{P}_f(Y) \cap \mathscr{P}_c(Y)$; etc.

Moreover, we also consider the family $\mathscr{D}(Y)$, introduced by Michael in [18], defined as

$$\mathscr{D}(Y) = \{H \in \mathscr{P}_c(Y)\colon\ H \supset I(\overline{H})\} \tag{2.1}$$

with $I(\overline{H}) = \{x \in \overline{H}\colon\ x \notin S,\ S \text{ supporting set for } \overline{H}\}$. We recall that a set S called a *supporting set for* $\overline{H}$ if it is a proper closed and convex subset of the (closed and convex) set $\overline{H}$ which satisfies the condition that for every segment $[x_1, x_2] \subset \overline{H}$ such that $]x_1, x_2[\, \cap\, S \neq \emptyset$, the whole segment $[x_1, x_2]$ is contained in S.

Recall also that for the family $\mathscr{D}(Y)$ the chain inclusion

$$\mathscr{P}_{fc}(Y) \subset \mathscr{D}(Y) \subset \mathscr{P}_c(Y) \tag{2.2}$$

holds (cf. [18]).

A multifunction $F\colon\ X \to \mathscr{P}(Y)$ is said to be *lower semicontinuous* at $x_0 \in X$ if for every open set $\Omega \subseteq Y$ with $F(x_0) \cap \Omega \neq \emptyset$ there exists a neighborhood V of x_0 such that $F(x) \cap \Omega \neq \emptyset$ for every $x \in V$ (see e.g. [13], [15]).

Let $[a,b]$ be an interval of the real line endowed with the usual Lebesgue measure λ defined on the Lebesgue σ-algebra Λ. A multifunction $F\colon [a,b] \to \mathscr{P}_f(Y)$ is said to be: *strongly measurable* if $F^-(C) = \{t \in [a,b]\colon F(t) \cap C \neq \emptyset\} \in \Lambda$ for every closed $C \subset Y$; *measurable* if $F^-(A) \in \Lambda$ for every open $A \subset Y$; *graph measurable* if $GrF = \{(t,y) \in [a,b] \times Y\colon y \in F(t)\} \in \Lambda \times \mathscr{B}(Y)$ (where $\mathscr{B}(Y)$ is the Borel σ-algebra on Y).

If Y is a separable Banach space the three definitions are equivalent (cf. [13], Theorem 2.1.35). Further, if the multifunction takes on compact values, they are also equivalent to the following property (cf. [15], Theorem 1.3.1): there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of step multifunctions such that $\lim\limits_{n\to+\infty} h(F_n(t), F(t)) = 0$ for a.e. $t \in [a,b]$ (where h is the Hausdorff metric on $\mathscr{P}_k(Y)$).

A multifunction $F\colon [a,b] \times X \to \mathscr{P}(Y)$ is said to have the *lower Scorza-Dragoni property* if

(l-SD) for every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset [a,b]$ such that $\lambda([a,b] \setminus K_\varepsilon) < \varepsilon$ and $F_{|K_\varepsilon \times X.}$ is lower semicontinuous;

whereas it is said to be a *Michael map* if it verifies the condition

(M) for every closed set $Z \subset [a,b]\times X$ such that $F_{|Z.}$ is lower semicontinuous, there exists a continuous selection of $F_{|Z.}$ (i.e. there exists a continuous function $f\colon Z \to Y$ such that $f(t,x) \in F(t,x)$ for every $(t,x) \in Z$).

Let $(E, \|\cdot\|)$ be a real Banach space. For $1 \leqslant p < +\infty$, $L^p([a,b],E)$ denotes all (equivalence classes) of Bochner integrable functions $x\colon [a,b] \to E$ such that $\|x\|_{L^p} = (\int_a^b \|x(t)\|^p \,\mathrm{d}t)^{1/p} < +\infty$.

Given a multifunction $F\colon [a,b] \to \mathscr{P}(E)$, we will consider the set $\mathscr{S}^1_F = \{f \in L^1([a,b],E)\colon f(t) \in F(t) \text{ for a.e. } t \in [a,b]\}$ and denote $\|F(t)\| = \sup\limits_{x\in F(t)} \|x\|$, $t \in [a,b]$.

Finally, let us recall that a two parameter family $\{T(t,s)\}_{(t,s)\in\Delta}$, $\Delta = \{(t,s) \in [a,b] \times [a,b]\colon s \leqslant t\}$, $T(t,s)\colon E \to E$ a bounded linear operator, is an *evolution system* if

j) $T(t,t) = I$, $t \in [a,b]$; $\quad T(t,r)T(r,s) = T(t,s)$, $a \leqslant s \leqslant r \leqslant t \leqslant b$;

jj) $(t,s) \mapsto T(t,s)$ is strongly continuous on Δ (i.e. the map $(t,s) \mapsto T(t,s)x$ is continuous on Δ for every $x \in E$).

Of course, there exists $D > 0$ such that

$$D = \sup_{(t,s)\in\Delta} \|T(t,s)\|_{\mathscr{L}(E)} \tag{2.3}$$

where $\mathscr{L}(E)$ is the space of bounded linear operators from E to itself.

3. Existence of mild solutions

In this section we consider the nonlocal Cauchy problem in a separable Banach space E

$$\text{(P)} \qquad \begin{cases} y'(t) \in A(t)y(t) + F(t, y(t)), \ t \in [0, b], \\ y(0) + \theta(y) = x_0 \end{cases}$$

where $\{A(t)\}_{t\in[0,b]}$ is a family of linear (not necessarily bounded) operators $A(t)\colon D(A) \subseteq E \to E$, with $D(A)$ not depending on t and dense in E, generating an evolution system $\{T(t,s)\}_{(t,s)\in\Delta}$; $F\colon [0,b] \times E \to \mathscr{P}(E)$ is a multifunction; $\theta\colon C([0,b], E) \to E$ is a function and $x_0 \in E$.

We recall that a family $\{A(t)\}_{t\in[0,b]}$ generates an evolution system $\{T(t,s)\}_{(t,s)\in\Delta}$ if on the region $D(A)$ each operator $T(t,s)$ is strongly differentiable (see, e.g. [16]) relative to t and s, while

$$\frac{\partial T(t,s)}{\partial t} = A(t)T(t,s) \qquad \text{and} \qquad \frac{\partial T(t,s)}{\partial s} = -T(t,s)A(s), \quad (t,s) \in \Delta.$$

A function $y \in C([0,b], E)$ is called a *mild solution* for (P) if

$$y(t) = T(t,0)(x_0 - \theta(y)) + \int_0^t T(t,s)f(s)\,\mathrm{d}s, \ t \in [0,b]$$

where $f \in \mathscr{S}^1_{F(\cdot,y(\cdot))}$.

The existence of mild solutions for problem (P) is provided by the following theorem.

Theorem 3.1. *Let E be a separable Banach space. Suppose that $F\colon [0,b] \times E \to \mathscr{P}(E)$ possesses properties* (l-SD) *and* (M). *Assume*

(F1) *there exists $m \in L^1([0,b];\mathbb{R})$ such that for every $x \in E$*

$$\|F(t,x)\| \leqslant m(t)(1 + \|x\|), \ \text{a.e. } t \in [0,b];$$

(F2) *there exists $h \in L^1([0,b];\mathbb{R})$ such that for every bounded $\Omega \subset E$*

$$\chi(F(t,\Omega)) \leqslant h(t)\chi(\Omega), \ \text{a.e. } t \in [0,b],$$

where χ is the Hausdorff measure of non-compactness;

(Θ1) *$\theta\colon C([0,b], E) \to E$ is a compact mapping such that there exist $\zeta, \eta > 0$ with*

$$\|\theta(y)\| \leqslant \zeta\|y\|_C + \eta, \ y \in C([0,b]; E).$$

Then problem (P) *has at least one mild solution on the interval* $[0,b]$.

P r o o f. We observe that the Lebesgue measure on the Borel σ-algebra of the interval $[0,b]$ is a Radon measure (cf. [13], Theorem A.2.67, and [14], Definition 7.6.8). Since the multifunction F verifies the hypotheses of the Selection Theorem in [8], we can say that F has a Carathéodory selection, i.e. there exists a function $f\colon [0,b]\times E \to E$ such that $f(t,\cdot)$ is continuous for every $t\in[0,b]$, $f(\cdot,x)$ is Borel-measurable for every $x\in E$, $f(t,x)\in F(t,x)$ for a.e. $t\in[0,b]$ and for every $x\in E$.

Now, defining the multifunction $G\colon [0,b]\times E\to \mathscr{P}_{kc}(E)$ as

$$G(t,x)=\{f(t,x)\},\quad (t,x)\in[0,b]\times E$$

and the mapping $g\colon C([0,b];E)\to E$ by

$$g(y)=x_0-\theta(y), \tag{3.1}$$

we consider the nonlocal Cauchy problem

$$\begin{cases} y'(t)\in A(t)y(t)+G(t,y(t)), & t\in[0,b],\\ y(0)=g(y). \end{cases} \tag{3.2}$$

First of all, we check that G satisfies hypotheses $(H1)$, $(H2)$, $(H3)$ of Theorem 3.1 in [2].

As for $(H1)$, we observe that obviously $G(t,\cdot)$ is upper semicontinuous; moreover, since $f(\cdot,x)$ is a Borel-measurable selector of $G(\cdot,x)$ and E is a separable Banach space, we have that the selector $f(\cdot,x)$ is also strongly measurable.

Then, $(F1)$ implies that G satisfies $(H2)$. Further, the monotonicity of the Hausdorff measure of non-compactness and $(F2)$ allow us to deduce that G satisfies $(H3)$.

Let $\tilde\zeta>\max\{\zeta;\ (1-D\|m\|_{L^1})/D\}$, where ζ is from $(\Theta1)$ and D is from (2.3). By (3.1), we have the estimate

$$\|g(y)\|\leqslant\|x_0\|+\tilde\zeta\|y\|_C+\eta,\ \ y\in C([0,b];E).$$

So, putting $c=\tilde\zeta$ and $d=\|x_0\|+\eta$, we can conclude that the function g verifies hypothesis $(H4)$ of Theorem 3.1 in [2].

Finally, $\tilde\zeta\neq(1-D\|m\|_{L^1})/D$ implies condition (3.1) of Theorem 3.1 in [2].

Therefore we can apply the theorem just mentioned (see also Theorem 3.1 in [23]) and claim that there exists a mild solution for the nonlocal Cauchy problem (3.2). This function is a mild solution for (P) too. $\square$

In Theorem 3.1, if the multifunction takes on values in the family $\mathscr{D}(E)$ (see (2.1)) then assumption (M) is easily verified (cf. [18], Theorem 3.1‴). Therefore we deduce the following

Corollary 3.1. *Let E be a separable Banach space. Suppose that $F\colon [0,b]\times E \to \mathscr{D}(E)$ satisfies hypotheses* (l-SD), (F1), (F2) *and* (Θ1).

Then problem (P) *has at least one mild solution on the interval* $[0,b]$.

Remark 3.1. Our existence results extend in a broad sense Theorem 3.1 in [2], Theorem 3.1 in [23] and Theorem 3.2 in [10]. This follows from the fact that the inclusion of $\mathscr{P}_{fc}(E)$ in $\mathscr{D}(E)$ is proper: this is obvious if E is a finite dimensional space since in this case $\mathscr{D}(E) = \mathscr{P}_c(E)$ (cf. [18]); whereas, if E is infinite dimensional, the strict inclusion is proved by Example 3.2 in [8].

4. Controllability

We will deal with the nonlocal Cauchy problem with controls

$$\text{(CP)}\qquad \begin{cases} y'(t) \in A(t)y(t) + F(t,y(t)) + Bu(t), & t\in[0,b],\\ y(0)+\theta(y) = x_0 \end{cases}$$

where E, $\{A(t)\}_{t\in[0,b]}$, $F\colon [0,b]\times E \to \mathscr{P}(E)$, $\theta\colon C([0,b],E)\to E$ and x_0 are as problem (P); U is a Banach space; $B\colon U\to E$ is a linear operator.

A function $y\in C([0,b],E)$ is called a *mild solution* for (CP) if

$$y(t) = T(t,0)(x_0-\theta(y)) + \int_0^t T(t,s)(f(s)+Bu(s))\,\mathrm{d}s, \quad t\in[0,b]$$

where $f\in \mathscr{S}^1_{F(\cdot,y(\cdot))}$ and $u\in L^2([0,b],U)$.

The nonlocal Cauchy problem (CP) is said to be *controllable on* $[0,b]$ if for every $x_1\in E$ there exists a mild solution for (CP) satisfying

$$\text{(4.1)}\qquad y(b)+\theta(y) = x_1.$$

A pair $(y_{x_1}, u_{x_1}) \in C([0,b],E)\times L^2([0,b],U)$ satisfying (CP) and (4.1) will be called a *solution of the controllability problem.*

In the sequel we will work in the following setting.

(A) the family $\{A(t)\}_{t\in[0,b]}$ generates an evolution system $\{T(t,s)\}_{(t,s)\in\Delta}$ such that for every (t,s) with $t-s>0$ the operator $T(t,s)$ is compact;

(B) the linear operator $B\colon U\to E$ is bounded and such that the operator $W\colon L^2([0,b],U)\to E$ defined by

$$Wu = \int_0^b T(b,s)Bu(s)\,\mathrm{d}s$$

is onto.

Remark 4.1. We wish to note that it is allowable to require the surjectivity of W in property (B). In fact, we can endow $\operatorname{Im} W$ with a suitable norm in such a way that it becomes a separable Banach space.

To show this, we consider the operator $\widetilde{W}\colon L^2([0,b],U)/\operatorname{Ker} W \to \operatorname{Im} \widetilde{W}$ defined by

$$\widetilde{W}[u] = Wu, \quad [u] \in L^2([0,b],U)/\operatorname{Ker} W$$

which is univocally determined, linear, bounded and invertible (see e.g. [19], §10.2).

Now we endow the set $\operatorname{Im} \widetilde{W}$ the norm

$$\|x\|_{\operatorname{Im} \widetilde{W}} = \|\widetilde{W}^{-1}x\|_{L^2/\operatorname{Ker} W},$$

where

$$\|[u]\|_{L^2/\operatorname{Ker} W} = \inf_{u\in[u]} \|u\|_{L^2}.$$

Taking into account that $(L^2([0,b];U)/\operatorname{Ker} W, \|\cdot\|_{L^2/\operatorname{Ker} W})$ is a Banach space (see e.g. [21], chap. II, Theorem 5.1), since $\widetilde{W}$ is an isometry we can conclude that also $(\operatorname{Im} \widetilde{W}, \|\cdot\|_{\operatorname{Im} \widetilde{W}})$ is a Banach space.

The separability of this space is a consequence of the homeomorphism chain

$$(\operatorname{Im} \widetilde{W}, \|\cdot\|_{\operatorname{Im} \widetilde{W}}) \simeq (L^2([0,b];U)/\operatorname{Ker} W, \|\cdot\|_{L^2/\operatorname{Ker} W}) \simeq (\operatorname{Im} W, \|\cdot\|)$$

and the last space is a separable space, E being separable.

Finally, since $\operatorname{Im} W = \operatorname{Im} \widetilde{W}$, we have proved that $(\operatorname{Im} W, \|\cdot\|_{\operatorname{Im} \widetilde{W}})$ is a separable Banach space (cf. [3]).

Remark 4.2. We notice that

$$\|[u]\|_{L^2/_{\operatorname{Ker} W}} = \min_{u\in[u]} \|u\|_{L^2}.$$

In fact, $[u]$ is a closed and convex subset of the Hilbert space $L^2([0,b];U)$. Hence, by Theorem 2.1.16 of [1], there exists a unique element $\bar{u} \in [u]$ such that

$$\|\bar{u}\|_{L^2} = \|[u]\|_{L^2/_{\operatorname{Ker} W}}. \tag{4.2}$$

Now, let us denote by M_1, M_2 the constants which bound B, $\widetilde{W}^{-1}$ respectively and let D be the positive number from (2.3). In the next theorem we will use

$$\begin{aligned} K_1 &= D(1 + M_1M_2\sqrt{b} + M_1M_2D\sqrt{b}), \\ K_2 &= D + M_1M_2D^2\sqrt{b}. \end{aligned} \tag{4.3}$$

Theorem 4.1. *Let U and E be Banach spaces, with E separable. Assume hypotheses* (A), (B) *and*

(Θ2) *the mapping $\theta\colon C([0,b],E) \to E$ is compact and there exists $\alpha \geqslant 0$ such that*

$$\limsup_{\|y\|\to+\infty} \frac{\|\theta(y)\|}{\|y\|} = \alpha.$$

Suppose that $F\colon [0,b]\times E \to \mathscr{P}(E)$ possesses properties (l-SD), (M) *and*

(Φ) *there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$, $\varphi_n \in L^1([0,b];\mathbb{R})$, such that*

$$\sup_{\|x\|\leqslant n} \|F(t,x)\| \leqslant \varphi_n(t), \quad \text{a.e. } t\in[0,b],\ n\in\mathbb{N}.$$

Moreover, put $\beta = \liminf\limits_{n\to+\infty} \frac{1}{n}\int_0^b \varphi_n(s)\,\mathrm{d}s$, and let

$$K_1\alpha + K_2\beta < 1. \tag{4.3}$$

Then (CP) *is controllable on* $[0,b]$.

P r o o f. As in Theorem 3.1, we can find a Carathéodory selector $g\colon [0,b]\times E \to E$ for the multifunction F.

Let us fix $x_1 \in E$ and consider the mapping $R\colon C([0,b];E) \to C([0,b];E)$, where, for every $y \in C([0,b];E)$, the function $R(y)$ is defined by

$$R(y)(t) = T(t,0)(x_0 - \theta(y)) + \int_0^t T(t,s)(g(s,y(s)) + Bu_y(s))\,\mathrm{d}s, \quad t\in[0,b] \tag{4.5}$$

where u_y is the representative of the class

$$\widetilde{W}^{-1}\biggl(x_1 - \theta(y) - T(b,0)(x_0-\theta(y)) - \int_0^b T(b,s)g(s,y(s))\,\mathrm{d}s\biggr)$$

verifying (4.2) (cf. Remark 4.2). In the sequel we will use the identification

$$u_y(t) = \widetilde{W}^{-1}\biggl(x_1 - \theta(y) - T(b,0)(x_0-\theta(y)) - \int_0^b T(b,s)g(s,y(s))\,\mathrm{d}s\biggr)(t), \quad t\in[0,b]. \tag{4.6}$$

We will show that R has a fixed point. The proof is given in several steps.

Step 1. Let us show that there exists $n_0 \in \mathbb{N}$ such that $R(B_{n_0}(0)) \subseteq B_{n_0}(0)$, where $B_{n_0}(0)$ is the ball of radius n_0 centered at 0 in the space $C([0,b];E)$.

Suppose by contradiction that for every $n \in \mathbb{N}$ there exists a function $y_n \in C([0,b];E)$ such that

$$\|y_n\|_C \leqslant n \quad \text{and} \quad \|R(y_n)\|_C > n. \tag{4.7}$$

From (Φ), by considering a subsequence if necessary, we have

$$\lim_{n\to+\infty} \frac{1}{n}\int_0^b \varphi_n(s)\,\mathrm{d}s = \beta. \tag{4.8}$$

Fixing $t \in [0,b]$, from (4.5) and (Φ), we have

$$\begin{aligned}
\|R(y_n)(t)\| &\leqslant \|T(t,0)x_0\| + \|T(t,0)\theta(y_n)\| \\
&\quad + \left\|\int_0^t T(t,s)g(s,y_n(s))\,\mathrm{d}s\right\| + \left\|\int_0^t T(t,s)Bu_{y_n}(s)\,\mathrm{d}s\right\| \\
&\leqslant D\big(\|x_0\| + \|\theta(y_n)\|\big) + \int_0^t D\big(\|g(s,y_n(s))\| + \|B\|_{\mathscr{L}(U;E)}\|u_{y_n}(s)\|\big)\,\mathrm{d}s \\
&\leqslant D\left(\|x_0\| + \|\theta(y_n)\| + \int_0^b \varphi_n(s)\,\mathrm{d}s + M_1\sqrt{b}\|u_{y_n}\|_{L^2}\right),
\end{aligned} \tag{4.9}$$

where M_1, D are the constants which have been introduced in order to define K_1, K_2 (see (4.3)) and $\mathscr{L}(U;E)$ is the space of bounded linear operators from U to E.

Moreover, from (4.6) and (4.7) we obtain the estimate

$$\|u_{y_n}\|_{L^2} \leqslant M_2\left(\|x_1\| + (1+D)\|\theta(y_n)\| + D\|x_0\| + D\int_0^b \varphi_n(s)\,\mathrm{d}s\right), \tag{4.10}$$

M_2 being the latter constant used in (4.3).

From (4.7), (4.9) and (4.10) we deduce

$$\begin{aligned}
n < \|R(y_n)\|_C &\leqslant (D + M_1M_2D^2\sqrt{b})\|x_0\| + M_1M_2D\sqrt{b}\|x_1\| \\
&\quad + (D + M_1M_2D\sqrt{b} + M_1M_2D^2\sqrt{b})\|\theta(y_n)\| \\
&\quad + (D + M_1M_2D^2\sqrt{b})\int_0^b \varphi_n(s)\,\mathrm{d}s.
\end{aligned}$$

Taking into account (4.3) and (4.4) and dividing by n, we get

$$1 < \frac{1}{n}\left(C + K_1\|\theta(y_n)\| + K_2\int_0^b \varphi_n(s)\,\mathrm{d}s\right) \tag{4.11}$$

where $C = (D + M_1M_2D^2\sqrt{b})\|x_0\| + M_1M_2D\sqrt{b}\|x_1\|$.

Now, if the set $\{y_n\colon\ n \in \mathbb{N}\}$ is bounded, then by (Θ2), the set $\{\theta(y_n)\colon\ n \in \mathbb{N}\}$ is bounded too. Hence

$$\lim_{n\to+\infty} \frac{\|\theta(y_n)\|}{n} = 0 \leqslant \alpha.$$

Then, passing to the limit for $n \to +\infty$ in (4.11), from (4.8) and (4.4) we obtain the contradiction

$$1 \leqslant K_1\alpha + K_2\beta < 1.$$

Otherwise, if $\{y_n\colon\ n \in \mathbb{N}\}$ is not bounded, there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to+\infty} \|y_{n_k}\| = +\infty$. Now, by (4.7) and ($\Theta$2), we deduce

$$\limsup_{k\to+\infty} \frac{\|\theta(y_{n_k})\|}{n_k} \leqslant \limsup_{\|y_{n_k}\|\to+\infty} \frac{\|\theta(y_{n_k})\|}{\|y_{n_k}\|} \leqslant \limsup_{\|y\|\to+\infty} \frac{\|\theta(y)\|}{\|y\|} = \alpha,$$

and so, by considering (4.11) relative to the subsequence, with the same reasoning as before, we obtain again the above contradiction.

Step 2. We prove that $R(B_{n_0}(0))$ is a relatively compact subset of $C([0,b];E)$. We start by showing that the set $R(B_{n_0}(0))(t)$ is relatively compact in E, for every $t \in [0,b]$.

Consider the set

$$S = \{g(\cdot, y(\cdot)) \in L^1([0,b];E)\colon\ y \in B_{n_0}(0)\}. \tag{4.12}$$

Since $L^1([0,b];E)$ is a separable space, $\overline{S}$ is a separable set. It is easy to see that there exists a countable subset of S

$$\{g_n(\cdot) = g(\cdot, y_n(\cdot))\colon\ n \in \mathbb{N}\} \tag{4.13}$$

which is dense in $\overline{S}$.

Let us define a multifunction $G_1\colon\ [0,b] \to \mathscr{P}(E)$ in the following way:

$$G_1(t) = \overline{\text{co}}\{g_n(t)\colon\ n \in \mathbb{N}\},\ t \in [0,b].$$

First we prove that G_1 is measurable. It is enough to show that (cf. [13], Proposition 2.2.3) there exists a countable set Γ of measurable selectors of G_1 such that

$$G_1(t) = \overline{\Gamma(t)},\ t \in [0,b].$$

To this aim, we define a countable set of functions

$$\Gamma = \left\{\gamma\colon\ [0,b] \to E\colon\ \gamma(t) = \sum_{n=1}^{+\infty} q_n g_n(t),\ t \in [0,b],\ (q_n)_n \in Q\right\},$$

where

$$Q=\Big\{(q_n)_{n\in\mathbb{N}}\colon \sum_{n=1}^{+\infty} q_n=1,\ q_n\in\mathbb{Q}_0^+,\ \exists\, k\in\mathbb{N}\colon\ q_n=0,\ n\geqslant k\Big\}.$$

Clearly, every $\gamma\in\Gamma$ is a measurable selector of G_1. Now, for a fixed $t\in[0,b]$, we prove that

$$\overline{\Gamma(t)}=G_1(t). \tag{4.14}$$

Obviously,

$$\overline{\Gamma(t)}\subset G_1(t). \tag{4.15}$$

To get the other inclusion it is sufficient to prove that $\mathrm{co}\{g_n(t)\colon n\in\mathbb{N}\}\subset\overline{\Gamma(t)}$.

Let $p(t)=\sum\limits_{n=1}^{+\infty}\alpha_n g_n(t)$, where $\alpha_n\in[0,1]$, $\sum\limits_{n=1}^{+\infty}\alpha_n=1$ and there exists $k\in\mathbb{N}$ such that $\alpha_n=0,\ n>k$.

We note that there exist k sequences $(q_m^1)_{m\in\mathbb{N}},\dots,(q_m^k)_{m\in\mathbb{N}}$ of rational non negative numbers converging respectively to $\alpha_1,\dots,\alpha_k$.

So we have $p(t)=\lim\limits_{m\to+\infty}\Big(\sum\limits_{n=1}^{k} q_m^n g_n(t)\Big)\in\overline{\Gamma(t)}$.

Since (4.14) is proved, we can conclude that G_1 is measurable.

We prove now that G_1 is integrably bounded. From (4.13) and (Φ), recalling that $y_n\in B_{n_0}(0)$, $n\in\mathbb{N}$, for almost every $t\in[0,b]$ we have

$$\|\mathrm{co}\{g_n(t)\colon n\in\mathbb{N}\}\|\leqslant\sup_{\|x\|\leqslant n_0}\|F(t,x)\|\leqslant\varphi_{n_0}(t).$$

Then it follows immediately that

$$\|G_1(t)\|\leqslant\varphi_{n_0}(t),\quad \text{a.e. } t\in[0,b]. \tag{4.16}$$

We denote $N_1=\{t\in[0,b]\colon \|G_1(t)\|>\varphi_{n_0}(t)\}$.

Now we set $K=\{Bu_y\colon y\in B_{n_0}(0)\}$, where u_y is the function defined in (4.6). Let us note that the function $Bu_y\colon [0,b]\to E,\ (Bu_y)(t)=Bu_y(t)$, is Bochner integrable. As for the set S defined in (4.12), it is possible to say that there exists a countable subset of K, $\{Bu_n\colon n\in\mathbb{N}\}$, dense in $\overline{K}$.

Let $G_2\colon [0,b]\to\mathscr{P}(E)$ be the multifunction defined by

$$G_2(t)=\overline{\mathrm{co}}\{Bu_n(t)\colon n\in\mathbb{N}\},\quad t\in[0,b].$$

By proceeding as for G_1, it is possible to claim that G_2 is measurable.

Furthermore, also G_2 is integrably bounded. In fact, first of all we observe that

$$\|\mathrm{co}\{Bu_n\colon\ n\in\mathbb{N}\}\|_{L^1}\leqslant\sup_{n\in\mathbb{N}}\|Bu_n\|_{L^1}\leqslant M_1\sup_{y\in B_{n_0}(0)}\|u_y\|_{L^2},\tag{4.17}$$

since $\|B\|_{\mathscr{L}(U;E)}\leqslant M_1$.

Now, since (4.10) is true also for every $y\in B_{n_0}(0)$, by applying (Θ2) we get

$$\begin{aligned}\|u_y\|_{L^2}&\leqslant M_2\Big(\|x_1\|+(1+D)\|\theta(y)\|+D\|x_0\|+D\int_0^b\varphi_{n_0}(s)\,\mathrm{d}s\Big)\\&\leqslant M_2\Big(\|x_1\|+(1+D)\|\theta(B_{n_0}(0))\|+D\|x_0\|+D\int_0^b\varphi_{n_0}(s)\,\mathrm{d}s\Big)\\&:=M_4.\end{aligned}\tag{4.18}$$

So from (4.17) and (4.18), the following estimate holds:

$$\|G_2\|_{L^1}\leqslant M_1M_4.$$

Then, putting $\psi(t)=M_1M_2/b$, $t\in[0,b]$, we can deduce

$$\|G_2(t)\|\leqslant\psi(t),\quad \text{a.e. } t\in[0,b],\tag{4.19}$$

i.e. we have the integrable boundedness of G_2.

Now, put $N_2=\{t\in[0,b]\colon\ \|G_2(t)\|>\psi(t)\}$ and consider the set $N=N_1\cup N_2$. For every $t\in[0,b]$, let $\Phi_t\colon\ [0,t]\to\mathscr{P}(E)$ be the multifunction defined by

$$\Phi_t(s)=\begin{cases}\overline{T(t,s)(G_1(s)+G_2(s))}, & s\in[0,t]\setminus N,\\ \{0\}, & s\in[0,t]\cap N.\end{cases}$$

From (4.16) and (4.19), taking into account (A), it is easy to see that Φ_t has compact and convex values and that $\|T(t,s)(G_1(s)+G_2(s))\|\leqslant D(\varphi_{n_0}(s)+\psi(s)),\ s\in[0,t]\setminus N$ (cf. (2.3)). The integrable boundedness of Φ_t immediately follows.

Now we shall prove that Φ_t is graph measurable.

The multifunctions $\widetilde{G}_i\colon\ [0,t]\times E\to\mathscr{P}(E)$, $i=1,2$, defined by

$$\widetilde{G}_i(s,x)=G_i(s),\quad (s,x)\in[0,t]\times E,$$

are obviously measurable, i.e. $\widetilde{G}_i^-(A)\in\Lambda\times\mathscr{B}(E)$ for every open $A\subset E$.

Hence, they have property (l-SD) (cf. [22], Theorem 2.3.2). Therefore for every $i=1,2$ and every $\varepsilon>0$ there exists a compact $K_\varepsilon^i\subset[0,t]$ with $\lambda([0,t]\setminus K_\varepsilon^i)<\varepsilon/2$

such that $\widetilde{G}_{i|K^i_\varepsilon\times E}$ is lower semicontinuous and hence, equivalently, $G_{i|K^i_\varepsilon}$ is lower semicontinuous.

Now we prove that the multifunction $\widetilde{\Phi}_t\colon [0,t]\times E\to\mathscr{P}_{kc}(E)$ defined as

$$\widetilde{\Phi}_t(s,x)=\Phi_t(s),\quad (s,x)\in[0,t]\times E$$

has property (l-SD).

To this aim, put $\Phi^*_t(s)=T(t,s)(G_1(s)+G_2(s))$, $s\in[0,t]$. By Proposition 1.2.6 of [13] and the continuity of $T(t,\cdot)$ in $[0,t]$, the multifunction $\Phi^*_t\colon [0,t]\to\mathscr{P}(E)$ is lower semicontinuous in $K_\varepsilon=K^1_\varepsilon\cap K^2_\varepsilon$. Hence, Proposition 1.2.38 of [13] implies that Φ_t is lower semicontinuous in K_ε and so we have lower semicontinuity of $\widetilde{\Phi}_t$ in $K_\varepsilon\times E$. Observing that $\lambda([0,t]\setminus K_\varepsilon)<\varepsilon$, we conclude that $\widetilde{\Phi}_t$ verifies property (l-SD).

Now, by Theorem 2.3.2 of [22] we get measurability of $\widetilde{\Phi}_t$; hence Φ_t is graph measurable (see [13], Proposition 2.1.7).

At this point all the assumptions of Theorem 2.5.23 in [13] are verified so that, following its proof, we get

$$\int_0^t\Phi_t(s)\,\mathrm{d}s\in\mathscr{P}_{kc}(E).$$

Consider now the set

$$H(t)=\left\{\int_0^t T(t,s)(\,g(s,y(s))+Bu_y(s))\,\mathrm{d}s\colon\ y\in B_{n_0}(0)\right\}.$$

It is easy to see that $H(t)\subseteq\int_0^t\Phi_t(s)\,\mathrm{d}s$, so we can deduce its relative compactness. Hence, since (4.5) implies the inclusion

$$R(B_{n_0}(0))(t)\subset T(t,0)[x_0-\theta(B_{n_0}(0))]+H(t)\tag{4.20}$$

and thanks to (A) and (Θ2), we can state that the set $R(B_{n_0}(0))(t)$ is relatively compact in E.

Next, we prove that $R(B_{n_0})$ is equicontinuous.

Fixed $y\in B_{n_0}$ and $\varepsilon>0$, let $t_1,t_2\in[0,b]$ with $0\leqslant t_1<t_2$. Put $\gamma\in]0,t_1[$. Using constants M_1,M_2,D (cf. (4.3) and (2.3)), we obtain the following estimate:

$$\begin{aligned}\|R(y)(t_2)-R(y)(t_1)\|&\\ &\leqslant\|T(t_2,0)-T(t_1,0)\|_{\mathscr{L}(E)}\|x_0-\theta(y)\|\\ &\quad+\int_{t_1}^{t_2}\|T(t_2,s)\|_{\mathscr{L}(E)}\|g(s,y(s))+Bu_y(s)\|\,\mathrm{d}s\\ &\quad+\int_0^{t_1}\|T(t_2,s)-T(t_1,s)\|_{\mathscr{L}(E)}\|g(s,y(s))+Bu_y(s)\|\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|T(t_2,0)-T(t_1,0)\|_{\mathscr{L}(E)}\|x_0-\theta(y)\| + \int_{t_1}^{t_2} D\varphi_{n_0}(s)\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} DM_1\|u_y(s)\|_U\,\mathrm{d}s + \int_0^{t_1-\gamma} \|T(t_2,s)-T(t_1,s)\|_{\mathscr{L}(E)}\varphi_{n_0}(s)\,\mathrm{d}s \\
&\quad + \int_0^{t_1-\gamma} \|T(t_2,s)-T(t_1,s)\|_{\mathscr{L}(E)}M_1\|u_y(s)\|_U\,\mathrm{d}s \\
&\quad + \int_{t_1-\gamma}^{t_1} 2D\varphi_{n_0}(s)\,\mathrm{d}s + \int_{t_1-\gamma}^{t_1} 2DM_1\|u_y(s)\|_U\,\mathrm{d}s,
\end{aligned}
$$

and using (4.18), we have

$$
\begin{aligned}
\|R(y)(t_2)-R(y)(t_1)\| &\leqslant \|T(t_2,0)-T(t_1,0)\|_{\mathscr{L}(E)}\|x_0-\theta(y)\| \\
&\quad + D\int_{t_1}^{t_2} \varphi_{n_0}(s)\,\mathrm{d}s + DM_1M_4\sqrt{t_2-t_1} \\
&\quad + \int_0^{t_1-\gamma} \|T(t_2,s)-T(t_1,s)\|_{\mathscr{L}(E)}\varphi_{n_0}(s)\,\mathrm{d}s \\
&\quad + M_1M_4\Big(\int_0^{t_1-\gamma} \|T(t_2,s)-T(t_1,s)\|^2_{\mathscr{L}(E)}\,\mathrm{d}s\Big)^{1/2} \\
&\quad + 2D\int_{t_1-\gamma}^{t_1} \varphi_{n_0}(s)\,\mathrm{d}s + 2DM_1M_4\sqrt{\gamma}.
\end{aligned}
$$

Put $k=\|x_0\|+\|\theta(B_{n_0}(0))\|+D+M_1DM_4+\int_0^b \varphi_{n_0}(s)\,\mathrm{d}s+M_1M_4\sqrt{b}+2D+2M_1DM_4$. There exists a constant $\delta_1(\varepsilon/7k)>0$ for which, if $\gamma<\min\{\delta_1(\varepsilon/7k),\ (\varepsilon/7k)^2\}$, we have

$$
(4.21)\qquad 2D\int_{t_1-\gamma}^{t_1} \varphi_{n_0}(s)\,\mathrm{d}s < \frac{\varepsilon}{7} \quad \text{and} \quad 2M_1DM_4\sqrt{\gamma} < \frac{\varepsilon}{7}.
$$

By virtue of jj) of the evolution systems and the summability of φ_{n_0}, there exists $\delta_2(\varepsilon/7k)>0$ such that, if t_1, t_2 verify $0<t_2-t_1<\delta_2(\varepsilon/7k)$, we have

$$
(4.22)\quad \|T(t_2,s)-T(t_1,s)\|_{\mathscr{L}(E)} < \frac{\varepsilon}{7k},\ s\in[0,t_1] \quad \text{and} \quad D\int_{t_1}^{t_2} \varphi_{n_0}(s)\,\mathrm{d}s < \frac{\varepsilon}{7}.
$$

Putting $\delta(\varepsilon)=\min\{(\varepsilon/7k)^2;\ \delta_1(\varepsilon/7k);\ \delta_2(\varepsilon/7k)\}$, from (4.21) and (4.22) we get

$$
\|R(y)(t_2)-R(y)(t_1)\| < \varepsilon, \quad 0<t_2-t_1<\delta(\varepsilon).
$$

At this point, by applying the Ascoli-Arzelà theorem, we have that the set $R(B_{n_0})$ is relatively compact in $C([0,b];E)$.

Step 3. Let us prove that $R_{|B_{n_0}(0)}\colon B_{n_0}(0) \to B_{n_0}(0)$ is continuous.

Fixing $y \in B_{n_0}(0)$, let $(y_n)_n$, $y_n \in B_{n_0}(0)$, be a sequence converging to y in $C([0,b];E)$. From (4.5), for every $t \in [0,b]$ and every $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\|R(y_n)(t) - R(y)(t)\| \leqslant \|T(t,0)(\theta(y_n) - \theta(y))\| \\
&\qquad + \int_0^t \|T(t,s)(g(s,y_n(s)) - g(s,y(s)))\,\mathrm{d}s + T(t,s)(Bu_{y_n}(s) - Bu_y(s))\|\,\mathrm{d}s \\
&\quad \leqslant D\|\theta(y_n) - \theta(y)\| + \int_0^b D\|g(s,y_n(s)) - g(s,y(s))\|\,\mathrm{d}s \\
&\qquad + \int_0^b D\|Bu_{y_n}(s) - Bu_y(s)\|\,\mathrm{d}s;
\end{aligned}$$

so, bearing in mind the fact that M_1 and M_2 bound the linear operators B and $\widetilde{W}^{-1}$ respectively, from (4.6) we obtain

$$\begin{aligned}
\|R(y_n)(t) - R(y)(t)\| &\leqslant D\|\theta(y_n) - \theta(y)\| + D\int_0^b \|g(s,y_n(s)) - g(s,y(s))\|\,\mathrm{d}s \\
&\quad + M_1M_2D\int_0^b \Big(\|\theta(y_n) - \theta(y)\| + D\|\theta(y_n) - \theta(y)\| \\
&\qquad + D\int_0^b \|g(r,y_n(r)) - g(r,y(r))\|\,\mathrm{d}r\Big)\,\mathrm{d}s \\
&\leqslant (D + bM_1M_2D + bM_1M_2D^2)\|\theta(y_n) - \theta(y)\| \\
&\quad + (D + bM_1M_2D^2)\int_0^b \|g(r,y_n(r)) - g(r,y(r))\|\,\mathrm{d}r.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\text{(4.23)}\qquad &\|R(y_n) - R(y)\|_{\mathscr{C}} \\
&\quad \leqslant (D + bM_1M_2D + bM_1M_2D^2)\|\theta(y_n) - \theta(y)\| \\
&\qquad + (D + bM_1M_2D^2)\int_0^b \|g(r,y_n(r)) - g(r,y(r))\|\,\mathrm{d}r, \quad n \in \mathbb{N}.
\end{aligned}$$

Hence, we can conclude that $(R(y_n))_{n\in\mathbb{N}}$ converges to $R(y)$ in $C([0,b];E)$.

Step 4. Now we are in position to apply the Schauder Theorem to the function $R_{|B_{n_0}(0)}$ and claim that there exists $\overline{y} \in C([0,b];E)$ such that

$$\overline{y}(t) = T(t,0)(x_0 - \theta(\overline{y})) + \int_0^t T(t,s)(g(s,\overline{y}(s)) + Bu_{\overline{y}}(s))\,\mathrm{d}s, \quad t \in [0,b].$$

Further, it is immediately seen that $\overline{y}$ also satisfies the terminal condition (4.1).

Therefore, the controllability of (CP) is proved. □

As for Corollary 3.1, in case that the multifunction takes on values in the family $\mathscr{D}(E)$, from Theorem 4.1 we can deduce

Corollary 4.1. *Let U and E be Banach spaces, with E separable. Assume hypotheses* (A), (B) *and* (Θ2) *hold. If $F\colon [0,b]\times E\to\mathscr{D}(E)$ has properties* (l-SD), (Φ) *and* (4.4), *then* (CP) *is controllable on* $[0,b]$.

Obviously (2.2) yields that the values of the multifunction in the previous corollary are convex.

Otherwise, if the values of the multifunction are compact and not necessarily convex, we can provide another controllability theorem where hypothesis (M) is removed.

To this aim, we recall beforehand the following lemma (cf. [20], 4.3 Fact 14).

Lemma 4.1. *Let E be a Banach space. For a multifunction $F\colon [0,b]\times E\to\mathscr{P}(E)$ the following conditions are equivalent:*

(I) *F has property* (l-SD);

(II) *F is almost lower semicontinuous, i.e. there exists a sequence of disjoint compact sets $(I_k)_{k\in\mathbb{N}}$, $I_k\subset[0,b]$, such that $\lambda\Big([0,b]\setminus\bigcup_{k=1}^{+\infty} I_k\Big)=0$ and $F_{|I_k\times E}$, $k\in\mathbb{N}$, is lower semicontinuous.*

Theorem 4.2. *Let U and E be Banach spaces, with E separable. Assume hypotheses* (A), (B) *and* (Θ2) *hold. If $F\colon [0,b]\times E\to\mathscr{P}_k(E)$ has properties* (l-SD), (Φ) *and* (4.4), *then* (CP) *is controllable on* $[0,b]$.

Proof. We consider the Nemitsky map $N\colon C([0,d];E)\to\mathscr{P}(L^1([0,b];E))$ defined by

$$N(y)=S^1_{F(\cdot,y(\cdot))},\quad y\in C([0,b];E).$$

First of all we note that N is well defined. In fact, from Lemma 3.1 in [7] and taking into account Lemma 4.1, we can say that for every $y\in C([0,b];E)$ there exists a (strongly) measurable selector $f_y\colon [0,b]\to E$ of $F(\cdot,y(\cdot))$. By (Φ) the selector f_y is Bochner integrable, hence $\mathscr{S}^1_{F(\cdot,y(\cdot))}\neq\emptyset$.

In order to prove that N has closed values, we fix $y\in C([0,b];E)$ and consider a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n\in N(y)$, converging to f in $L^1([0,b];E)$. Since the sequence also a.e. converges in $[0,b]$ to f and F has closed values, we can conclude that $f\in N(y)$.

Moreover, it is easy to see that, for a fixed $y\in C([0,b];E)$, $N(y)$ is a decomposable set, i.e. $f\chi_M+g\chi_{[0,b]\setminus M}\in N(y)$ for every $f,g\in N(y)$ and every measurable $M\subset[0,b]$.

Let us show that N is lower semicontinuous.

By virtue of Proposition 1.2.66 in [13], it is sufficient to prove that, for a fixed $\overline{y} \in C([0,b];E)$, for every $\varepsilon > 0$ there exists $\sigma(\varepsilon, \overline{y}) > 0$ such that for every $y \in C([0,b];E)$ with $\|y - \overline{y}\|_C \leqslant \sigma(\varepsilon, \overline{y})$ we have

$$N(\overline{y}) \subset B_\varepsilon(N(y)),$$

where $B_\varepsilon(N(y)) = \{z \in L^1([0,b];E)\colon\ \varrho_{L^1}(z, N(y)) \leqslant \varepsilon\}$ and ϱ_{L^1} is the distance from a point to a set in $L^1([0,b];E)$.

We start showing that for every $z \in L^1([0,b];E)$ we have

$$\varrho_{L^1}(z, N(y)) = \int_0^b \varrho_E(z(s), F(s, y(s)))\,\mathrm{d}s,\ \ y \in C([0,b];E) \tag{4.24}$$

where ϱ_E is defined analogously to ϱ_{L^1}.

In fact, using again Lemma 4.1, from Lemma 5.5.2 of [15] we can conclude that for every $y \in C([0,b];E)$ there exists a function $w \in N(y)$ such that

$$\int_0^b \varrho_E(z(s), F(s, y(s)))\,\mathrm{d}s = \int_0^b \|z(s) - w(s)\|\,\mathrm{d}s. \tag{4.25}$$

Since $w \in N(y)$, we can also write

$$\varrho_{L^1}(z, N(y)) \leqslant \int_0^b \|z(s) - w(s)\|\,\mathrm{d}s.$$

Now suppose, by contradiction, that strict inequality holds in the expression above. Then a function $v \in N(y)$ such that

$$\int_0^b \|z(s) - v(s)\|\,\mathrm{d}s < \int_0^b \|z(s) - w(s)\|\,\mathrm{d}s$$

had to exist and so, using (4.25) too, we would get

$$\int_0^b \|z(s) - v(s)\|\,\mathrm{d}s < \int_0^b \varrho_E(z(s), F(s, y(s)))\,\mathrm{d}s \leqslant \int_0^b \|z(s) - v(s)\|\,\mathrm{d}s.$$

Clearly it is a contradiction.

So we can conclude that $\varrho_{L^1}(z, N(y)) = \int_0^b \|z(s) - w(s)\|\,\mathrm{d}s$.

Hence, even (4.24) is satisfied.

Now, by applying Lemma 5.5.1 of [15], for a fixed $\varepsilon > 0$ there exists a $\sigma = \sigma(\varepsilon/b, \overline{y}) > 0$ such that for every $y \in C([0,b];E)$ with $\|y - \overline{y}\|_C \leqslant \sigma$ we can write

$$F(t, \overline{y}(t)) \subset B_{\varepsilon/b}(F(t, y(t))), \quad \text{a.e. } t \in [0,b]. \tag{4.26}$$

Fixing z in $N(\overline{y})$, from (4.24) and (4.26) we deduce that for every $y \in C([0,b];E)$ with $\|y-\overline{y}\|_C \leqslant \sigma$ the following estimate holds:

$$\varrho_{L^1}(z, N(y)) = \int_0^b \varrho_E(z(s), F(s,y(s)))\,\mathrm{d}s \leqslant \int_0^b \frac{\varepsilon}{b}\,\mathrm{d}s = \varepsilon.$$

Hence we get

$$N(\overline{y}) \subset B_\varepsilon(N(y)). \tag{4.27}$$

Now we are in position to use the Bressan-Colombo selection theorem (cf. [5]). Hence there exists a continuous function $r\colon C([0,b];E) \to L^1([0,b];E)$ such that $r(y) \in N(y)$ for every $y \in C([0,b];E)$.

Fixing $x_1 \in E$, we consider the function $\widehat{R}\colon C([0,b];E) \to C([0,b];E)$ defined by

$$\widehat{R}(y)(t) = T(t,0)(x_0 - \theta(y)) + \int_0^t T(t,s)(r(y)(s) + Bu_y(s))\,\mathrm{d}s, \quad t \in [0,b]$$

where for the representative u_y, chosen as in the proof of Theorem 4.1, we use the identification

$$u_y(t) = \widetilde{W}^{-1}\left(x_1 - \theta(y) - T(b,0)(x_0 - \theta(y)) - \int_0^b T(b,s)r(y)(s)\,\mathrm{d}s\right)(t), \quad t \in [0,b].$$

Proceeding as in Theorem 4.1, we can claim that there exists $n_0 \in \mathbb{N}$ such that $\widehat{R}(B_{n_0}(0)) \subseteq B_{n_0}(0)$, $\widehat{R}(B_{n_0}(0))$ is compact in $C([0,b];E)$ and $\widehat{R}_{|B_{n_0}(0)}$ is continuous.

Therefore we can apply the Schauder fixed point theorem and obtain the existence of $\overline{y} \in B_{n_0}(0)$ such that

$$\overline{y}(t) = T(t,0)(x_0 - \theta(\overline{y})) + \int_0^t T(t,s)(r(\overline{y})(s) + Bu_{\overline{y}}(s))\,\mathrm{d}s, \quad t \in [0,b].$$

Moreover, it is easy to see that $\overline{y}$ also verifies $\overline{y}(b) + \theta(\overline{y}) = x_1$.

Hence, we can conclude the controllability of problem (CP). $\square$

Remark 4.3. We wish to note that Theorem 4.1 covers a large class of multifunctions since no assumptions are required on the values of the multifunction.

Moreover, we observe that our results extend in a broad sense the analogous ones in [17] as the following example shows.

Example 4.1. Put $E = \mathbb{R}$, let us suppose that the constant in assumption $(\Theta 2)$ is $\alpha = 0$.

We consider the multifunction $F\colon [0,1]\times\mathbb{R}\to\mathscr{P}_{kc}(\mathbb{R})$ defined by

$$F(t,x)=\begin{cases}[0,-\beta|x|\log t], & (t,x)\in\,]0,1]\times\mathbb{R},\\ \{0\}, & \text{otherwise}\end{cases}$$

with $\beta>0$.

It is easy to check that F verifies (M) and (l-SD). Obviously (4.4) is true for β sufficiently small. Further, by considering

$$\psi_n(t)=\begin{cases}-n\beta\log t, & t\in\,]0,1],\\ 0, & t=0,\end{cases}\qquad n\in\mathbb{N},$$

we can see that F verifies also (Φ).

On the other hand, by the definition of F every sequence $(\varphi_n)_{n\in\mathbb{N}}$ of summable functions with $\sup\limits_{\|x\|\leqslant n}\|F(t,x)\|\leqslant\varphi_n(t)$, for a.e. $t\in[0,1]$, $n\in\mathbb{N}$, is such that

$$\liminf_{n\to+\infty}\frac{1}{n}\int_0^1\varphi_n(s)\,\mathrm{d}s\geqslant\beta\neq 0.$$

So the assumption required in [17] cannot be satisfied, which shows that our results improve also those in [17].

Remark 4.4. In order to obtain the controllability, with respect to Section 3 here we strengthen the hypothesis on the linear part of the differential inclusion by requiring property (A), but concerning the nonlinearity we omit hypothesis (F2) and substitute (F1) with the weaker (Φ).

The fact that (Φ) is weaker than (F1) is immediately seen by considering the sequence $(\varphi_n)_{n\in\mathbb{N}}$ defined by $\varphi_n(t)=m(t)(1+n)$, $t\in[0,b]$ (where m is from (F1)).

It should be clarified that conditions (A) and (B), assumed in several papers in order to obtain the controllability (see for instance the recent [9], [17]), implicitly imply that the Banach space E has finite dimension, as proved by Hernández and O'Regan in [12].

Therefore, for lower Scorza-Dragoni nonlinearities the controllability in infinite-dimensional Banach spaces is an open problem at present.

References

[1] *R. Abraham, J. E. Marsden and T. S. Ratiu*: Manifolds, Tensor Analysis, and Applications. Second Edition, Springer-Verlag, New York, 1988.

[2] *R. A. Al-Omair and A. G. Ibrahim*: Existence of mild solutions of a semilinear evolution differential inclusion with nonlocal conditions. Electron. J. Differential Equations *42* (2009), 11 pp.

[3] *K. Balachandran and J. P. Dauer*: Controllability of nonlinear systems in Banach spaces: a survey. J. Optim. Theory Appl. *115* (2002), 7–28.

[4] *S. Boulite, A. Idrissi and L. Maniar*: Controllability of semilinear boundary problems with nonlocal initial conditions. J. Math. Anal. Appl. *316* (2006), 566–578.

[5] *A. Bressan and G. Colombo*: Extensions and selections of maps with decomposable values. Studia Math. *90* (1988), 69–85.

[6] *L. Byszewski*: Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. *162* (1991), 494–505.

[7] *T. Cardinali and S. Panfili*: Global mild solutions for semilinear differential inclusions and applications to impulsive problems. PU.M.A. *19* (2008), 1–19.

[8] *T. Cardinali, F. Portigiani and P. Rubbioni*: Local mild solutions and impulsive mild solutions for semilinear Cauchy problems involving lower Scorza-Dragoni multifunctions. Topol. Methods Nonlinear Anal. *32* (2008), 247–259.

[9] *Y.-K. Chang, W.-T. Li and J. J. Nieto*: Controllability of evolution differential inclusions in Banach spaces. Nonlinear Anal. *67* (2007), 623–632.

[10] *L. Górniewicz, S. K. Ntouyas and D. O'Regan*: Existence results for first and second order semilinear differential inclusions with nonlocal conditions. J. Comput. Anal. Appl. *9* (2007), 287–310.

[11] *L. Górniewicz, S. K. Ntouyas and D. O'Regan*: Controllability of semilinear differential equations and inclusions via semigroup theory in Banach spaces. Rep. Math. Phys. *56* (2005), 437–470.

[12] *E. M. Hernández and D. O'Regan*: Controllability of Volterra-Fredholm type systems in Banach spaces. J. Franklin Inst. *346* (2009), 95–101.

[13] *S. Hu and N. S. Papageorgiou*: Handbook of Multivalued Analysis. Vol. I: Theory. Mathematics and its Applications, 419. Kluwer Academic Publishers, Dordrecht, 1997.

[14] *S. Hu and N. S. Papageorgiou*: Handbook of Multivalued Analysis. Vol. II: Applications. Kluwer Academic Publishers, Dordrecht, 2000.

[15] *M. Kamenskii, V. V. Obukhovskii and P. Zecca*: Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, de Gruyter Series in Nonlinear Analysis and Applications, 7. Berlin: de Gruyter, 2001.

[16] *S. G. Krein*: Linear Differential Equations in Banach Spaces. Amer. Math. Soc., Providence, 1971.

[17] *G. Li and X. Xue*: Controllability of evolution inclusions with nonlocal conditions. Appl. Math. Comput. *141* (2003), 375–384.

[18] *E. Michael*: Continuous selections I. Ann. Math. *63* (1956), 361–382.

[19] *H. L. Royden*: Real Analysis. Macmillan Publishing Company, New York, 1988.

[20] *H. J. Sussman*: Needle variations and almost lower semicontinuous differential inclusions. Set-Valued Analysis *10* (2002), 233–285.

[21] *A. E. Taylor and D. C. Lay*: Introduction to Functional Analysis. Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1986.

[22] *A. Tolstonogov*: Differential Inclusions in a Banach Space. Kluwer Academic Publishers, Dordrecht, 2000.

[23] *L. Zhu and G. Li*: On a nonlocal problem for semilinear differential equations with upper semicontinuous nonlinearities in general Banach spaces. J. Math. Anal. Appl. *341* (2008), 660–675.

Authors' addresses: Tiziana Cardinali, University of Perugia, Perugia, Italy, e-mail: tiziana@dmi.unipg.it; Francesco Portigiani, University of Modena-Reggio Emilia, Modena, Italy, e-mail: francesco.portigiani@unimore.it; Paola Rubbioni (corresponding author), University of Perugia, Perugia, Italy, e-mail: rubbioni@dmi.unipg.it.