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APPROXIMATION BY q-BERNSTEIN TYPE OPERATORS

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Abstract. Using the q-Bernstein basis, we construct a new sequence $\{L_n\}$ of positive linear operators in $C[0,1]$. We study its approximation properties and the rate of convergence in terms of modulus of continuity.

Keywords: q-integers, q-Bernstein operators, the Hahn-Banach theorem, modulus of continuity

MSC 2010: 41A25, 41A36

1. Introduction

Let $q > 0$. For each non-negative integer k, the q-integers $[k]$ and the q-factorials $[k]!$ are defined by

$$[k] = \begin{cases} 1 + q + \ldots + q^{k-1} & \text{if } k \geqslant 1, \\ 0 & \text{if } k = 0 \end{cases}$$

and

$$[k]! = \begin{cases} [1][2]\ldots[k] & \text{if } k \geqslant 1, \\ 1 & \text{if } k = 0. \end{cases}$$

For integers $0 \leqslant k \leqslant n$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}\ .$$

Following [4], the q-Bernstein operators $B_{n,q}\colon C[0,1] \to C[0,1]$ are introduced by

$$(B_{n,q}f)(x) \equiv B_{n,q}(f,x) = \sum_{k=0}^{n} f\Big(\frac{[k]}{[n]}\Big) p_{n,k}(q,x),$$

where $n = 1, 2, \ldots,$ $x \in [0, 1]$ and

$$p_{n,k}(q, x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k (1 - x)(1 - xq) \ldots (1 - xq^{n-k-1}), \tag{1.1}$$

$k = 0, 1, \ldots, n$, are the q-Bernstein basis polynomials (an empty product denotes 1). For $q = 1$, we recover the classical Bernstein operators.

Taking into account [4, (13)–(15)], we have

$$B_{n,q}(e_0, x) = 1, \tag{1.2}$$

$$B_{n,q}(e_1, x) = x \tag{1.3}$$

and

$$B_{n,q}(e_2, x) = x^2 + \frac{1}{[n]} x(1 - x), \tag{1.4}$$

where $e_i(x) = x^i$, $x \in [0, 1]$ and $i \in \{0, 1, 2\}$. Due to (1.4), it is worth mentioning that for a fixed value of q with $0 < q < 1$ we obtain

$$\lim_{n \to \infty} B_{n,q}(e_2, x) = x^2 + (1 - q)x(1 - x). \tag{1.5}$$

Further, let $q = q_n$ satisfy $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. Then $B_{n,q}(f, x)$ converges uniformly to $f(x)$ on $[0, 1]$ as $n \to \infty$ (see [4, Theorem 2]). Moreover, due to [4, (16)], we have

$$\|B_{n,q} f - f\| \leqslant \frac{3}{2} \omega(f, [n]^{-1/2}), \tag{1.6}$$

where $\|\cdot\|$ is the uniform norm on $C[0, 1]$ and the modulus of continuity of $f \in C[0, 1]$ is defined by

$$\omega(f, \delta) = \sup\{|f(u) - f(v)| \colon \ u, v \in [0, 1], |u - v| \leqslant \delta\}. \tag{1.7}$$

We mention, that an estimation of the rate of convergence of the q-Bernstein operators $B_{n,q}$ $(0 < q < 1)$ was also presented in [5, (17)]. The convergence properties of $B_{n,q}$ $(0 < q < 1)$ in the complex plane were studied in [3].

The goal of the paper is to construct a new non-trivial sequence $\{L_n\}$ of bounded positive linear operators in $C[0, 1]$ using the q-Bernstein basic polynomials (1.1), such that L_n has different properties from (1.3), (1.4) and (1.5), but $L_n(f, x)$ converges uniformly to $f(x)$ on $[0, 1]$ as $n \to \infty$. The rate of approximation $\|L_n f - f\|$ will be also estimated by the modulus of continuity (1.7).

2. The construction of L_n

Let us consider the following q-Bernstein type operators $L_n\colon C[0,1] \to C[0,1]$ defined by

$$(L_n f)(x) \equiv L_n(f,x) = \sum_{k=0}^{n} \lambda_{n,k}(q,f)\, p_{n,k}(q,x),$$

where $q \in (0,1)$, $x \in [0,1]$, $f \in C[0,1]$ and the bounded positive linear functionals $\lambda_{n,k}(q,\cdot) \in C[0,1]^*$ will be defined step by step as follows.

We set

$$\begin{gathered} \lambda_{n,0}(q,f) = f(0), \quad \lambda_{n,n}(q,f) = f(1), \\ \lambda_{n,k}(q,e_0) = 1, \quad \lambda_{n,k}(q,e_1) = \frac{[k-1]}{[n]} \end{gathered} \tag{2.1}$$

for $k = 1,2,\ldots,n-1$, and

$$\lambda_{n,k}(q,e_2) = \frac{[k][k-1]}{[n][n-1]} \tag{2.2}$$

for $k = 1,2,\ldots,n-1$, where $n \geqslant 2$.

Furthermore, $\lambda_{n,k}(q,\cdot)$, $k = 0,1,\ldots,n$, will be defined on the normed subspace $Y = \{\alpha e_0 + \beta e_1 + \gamma e_2\colon\ \alpha,\beta,\gamma \in \mathbb{R}\}$ of $C[0,1]$ as follows. For $P(x) = \alpha + \beta x + \gamma x^2$, $x \in [0,1]$, and $k = 0,1,\ldots,n$, we set

$$\lambda_{n,k}(q,P) = \alpha\lambda_{n,k}(q,e_0) + \beta\lambda_{n,k}(q,e_1) + \gamma\lambda_{n,k}(q,e_2).$$

We prove that $\lambda_{n,k}(q,\cdot) \in Y^*$ are bounded positive linear functionals, $k = 1,2,\ldots,n-1$. Obviously $\lambda_{n,k}(q,\cdot)$ are linear. Moreover, $\lambda_{n,k}(q,\cdot)$ are positive: if $P(x) \geqslant 0$ for $x \in [0,1]$, then we distinguish the following two cases:

a) $\gamma \geqslant 0$. Then $\lambda_{n,k}(q,P) \geqslant P([k-1]/[n]) \geqslant 0$, because $\lambda_{n,k}(q,e_2) \geqslant [k-1]^2/[n]^2$ for $k = 1,2,\ldots,n-1$.

b) $\gamma < 0$. Then, by $\lambda_{n,k}(q,e_2) \leqslant [k-1]/[n]$, $k = 1,2,\ldots,n-1$, we get

$$\begin{aligned} \lambda_{n,k}(q,P) &\geqslant \alpha + \beta\frac{[k-1]}{[n]} + \gamma\frac{[k-1]}{[n]} \\ &= \alpha\Big(1 - \frac{[k-1]}{[n]}\Big) + (\alpha+\beta+\gamma)\frac{[k-1]}{[n]} \\ &= P(0)\Big(1 - \frac{[k-1]}{[n]}\Big) + P(1)\frac{[k-1]}{[n]} \geqslant 0 \end{aligned}$$

for $k = 1,2,\ldots,n-1$.

Further, $\lambda_{n,k}(q,\cdot)$ are bounded on Y, $k=0,1,\dots,n$. Indeed, by positivity of $\lambda_{n,k}(q,\cdot)$ and (2.1) we have for all $P\in Y$ that

$$|\lambda_{n,k}(q,P)| \leqslant \lambda_{n,k}(q,|P|) \leqslant \lambda_{n,k}(q,\|P\|e_0) = \|P\|\lambda_{n,k}(q,e_0) = \|P\|.$$

Finally, we define $\lambda_{n,k}(q,\cdot)$, $k=1,2,\dots,n-1$, on the whole space $C[0,1]$. The real linear space $C[0,1]$ is an ordered Banach space with the uniform norm $\|\cdot\|$ and the natural order relation: $f\leqslant g$ if and only if $f(x)\leqslant g(x)$, $x\in[0,1]$. Using the notation $C[0,1]_+ = \{f\in C[0,1]\colon\ 0_{C[0,1]}\leqslant f\}$, we have $\{f\in C[0,1]\colon\ \|f-e_0\|<1\}\subset C[0,1]_+$. Thus $\operatorname{int} C[0,1]_+\neq\emptyset$ and $e_0\in Y\cap\operatorname{int} C[0,1]_+$. Now we can extend $\lambda_{n,k}(q,\cdot)$ onto the whole space $C[0,1]$ as bounded positive linear functionals, because of the following Hahn-Banach type theorem: if $(X,\leqslant)$ is an ordered normed space with $\operatorname{int}\{x\in X\colon\ 0_X\leqslant x\}\neq\emptyset$ and Y is a normed subspace of X such that $Y\cap\operatorname{int}\{x\in X\colon\ 0_X\leqslant x\}\neq\emptyset$, then every bounded positive linear functional $\lambda\colon\ Y\to\mathbb{R}$ can be extended to a bounded positive linear functional $\tilde{\lambda}\colon\ X\to\mathbb{R}$, i.e. $\tilde{\lambda}(x)=\lambda(x)$ for all $x\in X$. This result is a particular case of a more general theorem of [2, p. 82], where $\mathbb{R}$ is replaced by a complete vector lattice with identity element. We mention that the extension of bounded positive linear functionals was studied first in [1].

Consequently, L_n are positive linear operators. Moreover, $\|L_nf\|\leqslant\|f\|$ for all $f\in C[0,1]$, because the positivity of $\lambda_{n,k}(q,\cdot)$, (2.1) and (1.2) imply for $x\in[0,1]$ that

$$\begin{aligned}|L_n(f,x)| &\leqslant \sum_{k=0}^{n}|\lambda_{n,k}(q,f)|p_{n,k}(q,x) \leqslant \sum_{k=0}^{n}\lambda_{n,k}(q,|f|)p_{n,k}(q,x)\\ &\leqslant \sum_{k=0}^{n}\lambda_{n,k}(q,\|f\|e_0)p_{n,k}(q,x) = \|f\|\sum_{k=0}^{n}\lambda_{n,k}(q,e_0)p_{n,k}(q,x)\\ &= \|f\|\sum_{k=0}^{n}p_{n,k}(q,x) = \|f\|B_{n,q}(e_0,x) = \|f\|.\end{aligned}$$

Thus L_n are bounded operators, $n\geqslant 2$.

3. Main results

For the operators L_n introduced in Section 2 we have the following results.

Theorem 3.1. *The operators L_n ($n\geqslant 2$ and $0<q<1$) verify:*

a) $L_n(e_0,x)=1$, $x\in[0,1]$;
b) $0\leqslant x-L_n(e_1,x)\leqslant 1/[n]$, $x\in[0,1]$;

c) $L_n(e_2,x)=x^2$, $x\in[0,1]$.

For a fixed value $q\in(0,1)$ *we have*

d) $\lim\limits_{n\to\infty} L_n(e_1,x)=x-(1-q)q^{-1}(1-x)\Big\{1-\prod\limits_{s=1}^{\infty}(1-xq^s)\Big\}$, $x\in[0,1]$.

Proof. a) By (2.1) and (1.2), we have $L_n(e_0,x)=B_{n,q}(e_0,x)=1$.

b) Taking into account (2.1) and (1.3), we obtain

$$\begin{aligned}L_n(e_1,x)&=\sum_{k=1}^{n-1}\frac{[k-1]}{[n]}p_{n,k}(q,x)+p_{n,n}(q,x)\\&=\sum_{k=1}^{n-1}\frac{[k]-q^{k-1}}{[n]}p_{n,k}(q,x)+p_{n,n}(q,x)\\&=B_{n,q}(e_1,x)-\frac{1}{[n]}\sum_{k=1}^{n-1}q^{k-1}p_{n,k}(q,x)=x-\frac{1}{[n]}\sum_{k=1}^{n-1}q^{k-1}p_{n,k}(q,x).\end{aligned}$$

Hence, by (1.2),

$$0\leqslant x-L_n(e_1,x)=\frac{1}{[n]}\sum_{k=1}^{n-1}q^{k-1}p_{n,k}(q,x)\leqslant\frac{1}{[n]}B_{n,q}(e_0,x)=\frac{1}{[n]}.$$

c) By (2.2) and (1.2), we have

$$\begin{aligned}L_n(e_2,x)&=\sum_{k=1}^{n-1}\frac{[k][k-1]}{[n][n-1]}\,p_{n,k}(q,x)+p_{n,n}(q,x)\\&=\sum_{k=2}^{n-1}\begin{bmatrix}n-2\\k-2\end{bmatrix}x^k(1-x)(1-xq)\dots(1-xq^{n-k-1})+x^n\\&=x^2\sum_{k=0}^{n-2}p_{n-2,k}(q,x)=x^2B_{n-2,q}(e_0,x)=x^2.\end{aligned}$$

d) Using $[k]=1+q[k-1]$, $k\geqslant1$, we find

$$\begin{aligned}L_n(e_1,x)&=\sum_{k=1}^{n-1}\frac{[k-1]}{[n]}\,p_{n,k}(q,x)+p_{n,n}(q,x)=\sum_{k=1}^{n-1}\frac{[k]-1}{q[n]}\,p_{n,k}(q,x)+p_{n,n}(q,x)\\&=\frac{1}{q}\sum_{k=0}^{n}\frac{[k]}{[n]}\,p_{n,k}(q,x)+\Big(1-\frac{1}{q}\Big)p_{n,n}(q,x)-\frac{1}{q[n]}\sum_{k=1}^{n-1}p_{n,k}(q,x)\\&=\frac{1}{q}B_{n,q}(e_1,x)-\frac{1}{q[n]}B_{n,q}(e_0,x)+\frac{1}{q[n]}p_{n,0}(q,x)\\&\quad+\Big(1-\frac{1}{q}+\frac{1}{q[n]}\Big)p_{n,n}(q,x).\end{aligned}$$

Hence, by (1.3) and (1.2),

$$L_n(e_1,x)=\frac{1}{q}x-\frac{1}{q[n]}+\frac{1}{q[n]}(1-x)(1-xq)\dots(1-xq^{n-1})+\Big(1-\frac{1}{q}+\frac{1}{q[n]}\Big)x^n. \tag{3.1}$$

On the other hand, due to [6, (2.8)], we have

$$\begin{aligned}&\Big|(1-x)(1-xq)\dots(1-xq^{n-1})-\prod_{s=0}^{\infty}(1-xq^s)\Big|\\&=(1-x)(1-xq)\dots(1-xq^{n-1})\Big|1-\prod_{s=n}^{\infty}(1-xq^s)\Big|\\&\leqslant\frac{q^n}{q(1-q)}\ln\frac{1}{1-q}\end{aligned}$$

for $x\in[0,1]$. Hence

$$\lim_{n\to\infty}(1-x)(1-xq)\dots(1-xq^{n-1})=\prod_{s=0}^{\infty}(1-xq^s). \tag{3.2}$$

Now combining (3.1), (3.2) and $\lim\limits_{n\to\infty}[n]=1/(1-q)$, we obtain

$$\begin{aligned}\lim_{n\to\infty}L_n(e_1,x)&=\frac{1}{q}x-\frac{1-q}{q}+\frac{1-q}{q}\prod_{s=0}^{\infty}(1-xq^s)\\&=x-\frac{1-q}{q}(1-x)\Big\{1-\prod_{s=1}^{\infty}(1-xq^s)\Big\},\end{aligned}$$

which was to be proved. □

Theorem 3.2. *Let $q=q_n\in(0,1)$ satisfy $q_n\to1$ as $n\to\infty$. Then for each $f\in C[0,1]$, the sequence $\{L_n(f,x)\}$ converges uniformly to $f(x)$ on $[0,1]$ as $n\to\infty$. Moreover, for each $f\in C[0,1]$ and $n\geqslant2$ we have*

$$\|L_nf-f\|\leqslant\Big(\sqrt{2}+\frac{5}{2}\Big)\omega(f,[n]^{-1/2}). \tag{3.3}$$

Proof. For any fixed positive integer k, we have $[n]\geqslant[k]=1+q+\ldots+q^{k-1}$ when $n\geqslant k$. But $q=q_n\to1$ as $n\to\infty$, therefore $\liminf\limits_{n\to\infty}[n]\geqslant\lim\limits_{n\to\infty}[k]=k$. Since k has been chosen arbitrarily, it follows that $[n]\to\infty$ as $n\to\infty$. Then (3.3) implies that $\{L_n(f,x)\}$ converges uniformly to $f(x)$ on $[0,1]$ as $n\to\infty$. Thus it remains to prove (3.3).

Let $x\in[0,1]$ and $n\geqslant 2$. Then (2.1) and (1.6) imply that

$$
\begin{aligned}
(3.4)\quad |L_n(f,x)-f(x)| &\leqslant |L_n(f,x)-B_{n,q}(f,x)|+|B_{n,q}(f,x)-f(x)|\\
&\leqslant \sum_{k=1}^{n-1}\Big|\lambda_{n,k}(q,f)-f\Big(\frac{[k]}{[n]}\Big)\Big|p_{n,k}(q,x)+\frac{3}{2}\omega(f,[n]^{-1/2})\\
&\leqslant \sum_{k=1}^{n-1}\lambda_{n,k}\Big(q,\Big|f-f\Big(\frac{[k]}{[n]}\Big)e_0\Big|\Big)p_{n,k}(q,x)+\frac{3}{2}\,\omega(f,[n]^{-1/2}).
\end{aligned}
$$

Further, using the property $\omega(f,a\delta)\leqslant(a+1)\omega(f,\delta)$, $a>0$, we obtain

$$
\begin{aligned}
\Big|f(t)-f\Big(\frac{[k]}{[n]}\Big)\Big| &\leqslant \omega\Big(f,\Big|t-\frac{[k]}{[n]}\Big|\Big)\\
&\leqslant \Big([n]^{1/2}\Big|t-\frac{[k]}{[n]}\Big|+1\Big)\omega(f,[n]^{-1/2}).
\end{aligned}
$$

Then, by positivity of $\lambda_{n,k}(q,\cdot)$, we find

$$
\begin{aligned}
(3.5)\qquad &\lambda_{n,k}\Big(q,\Big|f-f\Big(\frac{[k]}{[n]}\Big)e_0\Big|\Big)\\
&\leqslant \Big\{[n]^{1/2}\lambda_{n,k}\Big(q,\Big|e_1-\frac{[k]}{[n]}\,e_0\Big|\Big)+1\Big\}\omega(f,[n]^{-1/2}).
\end{aligned}
$$

Because $\lambda_{n,k}(q,\cdot)$ are bounded linear functionals, we have $\lambda_{n,k}(q,f)=\int_0^1 f\,\mathrm{d}\mu_{n,k}$ for some positive measures $\mu_{n,k}$. Applying the Hölder inequality, (2.1) and (2.2), we get

$$
\begin{aligned}
\int_0^1\Big|t-\frac{[k]}{[n]}\Big|\,\mathrm{d}\mu_{n,k} &\leqslant \Big(\int_0^1\,\mathrm{d}\mu_{n,k}\Big)^{1/2}\Big(\int_0^1\Big(t-\frac{[k]}{[n]}\Big)^2\,\mathrm{d}\mu_{n,k}\Big)^{1/2}\\
&= (\lambda_{n,k}(q,e_0))^{1/2}\Big(\lambda_{n,k}\Big(q,\Big(e_1-\frac{[k]}{[n]}\,e_0\Big)^2\Big)\Big)^{1/2}\\
&= \Big(\frac{[k][k-1]}{[n][n-1]}-2\frac{[k-1]}{[n]}\frac{[k]}{[n]}+\frac{[k]^2}{[n]^2}\Big)^{1/2}\\
&= [n]^{-1/2}\Big(q^{n-1}\frac{[k][k-1]}{[n][n-1]}+q^{k-1}\frac{[k]}{[n]}\Big)^{1/2}\\
&\leqslant \sqrt{2}\,[n]^{-1/2}.
\end{aligned}
$$

Hence, for $k=1,2,\dots,n-1$ we have

$$
(3.6)\qquad \lambda_{n,k}\Big(q,\Big|e_1-\frac{[k]}{[n]}e_0\Big|\Big)\leqslant\sqrt{2}[n]^{-1/2}.
$$

Now combining (3.4), (3.5), (3.6) and (1.2), we obtain

$$|L_n(f,x) - f(x)| \leqslant \Big(\sqrt{2} + \frac{5}{2}\Big)\omega(f, [n]^{-1/2}),$$

which completes the proof of our theorem. □

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