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THE STRUCTURE OF DIGRAPHS ASSOCIATED WITH THE
CONGRUENCE $x^k \equiv y \pmod{n}$

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Abstract. We assign to each pair of positive integers n and $k \geq 2$ a digraph $G(n, k)$ whose set of vertices is $H = \{0, 1, \dots, n - 1\}$ and for which there is a directed edge from $a \in H$ to $b \in H$ if $a^k \equiv b \pmod{n}$. We investigate the structure of $G(n, k)$. In particular, upper bounds are given for the longest cycle in $G(n, k)$. We find subdigraphs of $G(n, k)$, called fundamental constituents of $G(n, k)$, for which all trees attached to cycle vertices are isomorphic.

Keywords: Sophie Germain primes, Fermat primes, primitive roots, Chinese Remainder Theorem, congruence, digraphs

MSC 2010: 11A07, 11A15, 05C20, 20K01

1. INTRODUCTION

In this paper, we construct a digraph associated with the congruence $x^k \equiv y \pmod{n}$. We will see that each component of this digraph contains a unique cycle. Our main result given in Theorem 6.1 is to partition this digraph into sets of components, called fundamental constituents, so that all trees attached to cycle vertices of a particular fundamental constituent of the digraph are isomorphic. In Theorem 9.2 we obtain new results on the length of the longest cycle in this digraph extending the results given in [7]. In Theorem 8.1, we obtain lower bounds for the number of cycles of length one, while in Theorem 8.2, we count the number of isolated cycles of length one. A major technique used in this paper is to decompose a digraph into a product of digraphs.

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The paper extends results given in the works [7], [10], [14], and [16], which provide an interesting connection between number theory, graph theory, and group theory. In the papers [10]–[13], we investigated properties of the iteration digraph representing a dynamical system occurring in number theory. For related results also see [1].

For $n \geq 1$ let

$$H = \{0, 1, \dots, n - 1\}$$

and let f be a map of H into itself. The *iteration digraph* of f is a directed graph whose vertices are elements of H and such that there exists exactly one directed edge from x to $f(x)$ for all $x \in H$. For a fixed integer $k \geq 2$ and for each $x \in H$ let $f(x)$ be the remainder of x^k modulo n , i.e.,

$$(1.1) \quad f(x) \in H \quad \text{and} \quad x^k \equiv f(x) \pmod{n}.$$

From here on, whenever we refer to the iteration digraph of f , we assume that the mapping f is as given in (1.1). Each pair of natural numbers n and $k \geq 2$ has a specific iteration digraph corresponding to it.

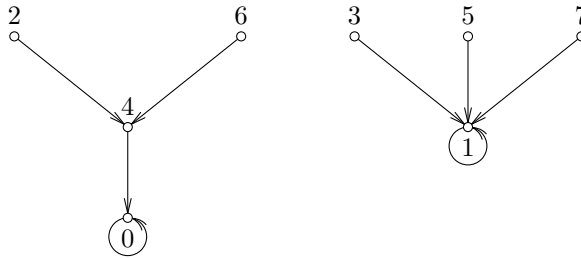


Figure 1. The iteration digraph $G(8, 2)$.

We identify the vertex a of H with its residue modulo n . We will also sometimes identify the vertex 0 with the integer n . For brevity we will make statements such as $\gcd(a, n) = 1$, treating the vertex a as a number. Moreover, when we refer, for instance, to the vertex a^k , we identify it with the remainder $f(a) \in H$ given by (1.1). For particular values of n and k , we denote the iteration digraph of f by $G(n, k)$, see Figures 1–3.

Let $\omega(n)$ denote the number of distinct primes dividing $n \geq 2$ and let the prime power factorization of n be given by

$$(1.2) \quad n = \prod_{i=1}^r p_i^{\alpha_i},$$

where $p_1 < p_2 < \dots < p_r$ are primes and $\alpha_i > 0$, i.e., $r = \omega(n)$. For $n = 1$, we set $\omega(1) = 0$.

A *component* of the iteration digraph is a subdigraph which is a maximal connected subdigraph of the associated nondirected graph.

The *indegree* of a vertex $a \in H$ of $G(n, k)$, denoted by $\text{indeg}_n(a)$, is the number of directed edges coming into a , and the *outdegree* of a is the number of directed edges leaving the vertex a . We will frequently simply write $\text{indeg}(a)$ when it is understood that a is a vertex in $G(n, k)$. By the definition of f , the outdegree of each vertex of $G(n, k)$ is equal to 1. It is obvious that $G(n, k)$ with n vertices also has exactly n directed edges. Thus, if $b_i, i = 1, 2, \dots, q$, denote the indegrees of all the vertices of $G(n, k)$ having positive indegree, then

$$\sum_{i=1}^q b_i = n.$$

It is clear that each component has exactly one cycle, since each vertex of the component has outdegree 1 and the component has only a finite number of vertices. It is also evident that cycle vertices have positive indegree. Cycles of length 1 are called *fixed points*.

Note that 0 and 1 are always fixed points of $G(n, k)$. Cycles of length t are called *t-cycles*. Let $A_t(G(n, k))$ denote the number of t -cycles in $G(n, k)$. Attached to each cycle vertex c of $G(n, k)$ is a tree $T(c)$ whose root is c and whose additional vertices are the noncycle vertices b for which $b^{k^i} \equiv c \pmod{n}$ for some $i \in \mathbb{N} = \{1, 2, \dots\}$, but $b^{k^{i-1}}$ is not congruent to a cycle vertex modulo n . Let $J(n, k)$ be a component in $G(n, k)$ and let c be a cycle vertex in $J(n, k)$. It is evident that b is a vertex in $J(n, k)$ if and only if $b^{k^h} \equiv c \pmod{n}$ for some positive integer h . The *height* of a vertex b in $G(n, k)$ is the least nonnegative integer i such that b^{k^i} is congruent modulo n to a cycle vertex in $G(n, k)$. Note that cycle vertices have height equal to 0.

Further, we specify two particular subdigraphs of $G(n, k)$. Let $G_1(n, k)$ be the induced subdigraph of $G(n, k)$ on the set of vertices which are coprime to n and $G_2(n, k)$ the induced subdigraph on the remaining vertices not coprime with n . If $n > 1$ we observe that $G_1(n, k)$ and $G_2(n, k)$ are disjoint, nonempty, and that $G(n, k) = G_1(n, k) \cup G_2(n, k)$, that is, no edge goes between $G_1(n, k)$ and $G_2(n, k)$. Since $\text{gcd}(a, n) = 1$ if and only if $\text{gcd}(a^k, n) = 1$, it follows that both $G_1(n, k)$ and $G_2(n, k)$ are unions of components of $G(n, k)$. For example, the second component of Figure 2 is $G_1(12, 2)$ whereas the remaining three components make up $G_2(12, 2)$.

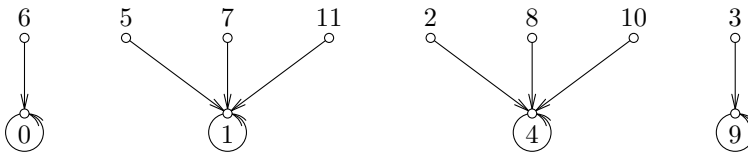


Figure 2. The iteration digraph $G(12, 2)$.

It is clear that 0 is always a fixed point of $G_2(n, k)$. If $n > 1$, then 1 and $n - 1$ are always vertices of $G_1(n, k)$. In Theorem 7.1, we show that if $G_2(n, k)$ contains a t -cycle, then $G_1(n, k)$ also contains a t -cycle. Theorem 7.6 determines the height of a vertex in $G_2(n, k)$.

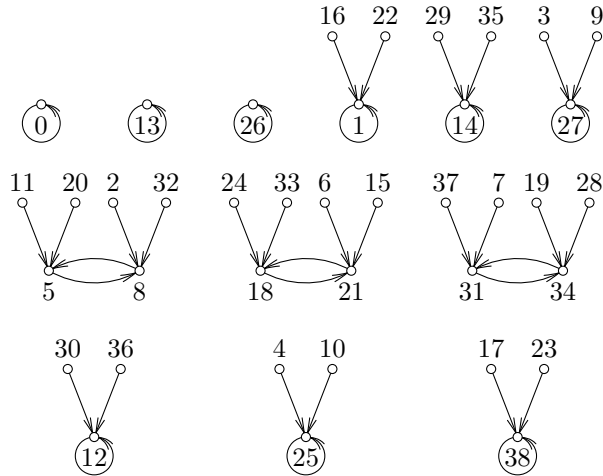


Figure 3. The iteration digraph $G(39, 3)$.

Let $N(n, k, a)$ denote the number of incongruent solutions of the congruence

$$x^k \equiv a \pmod{n}.$$

Then obviously

$$(1.3) \quad N(n, k, a) = \text{indeg}_n(a).$$

It follows from (1.3) and Theorem 2.20 in [9] that if n has the factorization given in (1.2), then

$$(1.4) \quad \text{indeg}_n(a) = N(n, k, a) = \prod_{i=1}^r N(p_i^{\alpha_i}, k, a) = \prod_{i=1}^r \text{indeg}_{q_i}(a),$$

where $q_i = p_i^{\alpha_i}$.

2. PROPERTIES OF THE CARMICHAEL LAMBDA-FUNCTION

Before proceeding further, we need to review some properties of the Carmichael lambda-function $\lambda(n)$. Its definition is a modification of the definition of the Euler totient function $\phi(n)$.

Definition 2.1. Let n be a positive integer. Then the *Carmichael lambda-function* $\lambda(n)$ is defined as follows (see [5, p. 21]):

$$\begin{aligned}\lambda(1) &= 1 = \phi(1), \\ \lambda(2) &= 1 = \phi(2), \\ \lambda(4) &= 2 = \phi(4), \\ \lambda(2^k) &= 2^{k-2} = \frac{1}{2}\phi(2^k) \quad \text{for } k \geq 3, \\ \lambda(p^k) &= (p-1)p^{k-1} = \phi(p^k) \quad \text{for any odd prime } p \text{ and } k \geq 1, \\ \lambda(p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}) &= \text{lcm}[\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \dots, \lambda(p_r^{k_r})],\end{aligned}$$

where p_1, p_2, \dots, p_r are distinct primes and $k_i \geq 1$ for all $i \in \{1, \dots, r\}$.

It immediately follows from Definition 2.1 that

$$\lambda(n) \mid \phi(n)$$

for all n and that $\lambda(n) = \phi(n)$ if and only if $n \in \{1, 2, 4, q^k, 2q^k\}$, where q is an odd prime and $k \geq 1$.

The following theorem generalizes the well-known Euler's theorem which says (see [5, p. 20]) that $a^{\phi(n)} \equiv 1 \pmod{n}$ if and only if $\gcd(a, n) = 1$. It shows that $\lambda(n)$ is the smallest possible universal order modulo n .

Theorem 2.2 (Carmichael). *Let $a, n \in \mathbb{N}$. Then*

$$a^{\lambda(n)} \equiv 1 \pmod{n}$$

if and only if $\gcd(a, n) = 1$. Moreover, there exists an integer g such that

$$\text{ord}_n g = \lambda(n),$$

where $\text{ord}_n g$ denotes the multiplicative order of g modulo n .

Proof. For a proof, see [5, p. 21]. □

3. RESULTS ON THE INDEGREE

We will need the following results concerning the indegrees of certain vertices in $G_1(n, k)$ and $G_2(n, k)$ in order to prove our main results.

Lemma 3.1. *Let n have the factorization given in (1.2) and let a be a vertex of positive indegree in $G_1(n, k)$. Then*

$$\text{indeg}(a) = N(n, k, a) = \prod_{i=1}^r \varepsilon_i \gcd(\lambda(p_i^{\alpha_i}), k),$$

where $\varepsilon_i = 2$ if $2 \mid k$ and $8 \mid p_i^{\alpha_i}$, and $\varepsilon_i = 1$ otherwise.

Proof. This is proved in [16, pp. 231–232]. □

Lemma 3.2. *Let p be a prime and let $\alpha \geq 1$ and $k \geq 2$ be integers. Then*

$$N(p^\alpha, k, 0) = p^{\alpha - \lceil \alpha/k \rceil}.$$

Proof. This follows from the fact that $a^k \equiv 0 \pmod{p^\alpha}$ if and only if $p^{\lceil \alpha/k \rceil} \mid a$. □

4. DIGRAPH PRODUCT

Let $n = n_1 n_2$, where $\gcd(n_1, n_2) = 1$. We show that we can represent $G(n, k)$ as a product of the two digraphs $G(n_1, k)$ and $G(n_2, k)$. By the Chinese Remainder Theorem, we can uniquely represent each vertex $a \in G(n, k)$ as the ordered pair (a_1, a_2) , where $0 \leq a_1 \leq n_1 - 1$, $0 \leq a_2 \leq n_2 - 1$, $a \equiv a_1 \pmod{n_1}$, and $a \equiv a_2 \pmod{n_2}$. For $a = (a_1, a_2)$ define

$$(4.1) \quad a^k = (a_1, a_2)^k = (a_1^k, a_2^k),$$

where we assume that a^k , a_1^k , and a_2^k are all reduced modulo n , n_1 and n_2 , respectively.

Let $G(n_1, k) \times G(n_2, k)$ denote the digraph whose vertices are the ordered pairs (a_1, a_2) , where $0 \leq a_1 \leq n_1 - 1$ and $0 \leq a_2 \leq n_2 - 1$. In addition, $\langle (a_1, b_1), (a_2, b_2) \rangle$ is a directed edge of $G(n_1, k) \times G(n_2, k)$ if and only if $a_2 \equiv a_1^k \pmod{n_1}$ and $b_2 \equiv b_1^k \pmod{n_2}$ (see [4]).

From (4.1) it follows that $G(n, k)$ is isomorphic to $G(n_1, k) \times G(n_2, k)$, i.e.,

$$G(n, k) \cong G(n_1, k) \times G(n_2, k)$$

and for simplicity we shall write further on

$$(4.2) \quad G(n, k) = G(n_1, k) \times G(n_2, k).$$

If n has the factorization given in (1.2), it follows from (4.2) that

$$G(n, k) = G(p_1^{\alpha_1}, k) \times G(p_2^{\alpha_2}, k) \times \dots \times G(p_r^{\alpha_r}, k).$$

5. RESULTS ON CYCLES AND COMPONENTS

Consider a digraph $G(n, k)$ and let

$$(5.1) \quad \lambda(n) = lw,$$

where l is the largest divisor of $\lambda(n)$ relatively prime to k . We will need the following theorems and lemmas to prove some of our major results.

Lemma 5.1. *There exists a t -cycle in $G_1(n, k)$ if and only if*

$$t = \text{ord}_d k$$

for some factor d of l . Moreover, $\text{ord}_d k$ is the length of the longest cycle in $G_1(n, k)$.

Proof. Both statements are proved in [16, pp. 232–233]. □

Corollary 5.2. *Every cycle in $G_1(n, k)$ is a fixed point if and only if $k \equiv 1 \pmod{l}$, where l is defined as in (5.1).*

Lemma 5.3. *Let c_1 and c_2 be any two cycle vertices in $G_1(n, k)$ and let $T(c_1)$ and $T(c_2)$ be the trees attached to c_1 and c_2 , respectively. Then $T(c_1) \cong T(c_2)$.*

Proof. This is proved in [16, p. 234]. □

Corollary 5.4. *Let $t \geq 1$ be a fixed integer. Then any two components in $G_1(n, k)$ containing t -cycles are isomorphic.*

Lemma 5.5. *The vertex c is a cycle vertex in $G_1(n, k)$ if and only if $\text{ord}_n c \mid l$, where l is defined as in (5.1). Moreover, any two vertices in the same cycle of $G_1(n, k)$ have the same order modulo n .*

Proof. These assertions are proved in [16, pp. 232–233]. □

By virtue of Lemma 5.5, we define the order of a cycle in $G_1(n, k)$ to be the order of any vertex in the cycle.

Lemma 5.6. *Let n have the factorization given in (1.2) and let t be a positive integer. Then*

$$(5.2) \quad A_t(G_1(n, k)) = \frac{1}{t} \left[\prod_{i=1}^r \delta_i \gcd(\lambda(p_i^{\alpha_i}), k^t - 1) - \sum_{\substack{d|t \\ d \neq t}} dA_d(G_1(n, k)) \right]$$

and

$$(5.3) \quad A_t(G(n, k)) = \frac{1}{t} \left[\prod_{i=1}^r (\delta_i \gcd(\lambda(p_i^{\alpha_i}), k^t - 1) + 1) - \sum_{\substack{d|t \\ d \neq t}} dA_d(G(n, k)) \right],$$

where $\delta_i = 2$ if $2 \mid k^t - 1$ and $8 \mid p_i^{\alpha_i}$, and $\delta_i = 1$ otherwise.

Proof. Both (5.2) and (5.3) are proved in [13]. □

Lemma 5.7. *If b is a noncycle vertex in $G_1(n, k)$ and c is a cycle vertex in $G_1(n, k)$, then bc is a noncycle vertex in $G_1(n, k)$.*

Proof. This is proved in [16, p. 234]. □

Lemma 5.8. *Let $c = (c_1, c_2)$ be a vertex in $G(n, k) = G(n_1, k) \times G(n_2, k)$, where $n = n_1 n_2$ and $\gcd(n_1, n_2) = 1$. Then c is a cycle vertex in $G(n, k)$ if and only if c_i is a cycle vertex in $G(n_i, k)$ for $i = 1, 2$. Moreover, if $c = (c_1, c_2)$ is a vertex in a t -cycle of $G(n, k)$ and c_i is a vertex in a t_i -cycle of $G(n_i, k)$ for $i = 1, 2$, then $t = \text{lcm}(t_1, t_2)$.*

Proof. These assertions are proved in [13]. □

Lemma 5.9. *Every vertex in $G_1(n, k)$ is a cycle vertex if and only if*

$$\gcd(\lambda(n), k) = 1.$$

Moreover, every vertex in $G_1(n, k)$ is a fixed point if and only if $k \equiv 1 \pmod{\lambda(n)}$. Further, every vertex in $G(n, k)$ is a fixed point if and only if n is square-free and $k \equiv 1 \pmod{\lambda(n)}$.

Proof. The first assertion is proved in [16, p. 232]. The other assertions now follow from Corollary 5.2 and Lemma 5.6. □

Lemma 5.10. *Let $b \in G_1(n, k)$ and suppose that $\text{ord}_n b = l'w'$, where $l' \mid l$ and $w' \mid w$ for l and w as defined in (5.1). Then the height h of b is equal to the least nonnegative integer such that $w' \mid k^h$. Furthermore, the height of any tree attached to a cycle vertex in $G_1(n, k)$ is the least integer h_1 such that $w \mid k^{h_1}$.*

Proof. These statements are proved in [16, pp. 234–235]. □

Lemma 5.11. *Let $n = n_1 n_2$, where $\gcd(n_1, n_2) = 1$. Let $D(n_1, k)$ be a union of components of $G(n_1, k)$ and let $R(n_2, k)$ be a union of components of $G(n_2, k)$. Then $D(n_1, k) \times R(n_2, k)$ is a union of components of $G(n, k) = G(n_1, k) \times G(n_2, k)$. Moreover, if*

$$R(n_2, k) = \bigcup_{i=1}^m R_i(n_2, k),$$

where $R_i(n_2, k)$ are distinct components of $G(n_2, k)$ for $i = 1, 2, \dots, m$, then

$$(5.4) \quad D(n_1, k) \times R(n_2, k) = \bigcup_{i=1}^m D(n_1, k) \times R_i(n_2, k),$$

where the union in (5.4) is a disjoint union.

Proof. These assertions are proved in [13]. □

As contrasted to the algebraic and elementary methods used in this paper to analyze the structure of $G(n, k)$, advanced analytic techniques have also been used in papers such as [2], [3], [6], [8], and [15] to obtain results related to the structure of $G(n, k)$.

In [2], the following result was proved concerning the average values of the number of cycle vertices and heights of vertices in $G_1(n, k)$, where p denotes a prime.

Theorem 5.12 (Chou and Shparlinski). *Let $T_0(p, k)$ denote the total number of cycle vertices in $G_1(p, k)$. Let $h_{p,k}(a)$ denote the height of the vertex a in $G_1(p, k)$. Let*

$$T(p, k) = \frac{1}{p-1} \sum_{a=1}^{p-1} h_{p,k}(a)$$

and let

$$S_0(k, N) = \frac{1}{\pi(N)} \sum_{p \leq N} T_0(p, k) \quad \text{and} \quad S(k, N) = \frac{1}{\pi(N)} \sum_{p \leq N} T(p, k),$$

where $\pi(N)$ denotes the number of primes not greater than N . Then for any integer $k \geq 2$, there are positive constants $C_1(k)$ and $C_2(k)$ such that the bounds

$$S_0 \sim C_1(k)N \quad \text{and} \quad S \sim C_2(k)$$

hold.

Theorem 5.12 generalizes Theorems 9 and 10 of [15] which treats only the case $k = 2$ and makes use of the Extended Riemann Hypothesis.

6. SUBDIGRAPHS FOR WHICH ALL TREES ATTACHED TO CYCLE VERTICES
ARE ISOMORPHIC

Let n have the factorization given by (1.2) and let \mathcal{P} be the set of primes dividing n . Let $\mathcal{P}_1 \cup \mathcal{P}_2$ be a partition of the set \mathcal{P} such that $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$. Let

$$(6.1) \quad m_1 = \prod_{p \in \mathcal{P}_1} p \quad \text{and} \quad m_2 = \prod_{p \in \mathcal{P}_2} p,$$

where $m_i = 1$ if $\mathcal{P}_i = \emptyset$. Let $G_{\mathcal{P}_i}^*(n, k)$, $i = 1, 2$, be the subdigraph of $G(n, k)$ induced by those vertices which are multiples of m_i and which are also relatively prime to m_j , where $j = 2/i$. Then $G_{\mathcal{P}_1}^*(n, k)$ and $G_{\mathcal{P}_2}^*(n, k)$ are called *fundamental constituents* of $G(n, k)$. The subdigraphs $G_{\mathcal{P}_1}^*(n, k)$ and $G_{\mathcal{P}_2}^*(n, k)$ were introduced by Wilson in [16].

Let $n = n_1 n_2$ have the factorization given in (1.2), where

$$(6.2) \quad n_1 = \prod_{p_i \in \mathcal{P}_1} p_i^{\alpha_i} \quad \text{and} \quad n_2 = \prod_{p_i \in \mathcal{P}_2} p_i^{\alpha_i}.$$

Let $L(n_2, k)$ denote the subdigraph of $G_2(n_2, k)$ induced by the vertices of $G_2(n_2, k)$ which are multiples of m_2 . Note that the only cycle vertex in $L(n_2, k)$ is the fixed point 0. It is clear that $G_{\mathcal{P}_2}^*(n, k) \cong G_1(n_1, k) \times L(n_2, k)$ and thus, we shall write

$$(6.3) \quad G_{\mathcal{P}_2}^*(n, k) = G_1(n_1, k) \times L(n_2, k).$$

If $\mathcal{P}_1 = \emptyset$, then $n_2 = n$ and $G_{\mathcal{P}_2}^*(n, k) \cong L(n, k)$. If $\mathcal{P}_2 = \emptyset$, then $n_1 = n$ and $G_{\mathcal{P}_2}^*(n, k) \cong G_1(n, k)$. Let p be a prime. Since $p \mid a^k$ if and only if $p \mid a$, it follows that $L(n_2, k)$ is a single component of $G(n, k)$. It further follows from (6.3) and Lemma 5.11 that $G_{\mathcal{P}_1}^*(n, k)$ and $G_{\mathcal{P}_2}^*(n, k)$ are disjoint unions of components of $G(n, k)$. It is evident that $G_2(n, k)$ is a disjoint union of $G_{\mathcal{P}_2}^*(n, k)$ as \mathcal{P}_2 ranges over all nonempty subsets of \mathcal{P} .

Figure 4 shows the fundamental constituents of $G(56, 2)$.

Let $J(n, k)$ be a component of $G(n, k)$ and let c be any cycle vertex in $G(n, k)$. Let \mathcal{P}_2 be the subset of primes in \mathcal{P} which divide c . Since a is a vertex of $J(n, k)$ if and only if $a^{k^h} \equiv c \pmod{c}$ for some positive integer h it follows that $J(n, k)$ is a subdigraph of $G_{\mathcal{P}_2}^*(n, k)$.

The following theorem shows that all trees attached to cycle vertices in a fundamental constituent of $G(n, k)$ are isomorphic. Its proof generalizes the method of proof by Wilson of Theorem 4 in [16].

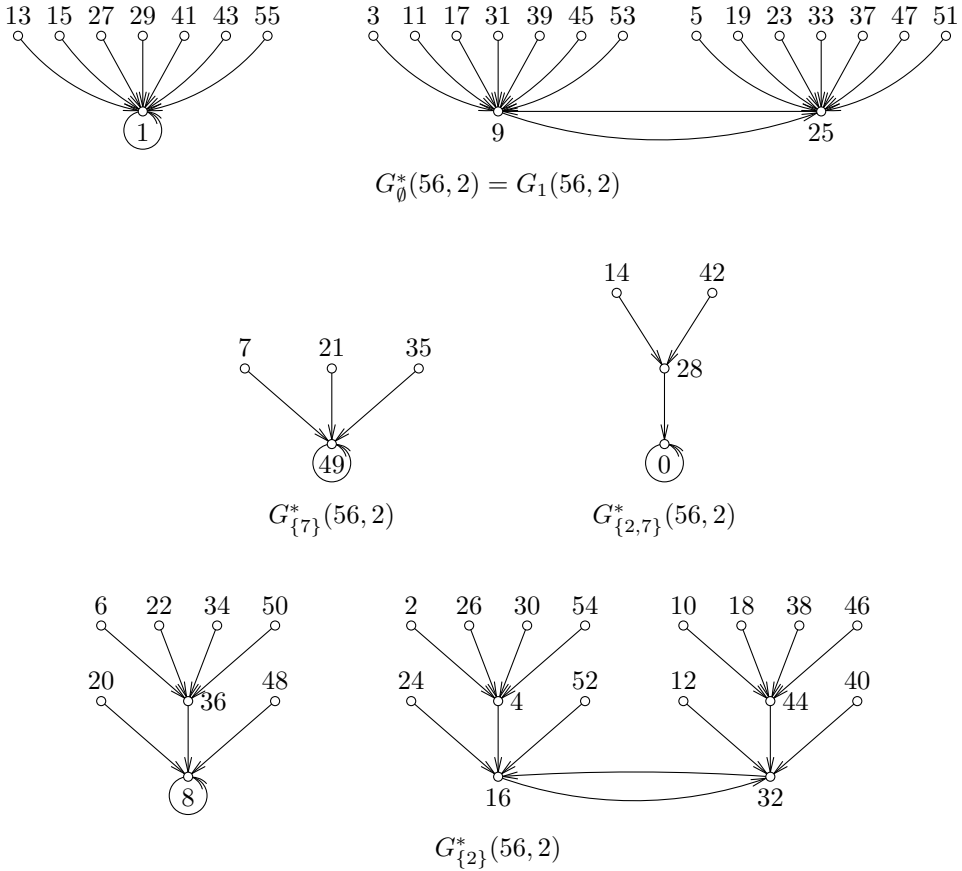


Figure 4. The four fundamental constituents of $G(56, 2)$.

Theorem 6.1. *Let n have the factorization given in (1.2) and let \mathcal{P} be the set of primes dividing n . Let a partition of \mathcal{P} be given by $\mathcal{P}_1 \cup \mathcal{P}_2$ such that $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$. Let c_1 and c_2 be two cycle vertices in $G_{\mathcal{P}_2}^*(n, k)$ and let $T(c_1)$ and $T(c_2)$ be the trees attached to c_1 and c_2 , respectively. Then $T(c_1) \cong T(c_2)$.*

Proof. If $\mathcal{P}_2 = \emptyset$, then $G_{\mathcal{P}_2}^*(n, k) = G_1(n, k)$, and the assertion follows from Lemma 5.3. Next suppose that $\mathcal{P}_2 = \mathcal{P}$. Then $n = n_2$, and $G_{\mathcal{P}_2}^*(n, k) = L(n, k)$. Since the only cycle vertex in $L(n, k)$ is the fixed point 0, there is only one tree in $G_{\mathcal{P}_2}^*(n, k)$, and the theorem holds trivially.

We now suppose that $\emptyset \neq \mathcal{P}_2 \neq \mathcal{P}$. Then

$$G_{\mathcal{P}_2}^*(n, k) = G_1(n_1, k) \times L(n_2, k),$$

where $n_1 > 1$ and $n_2 > 1$. By Lemma 5.8, we can write $c_1 = (d, 0)$, where d is a cycle vertex in $G_1(n_1, k)$ and 0 is the unique cycle vertex in $L(n_2, k)$. In particular,

$(1, 0)$ is a cycle vertex in $G_1(n_1, k) \times L(n_2, k)$ and is the unique cycle vertex in its component.

We complete the proof by showing that $T((1, 0)) \cong T((d, 0))$. Let (u, v) be a vertex in $T((1, 0))$. Suppose that (u, v) has height h in the tree $T((1, 0))$. Let d_h be the unique vertex in $G_1(n_1, k)$ which is in the same cycle as d and such that $d_h^{k^h} \equiv d \pmod{n_1}$, that is, d_h is the cycle vertex which is h vertices before the cycle vertex d . Note that $d_0 = d$. We define the mapping F from $T((1, 0))$ into $G_1(n_1, k) \times L(n_2, k)$ by

$$F((u, v)) = (ud_h, v).$$

We will show that F is a digraph isomorphism from $T((1, 0))$ onto $T((d, 0))$.

We first demonstrate that F is a mapping from $T((1, 0))$ into $T((d, 0))$ that sends vertices of height h into vertices of the same height h . If $(u, v) = (1, 0)$, then $F((u, v)) = (d, 0)$, and both the vertices $(1, 0)$ and $(d, 0)$ have height 0. Now suppose that (u, v) is not a cycle vertex. Then

$$[F((u, v))]^{k^h} = (ud_h, v)^{k^h} = (u^{k^h} d_h^{k^h}, v^{k^h}) = (1 \cdot d, 0) = (d, 0).$$

Moreover, if $0 \leq i < h$, then

$$(ud_h, v)^{k^i} = (u^{k^i} d_h^{k^i}, v^{k^i}),$$

where either u^{k^i} or v^{k^i} is a noncycle vertex. If u^{k^i} is a noncycle vertex, then $u^{k^i} d_h^{k^i}$ is a noncycle vertex by Lemma 5.7, since $d_h^{k^i}$ is a cycle vertex. It now follows by Lemma 5.8 that $(ud_h, v)^{k^i}$ is a noncycle vertex. Therefore, $F((u, v))$ is a vertex in $T((d, 0))$ that has height h .

We now show that F is a one-to-one mapping. Suppose that (u_1, v_1) and (u_2, v_2) have heights h_1 and h_2 , respectively in $T((1, 0))$ and

$$(6.4) \quad F((u_1, v_1)) = (u_1 d_{h_1}, v_1) = (u_2 d_{h_2}, v_2) = F((u_2, v_2)).$$

By our argument above, it then follows that $F((u_1, v_1))$ has height h_1 , while $F((u_2, v_2))$ has height h_2 . If $h_1 \neq h_2$, then $F((u_1, v_1)) \neq F((u_2, v_2))$, which is a contradiction. Hence, $h_1 = h_2$ and $d_{h_1} \equiv d_{h_2} \pmod{n_1}$. By (6.4), $v_1 \equiv v_2 \pmod{n_2}$. Since d_{h_1} is a vertex in $G_1(n_1, k)$, d_{h_1} is invertible modulo n_1 . It now follows from (6.4) that $u_1 \equiv u_2 \pmod{n_1}$, which implies that F is one-to-one.

We next show that F is onto. Let (u', v') be a vertex of height h in $T((d, 0))$. If $h = 0$, then $(u', v') = (d, 0)$ and $F((1, 0)) = (d, 0)$. We now assume that $h \geq 1$. Consider the vertex $(u' d_h^{-1}, v')$ in $G_1(n_1, k) \times L_2(n_2, k)$. We claim that $(u' d_h^{-1}, v')$ is

a vertex of height h in $T((1, 0))$. Since d_h is a cycle vertex, $d_h^{k^j} \equiv d_h \pmod{n_1}$ for some positive integer j . Then

$$(d_h^{-1})^{k^j} \equiv (d_h^{k^j})^{-1} \equiv d_h^{-1} \pmod{n_1},$$

and d_h^{-1} is also a cycle vertex. Note that

$$\begin{aligned} (u'd_h^{-1}, v')^{k^h} &= ((u')^{k^h} (d_h^{-1})^{k^h}, (v')^{k^h}) = ((u')^{k^h} (d_h^{k^h})^{-1}, (v')^{k^h}) \\ &= (dd^{-1}, 0) = (1, 0). \end{aligned}$$

If $0 \leq i < h$, then

$$(u'd_h^{-1}, v')^{k^i} = ((u')^{k^i} (d_h^{-1})^{k^i}, (v')^{k^i}),$$

where either $(u')^{k^i}$ or $(v')^{k^i}$ is a noncycle vertex. If $(u')^{k^i}$ is a noncycle vertex, then by Lemma 5.7, $(u')^{k^i} (d_h^{-1})^{k^i}$ is a noncycle vertex, since $(d_h^{-1})^{k^i}$ is a cycle vertex. Thus, $(u'd_h^{-1}, v')^{k^i}$ is a noncycle vertex, and hence $(u'd_h^{-1}, v')$ is a vertex in $T((1, 0))$ of height h . Now notice that

$$F((u'd_h^{-1}, v')) = (u'd_h^{-1}d_h, v') = (u', v'),$$

which implies that F is onto.

Finally, we show that F is edge-preserving. Suppose that $(u, v) \neq (1, 0)$ is a vertex in $T((1, 0))$ of height $h \geq 1$. Then $(u, v)^k$ is a vertex in $T((1, 0))$ of height $h - 1$ and

$$F((u, v)^k) = F((u^k, v^k)) = (u^k d_{h-1}, v^k) = (u^k d_h^k, v^k) = (u d_h, v)^k = [F((u, v))]^k.$$

The result now follows. □

Corollary 6.2. *Let $J(n, k)$ be a component in $G(n, k)$ and let c_1 and c_2 be any two cycle vertices in $J(n, k)$. Then $T(c_1) \cong T(c_2)$.*

P r o o f. This follows from Theorem 6.1 upon noting that $J(n, k)$ is a subdigraph of $G_{\mathcal{P}_2}^*(n, k)$ for some subset \mathcal{P}_2 of the set of primes dividing n . □

Corollary 6.3. *Let $n > 1$ be an integer and let \mathcal{P} be the set of primes dividing n . Let \mathcal{P}_2 be a subset of \mathcal{P} . Let t be a fixed positive integer. Then all components in $G_{\mathcal{P}_2}^*(n, k)$ having a t -cycle are isomorphic.*

Example 6.4. In Figure 4, we observe that trees attached to cycle vertices in the same fundamental constituent of $G(56, 2)$ are isomorphic, whereas trees attached to cycle vertices in different fundamental constituents are not isomorphic.

Example 6.5. From Figure 3 we can see that for the digraph $G(39, 3)$, the fundamental constituents $G_0^*(39, 3)$ and $G_{\{3\}}^*(39, 3)$ have isomorphic nontrivial trees attached to their cycle vertices, while the fundamental constituents $G_{\{13\}}^*(39, 3)$ (see the second and third components in Figure 3) and $G_{\{3,13\}}^*(39, 3)$ (see the first component in Figure 3) have the trivial tree attached to their cycle vertices.

7. POSSIBLE CYCLE LENGTHS AND HEIGHTS IN $G_2(n, k)$

Theorem 7.1. *If C is a t -cycle in $G_2(n, k)$, then there exists a t -cycle in $G_1(n, k)$.*

Proof. Since $G_2(n, k)$ is the disjoint union of the fundamental constituents $G_{\mathcal{P}_2}^*(n, k)$ of $G(n, k)$ as \mathcal{P}_2 ranges over the nonempty subsets of \mathcal{P} , the set of primes dividing n , we see that C is a cycle in some fundamental constituent $G_{\mathcal{P}_2}^*(n, k)$. Then

$$(7.1) \quad G_{\mathcal{P}_2}^*(n, k) = G_1(n_1, k) \times L(n_2, k),$$

where n_1 and n_2 are defined as in (6.2). Let c be a vertex in the t -cycle C . Noting that the only cycle vertex in $L(n_2, k)$ is the fixed point 0, we see by Lemma 5.8 that we can write $c = (c_1, 0)$, where c_1 is a vertex in a t_1 -cycle of $G_1(n_1, k)$. It further follows from Lemma 5.8 that $t = t_1 \cdot 1 = t_1$. Now consider the vertex $d = (c_1, 1)$ in $G_1(n, k) = G_1(n_1, k) \times G_1(n_2, k)$. Since c_1 is a cycle vertex in $G_1(n_1, k)$ and 1 is a fixed point in $G_1(n_2, k)$, we find that d is a cycle vertex in $G_1(n, k)$. By Lemma 5.8, we observe that d is part of a t -cycle also. \square

Corollary 7.2. *Every cycle in $G(n, k)$ is a fixed point if and only if $k \equiv 1 \pmod{l}$, where l is as defined in (5.1).*

Proof. The proof follows from Corollary 5.2 and Theorem 7.1. \square

Theorem 7.3. *Let n have the factorization given in (1.2). Suppose that $G_1(n, k)$ contains a t -cycle. Then the subdigraph $G_2(n, k)$ also contains a t -cycle if and only if there exist $i \in \{1, 2, \dots, r\}$ and an integer d relatively prime to $\lambda(n)$ such that $t = \text{ord}_d k$ and $d \mid \lambda(n/p_i^{\alpha_i})$.*

Proof. As noted earlier, $G_2(n, k)$ is a disjoint union of $G_{\mathcal{P}_2}^*(n, k)$ as \mathcal{P}_2 ranges over all nonempty subsets of \mathcal{P} . Let C be a t -cycle in $G_2(n, k)$. Then C is a t -cycle in $G_{\mathcal{P}_2}^*(n, k)$ for some nonempty subset \mathcal{P}_2 of \mathcal{P} . By (7.1)

$$G_{\mathcal{P}_2}^*(n, k) \cong G_1(n_1, k) \times L(n_2, k),$$

where $n_1 \mid (n/p_i^{k_i})$ for some $i \in \{1, 2, \dots, r\}$. Recall that the only cycle vertex in $L(n_2, k)$ is the fixed point 0. It now follows from Lemmas 5.8, 5.1, and 5.5 that if d is any positive integer for which $d \mid \lambda(n_1)$ and $\gcd(d, k) = 1$, then there exists a t -cycle in $G_{\mathcal{P}_2}^*(n, k)$ such that $t = \text{ord}_d k$. Since $\lambda(a) \mid \lambda(b)$ when $a \mid b$ by the property of the Carmichael-lambda function, the result now follows. \square

Example 7.4. Suppose that n has at least two distinct prime divisors. It was shown in Remark 3.6 of [11] that if $k = 2$, then $n = 203 = 7 \cdot 29$ is the least positive integer n for which there exists a positive integer t such that $G_1(n, k)$ has a t -cycle, but $G_2(n, k)$ does not have a t -cycle. In this case, $G_1(203, 2)$ has a 6-cycle, whereas $G_2(203, 2)$ does not have a 6-cycle. When $k = 3$ the least such integer n is $n = 115 = 5 \cdot 23$. In this instance, $G_1(115, 3)$ has a 10-cycle, while $G_2(115, 3)$ does not contain a 10-cycle. Note that $\lambda(115) = 44$. However, $44 \nmid \lambda(5) = 4$ and $44 \nmid \lambda(23) = 22$. Moreover, $\text{ord}_{44} 3 = 10$, whereas $\text{ord}_4 3 = 2$ and $\text{ord}_{22} 3 = 5$.

The next corollary is a partial converse of Theorem 7.1.

Corollary 7.5. *Let $B(G(n, k))$ denote the set of integers t such that $G(n, k)$ has a t -cycle. Suppose that n is a prime or a prime power. Then $B(G_1(n, k)) = B(G_2(n, k))$ if and only if $k \equiv 1 \pmod{l}$, where l is defined as in (5.1).*

Proof. By Theorem 7.3, the only cycle in $G_2(n, k)$ is the fixed point 0. The result now follows from Corollary 5.2. \square

Theorem 7.6. *Let $n > 1$ be as defined in (1.2) and let $a \in \{1, 2, \dots, n\}$ be an integer such that $a \in G_2(n, k)$ and*

$$a = b \prod_{i=1}^r p_i^{l_i},$$

where $l_i \geq 0$ and $\gcd(b, n) = 1$. For $i = 1, 2, \dots, r$, define m_i by

$$m_i = \begin{cases} 0 & \text{if } l_i = 0, \\ \alpha_i & \text{if } 1 \leq l_i \leq \alpha_i, \\ l_i & \text{if } l_i > \alpha_i. \end{cases}$$

Let

$$n_1 = \prod_{i=1}^r p_i^{\alpha_i - \min(m_i, \alpha_i)}.$$

Then $\gcd(n_1, a) = 1$. Let l and w be as given in (5.1) and let $\text{ord}_{n_1} a = l'w'$, where $l' \mid l$ and $w' \mid w$. Let $h(a)$ be the least nonnegative integer j such that $w' \mid k^j$. Then

the height of a is equal to

$$\max\left(\max_{1 \leq i \leq r} \left\lceil \log_k \frac{m_i}{l_i} \right\rceil, h(a)\right),$$

where we define $m_i/l_i = 1$ if $m_i = l_i = 0$.

Theorem 7.7. *Let $n > 1$ be as defined in (1.2). Let $e_i = n/p_i^{\alpha_i}$, $i = 1, 2, \dots, r$, and let $\lambda(e_i) = l_i w_i$. Let h_i be the least nonnegative integer such that*

$$w_i \mid k^{h_i}.$$

Let $g = \max_{1 \leq i \leq r} h_i$. Let h be the maximum height of any vertex in $G_2(n, k)$. Then

$$h = \max\left(\max_i \lceil \log_k \alpha_i \rceil, g\right).$$

Theorems 7.6 and 7.7 were proved for the case $k = 2$ in Theorems 3.10 and 3.14, respectively, of [11]. Moreover, the proofs of Theorems 7.6 and 7.7 are completely similar to the proofs of these theorems in [11] upon making use of Lemma 5.10 of our present paper.

8. RESULTS ON FIXED POINTS

As we mentioned earlier, fixed points are of interest, because any digraph $G(n, k)$ always has fixed points including 0 and 1. On the other hand, by Corollary 7.2, there exist digraphs $G(n, k)$ not having t -cycles for any $t > 1$.

We have the following two theorems on the number of fixed points and the number of isolated fixed points in $G(n, k)$. Note that an isolated fixed point is a fixed point with indegree 1.

Theorem 8.1. *Let $n > 1$.*

- (i) *If k is even, then $A_1(G(n, k)) \geq 2^{\omega(n)}$ and $A_1(G_1(n, k)) \geq 1$. In particular, if $k = 2$, then $A_1(G(n, k)) = 2^{\omega(n)}$ and $A_1(G_1(n, k)) = 1$.*
- (ii) *If $k \geq 3$ is odd and $2 \parallel n$, then $A_1(G(n, k)) \geq 2 \cdot 3^{\omega(n)-1}$ and $A_1(G_1(n, k)) \geq 2^{\omega(n)-1}$. In particular, if $k = 3$, then we have $A_1(G(n, k)) = 2 \cdot 3^{\omega(n)-1}$ and $A_1(G_1(n, k)) = 2^{\omega(n)-1}$.*
- (iii) *If $k \geq 3$ is odd and either n is odd or $4 \parallel n$, then $A_1(G(n, k)) \geq 3^{\omega(n)}$ and $A_1(G_1(n, k)) \geq 2^{\omega(n)}$. In particular, if $k = 3$, then $A_1(G(n, k)) = 3^{\omega(n)}$ and $A_1(G_1(n, k)) = 2^{\omega(n)}$.*

(iv) If $k \geq 3$ is odd and $8 \parallel n$, then $A_1(G(n, k)) \geq 5 \cdot 3^{\omega(n)-1}$ and $A_1(G_1(n, k)) \geq 4 \cdot 2^{\omega(n)-1}$. In particular, if $k = 3$, then $A_1(G(n, k)) = 5 \cdot 3^{\omega(n)-1}$ and $A_1(G_1(n, k)) = 4 \cdot 2^{\omega(n)-1}$.

Proof. The proof follows from Lemma 5.6. □

It was proved in [10] that if $k = 2$ then $G(n, k)$ has a nonzero isolated fixed point if and only if $n = 2m$, where m is an odd square-free integer. In this case, a is a nonzero isolated fixed point if and only if $a = m$. In Theorem 8.2, we extend this result by counting isolated fixed points in $G(n, k)$ for any $n > 1$ and any $k \geq 2$.

Theorem 8.2. *Let $n > 1$ have the factorization given in (1.2). The number of isolated fixed points in $G(n, k)$ is given by*

$$\prod_{i=1}^r [\delta(\gcd(\lambda(p_i^{\alpha_i}), k)) \cdot \delta_i \gcd(\lambda(p_i^{\alpha_i}), k - 1) + \delta(\alpha_i)],$$

where $\delta(m) = 1$ if $m = 1$ and $\delta(m) = 0$ otherwise, and δ_i is defined as in Lemma 5.6.

Proof. Let a be an isolated fixed point in $G(n, k)$. Then $\text{indeg}_n(a) = 1$. By (1.4), $\text{indeg}_n(a) = 1$ if and only if $\text{indeg}_{q_i}(a) = 1$ for $i = 1, 2, \dots, r$, where $q_i = p_i^{\alpha_i}$. Clearly, a is a fixed point in $G(n, k)$ if and only if a is a fixed point in $G(q_i, k)$ for $1 \leq i \leq r$. Suppose that $a \in G_1(q_i, k)$ for some i such that $1 \leq i \leq r$. Then by Lemma 3.1, $\text{indeg}_{q_i}(a) = 1$ if and only if $\varepsilon_i \gcd(\lambda(q_i), k) = 1$, where ε_i is defined as in Lemma 3.1. By Lemma 5.6, the number of fixed points in $G_1(q_i, k)$ is equal to $\delta_i \gcd(\lambda(q_i), k - 1)$, where δ_i is defined as in Lemma 5.6.

Now suppose that a is a fixed point in $G_2(q_i, k)$. This occurs if and only if $a \equiv 0 \pmod{q_i}$. Note that $\text{indeg}_{q_i}(0) = 1$ if and only if $\alpha_i = 1$. The result now follows. □

Remark 8.3. Note that by the proof of Theorem 8.2, the vertex 0 is an isolated fixed point of $G(n, k)$ if and only if n is square-free (see Figures 1–4).

9. LENGTH OF THE LONGEST CYCLE

In [7], the following theorem was proved giving an upper bound for the length of the longest cycle in $G(p, k)$ when $p > 5$ is a prime. We let $L(G(n, k))$ denote the length of the longest cycle in the digraph $G(n, k)$.

Theorem 9.1 (Lucheta et al.). *Let $p > 5$ be a prime. Then*

$$L(G(p, k)) \leq \frac{p-1}{2} - 1.$$

Moreover, if $(p-1)/2$ is also an odd prime, i.e., $(p-1)/2$ is a Sophie Germain prime, and k is a primitive root modulo $(p-1)/2$, then $L(G(p, k)) = (p-1)/2 - 1$. Furthermore, if $(p-1)/2$ is an odd prime and k is an odd primitive root modulo $(p-1)/2$, then $G(p, k)$ contains two cycles of length $(p-1)/2 - 1$.

Theorem 9.2 below extends Theorem 9.1 to digraphs $G(n, k)$ for any fixed positive integer n and an integer $k \geq 2$ which is allowed to vary. Improved bounds are also found for $L(G(n, k))$, and all values of k are determined for which $L(G(n, k)) \leq 2$ for all n .

Theorem 9.2. *Let $n \geq 1$ be a fixed integer. Then we have:*

- (i) $\max_{k \geq 2} L(G(n, k)) = \lambda(\lambda(n))$.
- (ii) *If k is a fixed integer and C is a t -cycle in $G(n, k)$, then $t \mid \lambda(\lambda(n))$.*
- (iii) *The digraph $G(n, k)$ contains only cycles of length 1 (fixed points) for all $k \geq 2$ if and only if n is one of the 8 positive divisors of 24.*
- (iv) $\max_{k \geq 2} L(G(n, k)) = 2$ if and only if n is one of the 136 positive divisors of $2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 = 131040$ which are not divisors of 24.
- (v) *If n is not a divisor of 24, then $\max_{k \geq 2} L(G(n, k))$ is an even integer.*
- (vi) *Suppose that $n > 5$. If it is not the case that n is a prime of the form $n = 2p^i + 1$, where p is an odd prime and $i \geq 1$, then*

$$(9.1) \quad \max_{k \geq 2} L(G(n, k)) < \frac{n}{4}.$$

If n is a prime of the form $2p^i + 1$, then

$$(9.2) \quad \max_{k \geq 2} L(G(n, k)) = p^{i-1}(p-1) = \frac{n-1}{2} - \frac{n-1}{2p} > \frac{n}{4}.$$

In particular, when n is a prime such that $n = 2p^i + 1$, then

$$(9.3) \quad \frac{n-1}{3} \leq \max_{k \geq 2} L(G(n, k)) \leq \frac{n-1}{2} - 1.$$

The upper bound in (9.3) is attained if and only if n is a prime of the form $2p+1$, i.e., p is a Sophie Germain prime, and the lower bound in (9.3) is attained when n is a prime of the form $2 \cdot 3^i + 1$, where $i \geq 1$.

Proof. (i) By Lemma 5.1 and Theorem 7.1, the longest cycle in $G(n, k)$ is equal to $\text{ord}_l k$, where l is the largest divisor of $\lambda(n)$ relatively prime to k . Clearly,

$$\text{ord}_l k \mid \lambda(l) \mid \lambda(\lambda(n)).$$

By Theorem 2.2, there exists a positive integer k such that $\gcd(k, \lambda(n)) = 1$ and $\text{ord}_{\lambda(n)} k = \lambda(\lambda(n))$. The assertion now follows.

(ii) By Lemma 5.1, there exists a divisor d of $\lambda(n)$ such that $\text{ord}_d k = t$. By Theorem 2.2 on λ , $t \mid \lambda(d)$. It follows from the definition of λ that if $m \mid n$, then $\lambda(m) \mid \lambda(n)$. Hence,

$$t \mid \lambda(d) \mid \lambda(\lambda(n)).$$

(iii) We note that $\lambda(m) = 1$ if and only if $m = 1$ or 2 . By the definition of $\lambda(n)$, we see that $\lambda(n) = 1$ or 2 if and only if n is a divisor of 24 . The result now follows from part (i).

(iv) Observe that $\lambda(m) = 2$ if and only if $m = 3, 4, 6, 8, 12$, or 24 . Using the definition of $\lambda(n)$, the result now easily follows.

(v) It follows from the properties of $\lambda(m)$ that $\lambda(m)$ is even if and only if $m \geq 3$. Our result now follows from the proof of part (iii).

(vi) First suppose that n is a prime of the form $2p^i + 1$. Then

$$\begin{aligned} (9.4) \quad \max_{k \geq 2} L(G(n, k)) &= \lambda(\lambda(2p^i + 1)) = \lambda(2p^i) = p^{i-1}(p - 1) \\ &= \frac{n - 1}{2} - \frac{n - 1}{2p}. \end{aligned}$$

The last inequality in (9.2) and the inequalities in (9.3) now follow immediately. It is easily seen that the upper bound in (9.3) is attained exactly when $n = 2p + 1$, whereas the lower bound in (9.3) is satisfied exactly when $n = 2 \cdot 3^i + 1$ for $i \geq 1$.

Now suppose that it is not the case that n is a prime of the form $2p^i + 1$. By part (i), it suffices to show that $\lambda(\lambda(n)) < n/4$. We make the following observations which derive from the definition of the Carmichael lambda-function. If $m \geq 2$ then $\lambda(m) < m$. If $2 \parallel m$ or $m = 4$, then $\lambda(m) \leq m/2$. Noting that $\lambda(m)$ is even for $m > 2$, we see that if $m > 4$ and $4 \mid m$, then $\lambda(m) \leq m/4$. Moreover, if m has $j \geq 2$ distinct prime divisors, then $\lambda(m) < m/2^{j-1}$.

We now suppose further that $4 \mid n$. Since $n > 5$ and $\lambda(n)$ is even, we see from our above comments that

$$\lambda(\lambda(n)) \leq \frac{\lambda(n)}{2} \leq \frac{n}{2 \cdot 4} = \frac{n}{8}.$$

Now assume that either $2 \parallel n$ or both n is odd and $\omega(n) \geq 2$. Since $n > 5$, we also have that $\omega(n) \geq 2$ if $2 \parallel n$. Then $\lambda(n) < n/2$ and $\lambda(n)$ is even. Hence,

$$\lambda(\lambda(n)) \leq \frac{\lambda(n)}{2} < \frac{n}{2 \cdot 2} = \frac{n}{4}.$$

We can now assume that n is odd and $\omega(n) = 1$. Suppose that $n = p^j$, where p is an odd prime and $j \geq 2$. Then

$$(9.5) \quad \lambda(\lambda(n)) = \lambda(\lambda(p^j)) = \lambda(p^{j-1}(p-1)).$$

If $p = 3$ and $j \geq 2$, then

$$\lambda(\lambda(n)) = 2 \cdot 3^{j-2} = \frac{2n}{9} < \frac{n}{4}.$$

Now suppose that $p \geq 5$ and $j \geq 2$. Then $\gcd(p, p-1) = 1$, $p-1$ is even, and $\lambda(p-1)$ is also even. From (9.5), we obtain

$$(9.6) \quad \begin{aligned} \lambda(\lambda(n)) &= \lambda(p^{j-1}(p-1)) \leq \text{lcm}(p^{j-2}(p-1), \lambda(p-1)) \\ &\leq \frac{1}{2} p^{j-2}(p-1) \frac{p-1}{2} < \frac{p^j}{4} = \frac{n}{4}. \end{aligned}$$

We finally assume that n is a prime. If $4 \mid n-1$, then $\lambda(n-1) \leq (n-1)/4$, since $n-1 > 4$. Hence,

$$\lambda(\lambda(n)) = \lambda(n-1) \leq \frac{n-1}{4} < \frac{n}{4}.$$

For our last case, we assume that $4 \nmid n-1$. Then $2 \parallel n-1$ and $\omega(n-1) = l \geq 3$, since $n-1$ is even, $n-1 > 4$, and n is not of the form $2p^i + 1$, where p is an odd prime and $i \geq 1$. Then

$$\lambda(\lambda(n)) = \lambda(n-1) \leq \frac{n-1}{2^{l-1}} \leq \frac{n-1}{4} < \frac{n}{4}.$$

Our result now follows. □

Remark 9.3. It is noted in [3, p. 1592] that Theorem 9.2 (i) holds.

For the next theorem we let S be the set consisting of natural numbers of the form $2^\alpha F_{m_1} \dots F_{m_j}$ for some $\alpha \geq 0$ and $j \geq 0$, where $F_{m_i} = 2^{2^{m_i}} + 1$ are distinct Fermat primes. If $j = 0$ then we set $n = 2^\alpha$. It is well known that $n \in S$ if and only if $\phi(n) = 2^i$ for some $i \geq 0$, where ϕ is Euler's totient function (see [5, pp. 34–35]). By a celebrated theorem due to Gauss, $n \in S$ for $n \geq 3$ if and only if the regular polygon with n sides has a Euclidean construction with ruler and compass.

Theorem 9.4. *Let $n \geq 1$ be a fixed integer. Then*

$$\max_{k \text{ even}} L(G(n, k)) = 1$$

if and only if $n \in S$.

Proof. Suppose that $n \in S$. Since $\lambda(n) \mid \phi(n)$, it follows that $\lambda(n) = 2^i$ for some $i \geq 0$. Thus if k is even, then 1 is the only divisor of $\lambda(n)$ which is relatively prime to k . It follows from Lemma 5.1 and Theorem 7.1 that the only cycles in $G(n, k)$ are fixed points.

Now suppose that $n \notin S$. Then there exists an odd prime p such that $p \mid \lambda(n)$. Clearly, there exists an even integer k such that $\gcd(k, p) = 1$ and $k \not\equiv 1 \pmod{p}$. Then $\text{ord}_p k \geq 2$ and the result follows from Lemma 5.1. \square

It follows from Lemma 5.1 that if $a \in G_1(n, k)$, then the length of the cycle in the same component as a is less than or equal to $\text{ord}_l k$, where l is defined as in (5.1) and depends on $\lambda(n)$. The following theorem, proved in [6] using analytic methods, gives lower bounds for $\text{ord}_l k$, which are valid for a positive proportion of integers n .

Theorem 9.5 (Kurlberg and Pomerance).

- (i) *Suppose $\varepsilon(x)$ tends to zero arbitrarily slowly as $x \rightarrow \infty$. Then $\text{ord}_l k \geq n^{1/2+\varepsilon(n)}$ for all but $o_\varepsilon(x)$ integers $n \leq x$.*
- (ii) *There is a positive constant γ such that $\text{ord}_l k \geq n^{1/2+\gamma}$ for a positive proportion of integers n .*
- (iii) *Assuming the Generalized Riemann Hypothesis, for each fixed $\varepsilon > 0$ we have $\text{ord}_l k > n^{1-\varepsilon}$ for all but $o_\varepsilon(x)$ integers $n \leq x$.*

The results in the paper [6] strengthen those given in [3].

As stated in Theorem 9.2 (i), $L(G(n, k)) \leq \lambda(\lambda(n))$. In [8], the following theorem is proved using analytic techniques regarding the order of $\lambda(\lambda(n))$.

Theorem 9.6 (Martin and Pomerance). *We have*

$$\lambda(\lambda(n)) = n \exp(-1(1 + o(1))(\log \log n)^2 \log \log \log n)$$

as $n \rightarrow \infty$ through a set of integers of asymptotic density 1.

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