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THE FOURIER INTEGRAL OPERATORS ON HARDY SPACES ASSOCIATED WITH HERZ SPACES

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Abstract. In this paper, it is proved that the Fourier integral operators of order m, with $-n < m \leqslant -(n-1)/2$, are bounded from three kinds of Hardy spaces associated with Herz spaces to their corresponding Herz spaces.

Keywords: Fourier integral operator, Hardy spaces, Herz spaces

MSC 2010: 47G10, 35S30, 42B30

1. Introduction

Fourier integral operators have in the last 50 years become an important tool in certain areas of analysis, and in particular in a variety of problems arising in partial differential equations. In its basic form, the Fourier integral operator of order m is given by

$$Tf(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi \mathrm{i}\Phi(x,\xi)} a(x,\xi) \hat{f}(\xi) \, \mathrm{d}\xi. \tag{1.1}$$

Here $\hat{f}$ denotes the Fourier transform of f; the functions a and Φ are defined as in [7]. That is, $a(x,\xi) \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}_n \setminus \{0\})$ and it has compact and connected support in x. The phase Φ is real-valued, homogeneous of degree 1 in ξ, and smooth in (x,ξ) for $\xi \neq 0$, on the support of a. We also assume that $0 \in \operatorname{supp}_x a$ and Φ satisfies the

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crucial nondegeneracy condition, that is, for $\xi \neq 0$,

$$\det\Big(\frac{\partial^2 \Phi}{\partial x_i \partial \xi_j}\Big) \neq 0 \tag{1.2}$$

on the support of a.

A well known result of Hörmander in [1] state that the Fourier integral operator T of order 0 is a bounded operator from L^2 to L^2. In [6], A. Seeger, C. D. Sogge and E. M. Stein showed that T of order $-(n-1)/2$ is bounded from H^1 to L^1, and then by complex interpolation they obtained the boundedness from L^p to L^p of T when $-(n-1)/2 < m \leqslant 0$ and $|1/2 - 1/p| \leqslant -m/(n-1)$. Recently, Marco M. P. and Silvia Secco proved the boundedness of T from $h^p(\mathbb{R}^n)$ to $h^p(\mathbb{R}^n)$ in [5].

On the other aspect, some new Hardy spaces HK_q associated with Herz spaces K_q are introduced by Shanzhen Lu and Dachun Yang [2], [8]. An interesting fact shown in [8] is that HK_q is the localization of H^1 at the origin. It is easy to see that the relation between HK_q and K_q is similar to the one between H^1 and L^1, and the relation between $K_q^{\alpha,p}$ and $K_q^{\alpha,p}$ is similar to the one between L^q and L^q.

In this paper we investigate the properties of the operator T defined by (1.1) on Herz spaces and Hardy spaces associated to Herz spaces. To state our results, let us introduce some definitions and facts.

Definition 1.1. Let $1 < q < \infty$ and $1/q + 1/q' = 1$. The Herz space $K_q(\mathbb{R}^n)$ consists of those functions $f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ for which

$$\|f\|_{K_q} := \sum_{k\in\mathbb{Z}} 2^{kn/q'} \cdot \|f\chi_k\|_{L^q} < \infty,$$

where χ_k denotes the characteristic function of C_k and $C_k = B_k \setminus B_{k-1}$, $B_k = \{x\colon |x| \leqslant 2^k\}$.

The Hardy space HK_q associated with Herz space K_q is defined by

$$HK_q = \{f \in L^1\colon\ Gf \in K_q\},$$

where Gf is the Grand maximal function of f and $q > 1$.

Definition 1.2. Let $0 < \alpha < \infty$, $0 < p < \infty$ and $1 < q < \infty$.

(i) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ consists of those functions $f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ for which

$$\|f\|_{\dot{K}_q^{\alpha,p}} := \Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \cdot \|f\chi_k\|^p_{L^q}\Big\}^{1/p} < \infty.$$

(ii) The non-homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ consists of those functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ for which

$$\|f\|_{K_q^{\alpha,p}} := \left\{ \|f\chi_{B_0}\|^p_{L^q} + \sum_{k=1}^{\infty} 2^{k\alpha p} \cdot \|f\chi_k\|^p_{L^q} \right\}^{1/p} < \infty,$$

where χ_k is as above.

Definition 1.3. Let $0 < \alpha < \infty$, $0 < p < \infty$ and $1 < q < \infty$.

(i) The homogeneous Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ associated with $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) \colon\ Gf \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|Gf\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$.

(ii) The non-homogeneous Hardy space $HK_q^{\alpha,p}(\mathbb{R}^n)$ associated with $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$HK_q^{\alpha,p}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) \colon\ Gf \in K_q^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover, we define $\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)} = \|Gf\|_{K_q^{\alpha,p}(\mathbb{R}^n)}$, where Gf is the Grand maximal function of f. Now, we can state our results as follows.

Theorem 1.1. *Let $1 < q < \infty$ and let T be the Fourier integral operator, as in* (1.1), *whose symbol a is of order m, with $-n < m \leqslant -(n-1)/2$. If q satisfies*

$$\frac{1}{q} \geqslant 2 + \frac{2(m-1)}{(n+1)}, \tag{1.3}$$

then T, originally defined on S, extends to a bounded operator from HK_q to K_q, that is

$$\|Tf\|_{K_q} \leqslant C \cdot \|f\|_{HK_q}.$$

Theorem 1.2. *Let $0 < p < \infty$, $1 < q < \infty$ and $0 < \alpha < n(1 - 1/q)$. Let T be the Fourier integral operator as in* (1.1), *whose symbol a is of order m, with $-n < m \leqslant -(n-1)/2$. If q satisfies* (1.3) *in Theorem* 1.1, *then T, originally defined on S, extends to a bounded operator on $\dot{K}_q^{\alpha,p}$ and $K_q^{\alpha,p}$, respectively.*

Theorem 1.3. *Let* $0 < p < \infty$, $1 < q < \infty$ *and* $\alpha \geqslant n(1-1/q)$. *Let* T *be the Fourier integral operator as in* (1.1), *whose symbol* a *is of order* m, *with* $-n-s-1 < m \leqslant -(n-1)/2 - s - 1$. *If* q *satisfies*

$$\frac{1}{q} \geqslant 2 + \frac{2(m+s)}{(n+1)}, \tag{1.4}$$

then T, *originally defined on* S, *extends to a bounded operator from* $H\dot{K}_q^{\alpha,p}$ *to* $\dot{K}_q^{\alpha,p}$ *and* $HK_q^{\alpha,p}$ *to* $K_q^{\alpha,p}$, *respectively.*

Obviously, if $\alpha = n(1-1/q)$ and $p=1$, then we denote $K_q^{\alpha,p}$ by K_q, so Theorem 1.1 is a special case of Theorem 1.3. As concerns the proofs of Theorems 1.2 and 1.3, we only prove the homogeneous case.

The paper is organized as follows. In the next section, we list some definitions and lemmas which will be used throughout the paper. In Section 3 we give the estimate of kernel. In Section 4, we present the proof of Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are given in Section 5 and we conclude the paper in Section 6.

2. DEFINITIONS AND LEMMAS

Let us first introduce some lemmas on the characterization of HK_q.

Definition 2.1. Let $1 < q < \infty$. A function a on $\mathbb{R}^n$ is called a central $(1,q)$-atom if

(1) $\operatorname{supp} a \subset B$, where B is a ball centered at the origin;
(2) $\|a\|_q \leqslant |B|^{1/q-1}$;
(3) $\int a(x)\,\mathrm{d}x = 0$.

In [2], [8], S. Z. Lu and D. C. Yang have proved the following result.

Lemma 2.1. *Let* $f \in L^1(\mathbb{R}^n)$ *and* $1 < q < \infty$. *Then* $f \in HK_q(\mathbb{R}^n)$ *if and only if* f *can be represented as*

$$f(x) = \sum_{l=-\infty}^{\infty} \lambda_l a_l(x),$$

where each a_l *is a central* $(1,q)$*-atom which satisfies* $\operatorname{supp} a_l \subset B(0,2^l)$, *and* $\sum\limits_{l=-\infty}^{+\infty} |\lambda_l| < \infty$. *Moreover,*

$$\|f\|_{HK_q} \sim \inf\Big\{\sum_{l=-\infty}^{\infty} |\lambda_l|\Big\},$$

where the infimum is taken over all decompositions of f *as above.*

Now let us introduce some lemmas on the characterization of $HK_q^{\alpha,p}$ and $K_q^{\alpha,p}$.

Definition 2.2. Let $0<\alpha<\infty$ and $1<q<\infty$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is called a central (α,q)-atom, if

(1) $\operatorname{supp} a\subset B(0,r)$;

(2) $\|a\|_q\leqslant |B|^{-\alpha/n}$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is called a central (α,q)-atom of restrictive type, if

(1) $\operatorname{supp} a\subset B(0,r)$, $r\geqslant 1$;

(2) $\|a\|_q\leqslant |B|^{-\alpha/n}$, where $B(0,r)=\{x\in\mathbb{R}^n\colon\ |x|<r\}$.

Lemma 2.2. *Let $0<\alpha<\infty$, $0<p<\infty$ and $1<q<\infty$. Then $f\in\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ if and only if f can be represented as*

$$f(x)=\sum_{l=-\infty}^{+\infty}\lambda_l a_l(x),$$

where each a_l is a central (α,q)-atom with support B_l, and $\sum\limits_{l=-\infty}^{+\infty}|\lambda_l|^p<\infty$. Moreover,

$$\|f\|_{\dot{K}^{\alpha,p}_q}\sim\inf\biggl(\sum_{l=-\infty}^{+\infty}|\lambda_l|^p\biggr)^{1/p},$$

where the infimum is taken over all decompositions of f as above.

We have found the proof of Lemma 2.2 in [3].

Definition 2.3. Let $n(1-1/q)\leqslant\alpha<\infty$, $1<q<\infty$ and let s be a non-negative integer, $s\geqslant\lfloor\alpha+n(1/q-1)\rfloor$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is called a central (α,q,s)-atom, if it satisfies

(1) $\operatorname{supp} a\subset B(0,r)$;

(2) $\|a\|_q\leqslant |B|^{-\alpha/n}$ where $B(0,r)=\{x\in\mathbb{R}^n\colon\ |x|<r\}$;

(3) $\int a(x)x^\beta\,\mathrm{d}x=0$, $|\beta|\leqslant s$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha,q,s)_0$-atom, if it satisfies (1) through (3), and $a(x)=0$ on some neighborhood of 0;

(iii) A function $a(x)$ on $\mathbb{R}^n$ is called a central (α,q,s)-atom of restrictive type, if it satisfies (1) with $r\geqslant 1$, (2) and (3).

Lemma 2.3. *Let $n(1-1/q)\leqslant\alpha<\infty$, $0<p<\infty$ and $1<q<\infty$. Then $f\in H\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ if and only if f can be represented as*

$$f(x)=\sum_{l=-\infty}^{+\infty}\lambda_l a_l(x),$$

where each a_l *is a central* (α, q)*-atom with support* B_l*, and* $\sum\limits_{l=-\infty}^{+\infty} |\lambda_l|^p < \infty$*. Moreover,*

$$\|f\|_{H\dot{K}_q^{\alpha,p}} \sim \inf \left(\sum_{l=-\infty}^{+\infty} |\lambda_l|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f *as above.*

We have found the proof of Lemma 2.3 in [4].

For the nonhomogeneous spaces $K_q^{\alpha,p}$ and $HK_q^{\alpha,p}$, there are results similar to Lemma 2.2 and Lemma 2.3. However, the central atom must be replaced by the central atom of restrictive type.

Lemma 2.4. *Let* $p \in (0, \infty)$*,* $q \in (1, \infty)$*,* $\alpha \in [n(1 - 1/q), \infty)$ *and let* s *be a nonnegative integer,* $s \geqslant \lfloor \alpha - n(1 - 1/q) \rfloor$*. Then* $\dot{F}_p^{\alpha,q,s}$ *and* $F_p^{\alpha,q,s}$ *are dense in* $H\dot{K}_p^{\alpha,q,s}$ *and* $HK_p^{\alpha,q,s}$ *respectively, where*

$$\dot{F}_p^{\alpha,q,s} \equiv \left\{ f = \sum_{j=1}^{m} \lambda_j a_j \colon\ m \in \mathrm{N}, \{a_j\}_{j=1}^{m} \text{ are central } (\alpha, q, s)_0\text{-atoms} \right\},$$

and

$$\|f\|_{\dot{F}_p^{\alpha,q,s}} \equiv \inf \left\{ \left(\sum_{j=1}^{m} |\lambda_j|^p \right)^{1/p} \colon\ f \in \dot{F}_p^{\alpha,q,s} \right\}.$$

We also introduce $F_p^{\alpha,q,s}$ *and* $\|f\|_{F_p^{\alpha,q,s}}$ *as in the above two formulas just replacing central* (α, q, s)*-atoms by central* (α, q, s)*-atoms of restrictive type.*

Lemma 2.5. *Let* $p \in (0, 1]$*,* $r \in [p, 1]$*,* $q \in (1, \infty)$*,* $\alpha \in [n(1 - 1/q), \infty)$ *and let* s *be a nonnegative integer,* $s \geqslant \lfloor \alpha - n(1 - 1/q) \rfloor$*.*

(i) *If* T *is a linear operator defined on* $\dot{F}_p^{\alpha,q,s}$ *such that*

$$S \equiv \mathit{sup}\{\|Ta\|_{B_r} \colon\ a \text{ is any central } (\alpha, q, s)_0 \text{ atom }\} < \infty$$

then T *uniquely extends to be a bounded* B_r*-sublinear operator from* $H\dot{K}_p^{\alpha,q}$ *to* B_r*.*

(ii) *If* T *is a linear operator defined on* $F_p^{\alpha,q,s}$ *such that*

$$S \equiv \sup\{\|Ta\|_{B_r} \colon\ a \text{ is any central } (\alpha, q, s)_0 \text{ atom of restrictive type}\} < \infty$$

then T *uniquely extends to a bounded* B_r*-sublinear operator from* $HK_p^{\alpha,q}$ *to* B_r*.*

Here B_r *is an* r*-quasi-Banach space and quasi-Banach spaces* $L^p(\mathbb{R}^n)$*,* $H^p(\mathbb{R}^n)$*,* $K_p^{\alpha,q}$*,* $\dot{K}_p^{\alpha,q}$*,* $HK_p^{\alpha,q}$ *and* $H\dot{K}_p^{\alpha,q}$ *with* $p \in (0, 1)$ *are typically* p*-quasi-Banach spaces.*

The proofs of the above two lemmas are in [9].

3. THE ESTIMATION OF KERNEL

By the definition of T, we see that its kernel is given by

$$K(x,y)=\int_{\mathbb{R}^n}\mathrm{e}^{2\pi\mathrm{i}(\Phi(x,\xi)-y\cdot\xi)}a(x,\xi)\,\mathrm{d}\xi. \tag{3.1}$$

As usual, we first construct an exceptional set B_l^* for every ball $B_l (l\leqslant 0)$, which is something like

$$B_l^*=\{x\colon\ \operatorname{dist}(\Sigma_x,B_l)\leqslant c2^l\},$$

where $\Sigma_x=\{y\colon\ y=\Phi_\xi(x,\xi)\text{ for some }\xi\}$, are the singular sets of T. For every positive integer j we choose atom vectors $\{\xi_j^\nu\}$, $\nu=1,\dots,N(j)$, such that $|\xi_j^\nu-\xi_j^{\nu'}|\geqslant 2^{-j/2}$ for different ν, ν'. Then for every $\xi\in S^{n-1}$ there exists a ξ_j^ν such that $|\xi-\xi_j^\nu|<2^{-j/2}$. Observe that

$$N(j)\sim 2^{j(n-1)/2}.$$

For $B(0,2^l)$ with radius $2^l\leqslant 1$ we define the "rectangles"

$$\widetilde{\mathbb{R}}_j^\nu=\{y;|y-0|\leqslant\bar{c}\cdot 2^{-j/2},|\pi_j^\nu(y-0)|\leqslant\bar{c}\cdot 2^{-j}\},$$

where π_j^ν is the orthogonal projection in the direction ξ_j^ν and $\bar{c}$ is a large constant (independent of j). Obviously, $|\widetilde{\mathbb{R}}_j^\nu|\sim 2^{j(n+1)/2}$. Next, the mapping

$$x\mapsto y=\Phi_\xi(x,\xi)$$

has for each ξ a nonvanishing Jacobian, by virtue of (1.2). So we take $\mathbb{R}_j^\nu$ to be the inverse under Φ_ξ, with $\xi=\xi_j^\nu$ from the rectangle $\widetilde{\mathbb{R}}_j^\nu$:

$$\mathbb{R}_j^\nu=\{x;|\Phi_\xi(x,\xi_j^\nu)-0|\leqslant\bar{c}\cdot 2^{-j/2},|\pi_j^\nu(\Phi_\xi(x,\xi_j^\nu)-0)|\leqslant\bar{c}\cdot 2^{-j}\}. \tag{3.2}$$

Now let $B_l^*=\bigcup\limits_{2^{-j}\leqslant 2^l}\bigcup\limits_\nu\mathbb{R}_j^\nu$. Since $\widetilde{\mathbb{R}}_j^\nu$ is compact and $\Phi(x,\xi)$ satisfies the crucial nondegeneracy condition, $\mathbb{R}_j^\nu$ is also compact. So there exists $\{x_i\}_{i=1}^n\subset\mathbb{R}_j^\nu$ such that $\mathbb{R}_j^\nu\subset\bigcup\limits_{i=1}^n U(x_i)$, where $U(x_i)$ is the neighborhood of x_i. For every $x\in\mathbb{R}_j^\nu$ there exists x_{i_0} such that $x\in U(x_{i_0})$, while for all $x\in U(x_{i_0})$ we have

$$\Phi_\xi(x,\xi_j^\nu)=\Phi_\xi(x_{i_0},\xi_j^\nu)+\Phi_{\xi x}(x',\xi_j^\nu)(x-x_{i_0}),$$

where x' is a point on the segment between x and x_{i_0}. Because $\Phi(x,\xi)$ satisfies the crucial nondegeneracy condition, there exist C_1,C_2 such that $C_1\leqslant\|\Phi_{\xi x}(x,\xi_j^\nu)\|\leqslant$

C_2, and because of the relation between $\widetilde{\mathbb{R}}_j^\nu$ and $\mathbb{R}_j^\nu$, there exist $y = \Phi_\xi(x, \xi_j^\nu) \in \widetilde{\mathbb{R}}_j^\nu$ and $y_{i_0} = \Phi_\xi(x_{i_0}, \xi_j^\nu) \in \widetilde{\mathbb{R}}_j^\nu$. So we have

$$|x| \leqslant |x_{i_0}| + c \cdot |y - y_{i_0}|,$$

and because of the proposition on finite covering, for all $x \in \mathbb{R}_j^\nu$, we have

$$|x| \leqslant \max_{1 \leqslant i \leqslant n} \{|x_i| + c \cdot |y - y_i|\},$$

where $y_i = \Phi_\xi(x_i, \xi_j^\nu) \in \widetilde{\mathbb{R}}_j^\nu$. Since $0 \in \operatorname{supp}_x a$, we have

$$\Phi_\xi(x_i, \xi_j^\nu) = \Phi_\xi(0, \xi_j^\nu) + \Phi_{\xi x}(x'', \xi_j^\nu)(x_i - 0),$$

where x'' is a point on the segment between x_i and 0. If $\operatorname{supp}_x a$ is a multiply connected domain, then use finite Talor expansion. Since the mapping

$$\Phi_\xi(0, \cdot)\colon\ |\xi| = 1 \to \Sigma_0$$

is smooth and the atom sphere is compact, we know Σ_0 is compact. So there exists a const r, which is independent of l, j, such that $\Sigma_0 \subset B_r$. That is, for any j, ν, we have

$$|\Phi_\xi(0, \xi_j^\nu)| \leqslant 2^r. \tag{3.3}$$

So for all $x \in \mathbb{R}_j^\nu$ there exists a constant k_0 such that

$$\begin{aligned} |x| &\leqslant \max_{1 \leqslant i \leqslant n} \{c|y_i| + 2^r + |y - y_i|\} \\ &\leqslant c \cdot |y| + 2^r \leqslant \bar{c} 2^{-j/2} + 2^r \\ &\leqslant 2^{-j/2 + k_0} + 2^r \\ &\leqslant 2^{k_1}(2^{-j/2} + 1), \end{aligned}$$

where $k_1 = \max\{k_0, r\}$. Then we have

$$B_l^* = \bigcup_{2^{-j} \leqslant 2^l} \bigcup_{\nu} \mathbb{R}_j^\nu \subset \bigcup_{2^{-j} \leqslant 2^l} B(0, 2^{k_1}(2^{-j/2} + 1)) \subset B(0, 2^{k_1}(2^{l/2} + 1));$$

since $l \leqslant 0$, we obtain

$$B_l^* \subset B_{k_1 + 1}, \tag{3.4}$$

where k_1 is a constant and is independent of l.

We recall the atom vectors $\{\xi_j^\nu\}$ used above; they give an essentially uniform grid on the atom sphere, with separation $2^{-j/2}$. Let

$$\Gamma_j^\nu = \{\xi\colon\ |\xi/|\xi| - \xi_j^\nu| \leqslant 2\cdot 2^{-j/2}\}.$$

For every j we shall choose C^∞ functions χ_j^ν, $\nu = 1,\ldots,N(j)$, each homogeneous of degree 0 in ξ and supported in Γ_j^ν, with

$$\sum_\nu \chi_j^\nu = 1 \qquad \text{for all } \xi \neq 0 \text{ and all } j. \tag{3.5}$$

So it is easy to obtain the refined Littlewood-Paley decomposition

$$1 = \widehat{\Psi}_0(\xi) + \sum_{j=1}^\infty \sum_\nu \chi_j^\nu(\xi)\cdot\widehat{\Psi}_j(\xi)$$

where $\widehat{\Psi}_j(\xi) = \eta(2^{-j}\xi) - \eta(2^{-j+1}\xi)$, and $\eta \in C^\infty$,

$$\eta(\xi) = \begin{cases} 1, & |\xi| \leqslant 1,\\ 0, & |\xi| > 2.\end{cases}$$

With this decomposition, we define operators T_j^ν by

$$T_j^\nu f(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi \mathrm{i}\Phi(x,\xi)} a_j^\nu(x,\xi)\hat{f}(x)\,\mathrm{d}\xi, \tag{3.6}$$

where $a_j^\nu(x,\xi) = \chi_j^\nu(\xi)\cdot\widehat{\Psi}_j(\xi)\cdot a(x,\xi)$. We also define the corresponding operators T_j, using symbols $a_j(x) = \widehat{\Psi}_j(\xi)\cdot a(x,\xi)$. Clearly

$$T_j = \sum_\nu T_j^\nu. \tag{3.7}$$

We let K_j denote the kernel of the operator T_j. For $y \in B(0,2^l)$, the key estimates we shall derive are

$$\left(\int_{C_k} |K_j(x,y)|^q\,\mathrm{d}x\right)^{1/q} \leqslant C2^{j(n+m-(n+1)/2q)} \quad \text{for all } k \in \mathbb{Z}, \tag{3.8}$$

$$\left(\int_{C_k} |K_j(x,y)|^q\,\mathrm{d}x\right)^{1/q} \leqslant C2^{j(n+m-1/2q-N)}, \quad k > k_1 + 1, \tag{3.9}$$

$$\left(\int_{C_k} |D_y^\gamma K_j(x,y)|^q\,\mathrm{d}x\right)^{1/q} \leqslant C2^{j(n+m+s+1-(n+1)/2q)} \quad \text{for all } k \in \mathbb{Z}, \tag{3.10}$$

$$\left(\int_{C_k} |D_y^\gamma K_j(x,y)|^q\,\mathrm{d}x\right)^{1/q} \leqslant C2^{j(n+m+s+1-1/2q-N)}, \quad k > k_1 + 1, \tag{3.11}$$

where the bound C is independent of j,k, $\gamma=(\gamma_1,\dots,\gamma_n)$ is a multi-index with $|\gamma|=\sum\limits_{k=1}^{n}\gamma_k=s+1$ and $D_y^\gamma=\sum\limits_{|\gamma|=s+1}D_{y_1}^{\gamma_1}\dots D_{y_n}^{\gamma_n}$, $y=(y_1,\dots,y_n)$.

Because of (3.7), it suffices to derive similar estimates for the kernels of the T_j^ν's, which we denote by K_j^ν. By the definition of T_j^ν, we see that its kernel is given by

$$K_j^\nu(x,y)=\int_{\mathbb{R}^n}\mathrm{e}^{2\pi\mathrm{i}(\Phi(x,\xi)-y\cdot\xi)}a_j^\nu(x,\xi)\,\mathrm{d}\xi.$$

From the discussion by E. M. Stein in [7], we know that

$$\begin{aligned}(3.12)\qquad |K_j^\nu(x,y)|\leqslant C2^j\cdot 2^{j(n-1)/2+mj}&\\ \times\{1+2^j\cdot|(\Phi_\xi(x,\xi_j^\nu)-y)_1|+2^{j/2}\cdot|(\Phi_\xi(x,\xi_j^\nu)-y)'|\}^{-2N}&,\end{aligned}$$

where $(\cdot)_1$ indicates the component in the direction ξ_j^ν, and $(\cdot)'$ denotes the orthogonal component.

So for $q>1$,

$$\begin{aligned}&\Big(\int_{C_k}|K_j^\nu(x,y)|^q\,\mathrm{d}x\Big)^{1/q}\\ &\leqslant C2^{j((n+1)/2+m)}\cdot\Big(\int_{C_k}(1+2^j|(\Phi_\xi(x,\xi_j^\nu)-y)_1|+2^{j/2}|(\Phi_\xi(x,\xi_j^\nu)-y)'|)^{-2Nq}\,\mathrm{d}x\Big)^{1/q}\\ &\leqslant C\cdot 2^{j((n+1)/2+m-(n+1)/2q)}\Big(\int_{\mathbb{R}^n}(1+|z|)^{-2Nq}\,\mathrm{d}z\Big)^{1/q}\\ &\leqslant C\cdot 2^{j((n+1)/2+m-(n+1)/2q)},\end{aligned}$$

if we choose $-2Nq<-n$.

Because B_k is compact, similarly to the consideration concerning $B_l^*\subset B_{k_1+1}$, we have that the mapping $x\mapsto z=\Phi_\xi(x,\xi_j^\nu)$ projects C_k into $\{x\colon\ 2^{k_1}+2^{k-1}<|x|\leqslant 2^{k_1}+2^k\}$, so the mapping

$$x\mapsto z=(2^j(\Phi_\xi(x,\xi_j^\nu))_1,2^{j/2}(\Phi_\xi(x,\xi_j^\nu))')$$

projects C_k into $D_{k,j}=\{x\colon\ (2^{k_1}+2^{k-1})2^{j/2}<|x|\leqslant(2^{k_1}+2^k)2^j\}$. When $k>k_1+1$, we know $C_k\subset^c B_l^*$, thus $|\Phi_\xi(x,\xi_j^\nu)-y|\sim|\Phi_\xi(x,\xi_j^\nu)|$, so we have

$$\begin{aligned}&\Big(\int_{C_k}|K_j^\nu(x,y)|^q\,\mathrm{d}x\Big)^{1/q}\\ &\leqslant C2^{j((n+1)/2+m)}\cdot\Big(\int_{C_k}(1+2^j|(\Phi_\xi(x,\xi_j^\nu)-y)_1|+2^{j/2}|(\Phi_\xi(x,\xi_j^\nu)-y)'|)^{-2Nq}\,\mathrm{d}x\Big)^{1/q}\\ &\leqslant C2^{j((n+1)/2+m-(n+1)/2q)}\Big(\int_{D_{k,j}}(1+|z|)^{-2Nq}\,\mathrm{d}z\Big)^{1/q},\end{aligned}$$

and taking N so large that $-2Nq < -n$, we have

$$
\begin{aligned}
\left(\int_{C_k} |K_j^{\nu}(x,y)|^q \,\mathrm{d}x\right)^{1/q} \\
&\leqslant C \cdot 2^{j((n+1)/2+m-(n+1)/2q)} \cdot ((2^{k_1}+2^{k-1}) \cdot 2^{j/2})^{(-2N+n/q)} \\
&\leqslant C \cdot 2^{j((n+1)/2+m-1/2q-N)} \cdot 2^{(k_1+1)(-2N+n/q)} \\
&\leqslant C \cdot 2^{j((n+1)/2+m-1/2q-N)};
\end{aligned}
$$

the second inequality follows by virtue of $2^{k_1+1} < 2^{k_1} + 2^{k-1} < 2^k$ when $k > k_1 + 1$.

Since $N(j) \sim 2^{j(n-1)/2}$, (3.8) and (3.9) are obtained.

A similar estimate holds for $D_y^{\gamma} K_j^{\nu}(x,y)$, once we observe that the differentiation in y introduces factors bounded by 2^j. As a result, (3.10) and (3.11) are obtained.

4. BOUNDEDNESS ON HK_q

Now let us present the proof of Theorem 1.1.

By Lemma 2.4 and lemma 2.5, it suffices to show that

$$\|Ta_l\|_{K_q} \leqslant C \tag{4.1}$$

for all $(1,q)$-atom a_l associated with the ball $B(0,2^l)$ and the C is independent of l.

To begin with, if the radius of B_l exceeds 1, the estimate of (4.1) is easy, because of our assumption that the symbol of T has compact support in the variable x: hence there exists a K_0 such that $k > K_0$ implies $Ta_l = 0$. So

$$
\begin{aligned}
\|Ta_l\|_{K_q} &= \sum_{k=-\infty}^{K_0} 2^{kn/q'} \|Ta_l\chi_k\|_{L^q} \\
&\leqslant C \cdot \sum_{k=-\infty}^{K_0} 2^{kn/q'} \|a_l\|_{L^q} \\
&\leqslant C \cdot 2^{ln(1/q-1)} \sum_{k=-\infty}^{K_0} 2^{kn/q'} \\
&\leqslant C;
\end{aligned}
$$

the first inequality follows from the L^q boundedness property already obtained by A. Seeger in [6].

Next, assuming that $l < 0$, we have

$$\begin{aligned}\|Ta_l\|_{K_q} &= \sum_{k=-\infty}^{k_1+1} 2^{kn/q'} \|Ta_l\chi_k\|_{L^q} + \sum_{k=k_1+2}^{K_0} 2^{kn/q'} \|Ta_l\chi_k\|_{L^q} \\ &\leqslant \sum_{k=-\infty}^{k_1+1} 2^{kn/q'} \sum_{j=1}^{\infty} \|T_ja_l\chi_k\|_{L^q} + \sum_{k=k_1+2}^{K_0} 2^{kn/q'} \sum_{j=1}^{\infty} \|T_ja_l\chi_k\|_{L^q} \\ &= I + II.\end{aligned}$$

By using the Minkowski inequality, we arrive at

$$\begin{aligned}\|T_ja_l\chi_k\|_{L_q} &= \left(\int_{C_k} \left| \int_{B_l} K_j(x,y)a_l(y)\,\mathrm{d}y \right|^q \mathrm{d}x\right)^{1/q} \\ &\leqslant \int_{B_l} \left(\int_{C_k} |K_j(x,y)|^q \,\mathrm{d}x\right)^{1/q} |a_l(y)|\,\mathrm{d}y \\ &\leqslant \sup_{y\in B_l} \left(\int_{C_k} |K_j(x,y)|^q \,\mathrm{d}x\right)^{1/q} \cdot \|a_l\|_1.\end{aligned} \tag{4.2}$$

By (3.8), (4.2) and $\|a_l\|_1 \leqslant C$ we get

$$\begin{aligned}II &\leqslant \sum_{k=-\infty}^{k_1+1} 2^{kn(1-1/q)} \cdot \sum_{j=1}^{\infty} \|T_ja_l\chi_k\|_{L^q} \\ &\leqslant \sum_{k=-\infty}^{k_1+1} 2^{kn(1-1/q)} \cdot \sum_{j=1}^{\infty} 2^{j(m+n-(n+1)/2q)}, \\ &\leqslant C,\end{aligned}$$

when m satisfies $m \leqslant (n+1)/2q - n$.

By (3.9), (4.2) and $\|a_l\|_1 \leqslant C$, we get

$$\begin{aligned}II &\leqslant \sum_{k=k_1+2}^{K_0} 2^{kn(1-1/q)} \cdot \sum_{j=1}^{\infty} \|T_ja_l\chi_k\|_{L^q} \\ &\leqslant \sum_{k=k_1+2}^{K_0} 2^{kn(1-1/q)} \cdot \sum_{j=1}^{\infty} 2^{j(m+n-1/2q-N)}, \\ &\leqslant C,\end{aligned}$$

when we choose N such that $N > \max\{m+n-1/2q, n/2q\}$.

Because $q > 1$, we have $(n+1)/2q - n < -(n+1)/2$. So by the properties of symbols that if $\alpha < \beta$, $a \in S^{\alpha}$ implies $a \in S^{\beta}$, we know that when $m \leqslant (n+1)/2q - n$, the (q,q) boundedness of the *Fourier integral operators* is certain. So when $-n < m \leqslant -(n-1)/2$ and q satisfies

$$\frac{1}{q} \geqslant 2 + \frac{2(m-1)}{(n+1)},$$

we get (4.1). So Theorem 1.1 is proved.

5. BOUNDEDNESS ON $K_q^{\alpha,p}$ AND $HK_q^{\alpha,p}$

Now, let us deal with the boundedness on $K_q^{\alpha,p}$.

Suppose $f \in \dot{K}_q^{\alpha,p}$. By Lemma 2.3, $f(x) = \sum\limits_{l\in\mathbb{Z}} \lambda_l a_l(x)$, where $a_l(x)$ is a central (α,q)-atom with the support B_l, and

$$\sum_{l\in\mathbb{Z}} |\lambda_l|^p \leqslant C\|f\|^p_{\dot{K}_q^{\alpha,p}}.$$

By Definition 1.2 we have

$$\begin{aligned}
\|Tf\|^p_{\dot{K}_q^{\alpha,p}} &= \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\|Tf\chi_k\|^p_{L^q}\\
&= \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\big(\sum_{l>0} + \sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\big)^p\\
&\leqslant C\cdot\Big\{\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\Big(\sum_{l>0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p + \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\Big(\sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p\Big\}\\
&= C(I_1 + I_2).
\end{aligned}$$

Let us first estimate I_1. Using the fact that T maps L^q into L^q and the assumption that the symbol of T has compact support in the variable x, we have

$$\begin{aligned}
I_1 &\leqslant C\sum_{k=-\infty}^{K_0} 2^{k\alpha p}\sum_{l>0}|\lambda_l|^p\|a_l\|^p_{L^q}\\
&\leqslant \begin{cases} C\sum\limits_{k=-\infty}^{K_0} 2^{k\alpha p}\sum\limits_{l>0}|\lambda_l|^p 2^{-l\alpha p}, & 0<p\leqslant 1\\[2ex] C\sum\limits_{k=-\infty}^{K_0} 2^{k\alpha p}\Big(\sum\limits_{l>0}|\lambda_l|^p 2^{-l\alpha p/2}\Big)\Big(\sum\limits_{l>0} 2^{-l\alpha p'/2}\Big)^{p/p'}, & p>1\end{cases}\\
&\leqslant C\sum_{l\in\mathbb{Z}}|\lambda_l|^p.
\end{aligned}$$

To estimate I_2, by Hölder's inequality and the proposition of (α, q)-atom we get

$$\int |a_l(y)|\,\mathrm{d}y \leqslant \left(\int |a_l(y)|^q\,\mathrm{d}y\right)^{1/q} \cdot |B_l|^{1-1/q} \leqslant C2^{l(n-n/q-\alpha)}. \tag{5.1}$$

By the Minkowski inequality and the above we have

$$\|Ta_l\chi_k\|_{L^q} \leqslant C2^{l(n-n/q-\alpha)} \sup_{y\in B_l} \left(\int_{C_k} |K(x,y)|^q\,\mathrm{d}x\right)^{1/q}. \tag{5.2}$$

I_2 is dominated by

$$\begin{aligned} I_2 &= \sum_{k=-\infty}^{k_1+1} 2^{k\alpha p}\Bigl(\sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Bigr)^p + \sum_{k=k_1+2}^{K_0} 2^{k\alpha p}\Bigl(\sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Bigr)^p \\ &= J_1 + J_2. \end{aligned}$$

For J_1, using (5.2) and (3.8) we have

$$J_1 \leqslant C\sum_{k=-\infty}^{k_1+1} 2^{k\alpha p}\Bigl(\sum_{j=1}^{\infty} 2^{j(m+n-(n+1)/2q)}\Bigr)^p \Bigl(\sum_{l\leqslant 0}|\lambda_l| 2^{l(n-n/q-\alpha)}\Bigr)^p.$$

By condition (1.3), J_1 can be controlled by

$$J_1 \leqslant \begin{cases} C\cdot \sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(n-n/q-\alpha)}, & 0<p\leqslant 1,\\ C\cdot \Bigl(\sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(n-n/q-\alpha)/2}\Bigr)\Bigl(\sum\limits_{l\leqslant 0} 2^{lp'(n-n/q-\alpha)/2}\Bigr)^{p/p'}, & p>1. \end{cases}$$

As $\alpha < n(1-1/q)$, we obtain

$$J_1 \leqslant C\sum_{l\in\mathbb{Z}}|\lambda_l|^p.$$

For J_2, using (5.2) and (3.9) we have

$$J_2 \leqslant C\sum_{k=k_1+2}^{k_0+1} 2^{k\alpha p}\Bigl(\sum_{j=1}^{\infty} 2^{j[m+n-1/2q-N]}\Bigr)^p \Bigl(\sum_{l\leqslant 0}|\lambda_l| 2^{l(n-n/q-\alpha)}\Bigr)^p.$$

Choosing N so that $N > \max\{n/2q, m+n-1/2q\}$, we have

$$J_2 \leqslant \begin{cases} C\cdot \sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(n-n/q-\alpha)}, & 0<p\leqslant 1,\\ C\cdot \Bigl(\sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(n-n/q-\alpha)/2}\Bigr)\Bigl(\sum\limits_{l\leqslant 0} 2^{lp'(n-n/q-\alpha)/2}\Bigr)^{p/p'}, & p>1. \end{cases}$$

Since $\alpha < n(1-1/q)$, it is easy to see that

$$J_2 \leqslant C \sum_{l\in\mathbb{Z}} |\lambda_l|^p.$$

Therefore

$$\|Tf\|^p_{\dot{K}_q^{\alpha,p}} \leqslant C \sum_{l\in\mathbb{Z}} |\lambda_l|^p.$$

So the boundedness on $K_q^{\alpha,p}$ is obtained.

Now let us consider the boundedness on $HK_q^{\alpha,p}$.

Suppose $f \in H\dot{K}_q^{\alpha,p}$. By Lemma 2.3, $f(x) = \sum\limits_{l\in\mathbb{Z}} \lambda_l a_l(x)$, where $a_l(x)$ is a central (α,q)-atom with the support B_l, and

$$\sum_{l\in\mathbb{Z}} |\lambda_l|^p \leqslant c\|f\|^p_{H\dot{K}_q^{\alpha,p}}\,.$$

By Definition 1.2, we have

$$\begin{aligned}
\|Tf\|^p_{\dot{K}_q^{\alpha,p}} &:= \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\|Tf\chi_k\|^p_{L^q}\\
&= \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\Big(\sum_{l>0} + \sum_{l\leqslant 0} |\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p\\
&\leqslant C\cdot\Big\{\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\Big(\sum_{l>0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p + \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\Big(\sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p\Big\}\\
&= C(I_1 + I_2).
\end{aligned}$$

Similarly to the estimate of I_1 in Theorem 1.2 we have $I_1 \leqslant C\sum\limits_{l} |\lambda_l|^p$.

For the estimate of I_2, suppose $n+s \leqslant \alpha + n/q < n+s+1$, and let γ be a multi-index with $|\gamma| = s+1$. we have

$$\begin{aligned}
\|T_j a_l\chi_k\|_{L^q} &= \Big(\int_{C_k}\Big|\int K_j(x,y)\cdot a_l(y)\,\mathrm{d}y\Big|^q\,\mathrm{d}x\Big)^{1/q}\\
&\leqslant C\Big(\int_{C_k}\Big|\int D_y^\gamma K_j(x,y)\cdot a_l(y)y^\gamma\,\mathrm{d}y\Big|^q\,\mathrm{d}x\Big)^{1/q}\\
&\leqslant C2^{l(s+1)}\cdot \sup\nolimits_{y\in B_l}\Big(\int_{C_k}|D_y^\gamma K_j(x,y)|^q\,\mathrm{d}x\Big)^{1/q}\cdot\int|a_l(y)|\,\mathrm{d}y,
\end{aligned}$$

by virtue of the vanishing moment condition of the atom and the Minkowski inequality. And then using (3.10), (3.11) and (5.1), we can dominate $\|T_j a_l\chi_k\|_{L^q}$ by

$$\|T_j a_l\chi_k\|_{L^q} \leqslant C2^{l(s+1+n-n/q-\alpha)}2^{j(s+1+m+n-(n+1)/2q)}, \quad k\in\mathbb{Z}, \tag{5.3}$$

and

$$(5.4) \qquad \|T_j a_l \chi_k\|_{L^q} \leqslant C 2^{l(s+1+n-n/q-\alpha)} 2^{j(s+1+m+n-1/2q-N)}, \quad k > k_1+1.$$

Now let us estimate I_2. Write

$$\begin{aligned} I_2 &= \sum_{k=-\infty}^{k_1+1} 2^{k\alpha p}\Big(\sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p + \sum_{k=k_1+2}^{K_0} 2^{k\alpha p}\Big(\sum_{l\leqslant 0}|\lambda_l|\|Ta_l\chi_k\|_{L^q}\Big)^p \\ &= J_1 + J_2. \end{aligned}$$

For J_1, using (5.3) we have

$$J_1 \leqslant C \sum_{k=-\infty}^{k_1+1} 2^{k\alpha p}\Big(\sum_{j=1}^{\infty} 2^{j(s+1+m+n-(n+1)/2q)}\Big)^p \Big(\sum_{l\leqslant 0}|\lambda_l| 2^{l(s+1+n-n/q-\alpha)}\Big)^p.$$

By the fact that $m \leqslant (n+1)/2q - n - s - 1$, we have

$$J_1 \leqslant \begin{cases} C\cdot \sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(s+1+n-n/q-\alpha)}, & 0<p\leqslant 1,\\ C\cdot \Big(\sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(s+1+n-n/q-\alpha)/2}\Big)\Big(\sum\limits_{l\leqslant 0} 2^{lp'(s+1+n-n/q-\alpha)/2}\Big)^{p/p'}, & p>1. \end{cases}$$

The assumption $\alpha + n/q < n+s+1$ implies

$$J_1 \leqslant C\sum_{l\in\mathbb{Z}}|\lambda_l|^p.$$

For J_2, using (5.4) we have

$$J_2 \leqslant C \sum_{k=k_1+2}^{K_0} 2^{k\alpha p}\Big(\sum_{j=1}^{\infty} 2^{j(s+1+m+n-1/2q-N)}\Big)^p \Big(\sum_{l\leqslant 0}|\lambda_l| 2^{l(s+1+n-n/q-\alpha)}\Big)^p.$$

Choosing N such that $N > \max\{n/2q, s+1+m+n-1/2q\}$, we have

$$J_2 \leqslant \begin{cases} C \sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(s+1+n-n/q-\alpha)}, & 0<p\leqslant 1,\\ C\Big(\sum\limits_{l\leqslant 0}|\lambda_l|^p 2^{lp(s+1+n-n/q-\alpha)/2}\Big)\Big(\sum\limits_{l\leqslant 0} 2^{lp'(s+1+n-n/q-\alpha)/2}\Big)^{p/p'}, & p>1. \end{cases}$$

By the assumption that $\alpha + n/q < n+s+1$, we get

$$J_2 \leqslant C\sum_{l\in\mathbb{Z}}|\lambda_l|^p.$$

Thus we get

$$\|Tf\|^p_{\dot{K}^{\alpha,p}_q} \leqslant C \sum_{l\in\mathbb{Z}} |\lambda_l|^p.$$

so we obtain the boundedness on $HK^{\alpha,p}_q$,

6. CONCLUSIONS

In this paper, the main result shows that a classical Fourier T is bounded from HK_q to K_q when the order m of T satisfies

$$-n < m \leqslant -\frac{n-1}{2}, \qquad \text{and} \qquad \frac{1}{q} \geqslant 2 + \frac{2(m-1)}{n+1}.$$

Also the boundedness from the other two kinds of Herz spaces $K^{\alpha,p}_q(\dot{K}^{\alpha,p}_q)$ and $HK^{\alpha,p}_q(H\dot{K}^{\alpha,p}_q)$ to Herz spaces $K^{\alpha,p}_q(\dot{K}^{\alpha,p}_q)$ is obtained. A natural problem is that whether the boundedness of T from Hardy spaces associated with Herz spaces into themselves is valid, on which we will keep an eye in the future.

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References

[1] *L. Hörmander*: Fourier integral operators I. Acta Math. *127* (1971), 79–183.

[2] *S. Lu and D. Yang*: The Littlewood-Paley function and φ-transform characterizations of a new Hardy space HK_2 associated with the Herz space. Stud. Math. *101* (1992), 285–298.

[3] *S. Lu and D. Yang*: The decomposition of the weighted Herz spaces on R_n and its applications. Sci. China Ser. A *38* (1995), 147–158.

[4] *S. Lu and D. Yang*: The weighted Herz-type Hardy space and its applications. Sci. China Ser. A *38* (1995), 661–673.

[5] *M. P. Marco and S. Silvia*: Boundedness of Fourier integral operators on Hardy spaces. Proc. Edinb. Math. Soc. *51* (2008), 443–463.

[6] *A. Seeger, C. Sogge and E. Stein*: Regularity properties of Fourier integral operators. Ann. Math. *134* (1991), 231–251.

[7] *E. Stein*: Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, 1993.

[8] *D. Yang*: The real-variable characterizations of Hardy spaces $HK_p(R^n)$. Adv. Math. *24* (1995), 63–73.

[9] *Y. Zhou*: Boundedness of sublinear operators in Herz-type Hardy spaces. Taiwannse J. Math. *13* (2009), 983–996.

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