Abstract. In this paper, we study the existence of the $n$-flat preenvelope and the $n$-FP-injective cover. We also characterize $n$-coherent rings in terms of the $n$-FP-injective and $n$-flat modules.

Keywords: $n$-flat module, $n$-FP-injective module, $n$-coherent ring, cotorsion theory

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1. Introduction

We use $R$-$\text{Mod}$ (resp., $\text{Mod}-R$) to denote the category of all left (resp., right) $R$-modules. For any $R$-module $M$, $\text{pd}_R M$ (resp., $\text{id}_R M$, $\text{fd}_R M$) denotes the projective (resp., injective, flat) dimension. The character module $\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by $M^+$.

Coherent rings have been characterized in various ways. The deepest result is the one due to Chase [2] which claims that the ring $R$ is left coherent if and only if products of flat right $R$-modules are again flat if and only if products of copies of $R$ are flat right $R$-modules. Lee [6] introduced the notions of left $n$-coherent and $n$-coherent rings and characterized them in various ways, using $n$-flat and $n$-FP-injective modules. In this paper we continue the study of $n$-coherent rings.

A ring $R$ is called left $n$-coherent (for integers $n > 0$ or $n = \infty$) if every finitely generated submodule of a free left $R$-module whose projective dimension is $\leq n - 1$ is finitely presented. Accordingly, all rings are left 1-coherent, and the left coherent rings are exactly those which are $d$-coherent ($d$ denotes the left global dimension of $R$). In particular, left $\infty$-coherent rings are left coherent. A right $R$-module $M$ will
be called \( n \)-flat if \( \text{Tor}_1^R(M, N) = 0 \) holds for all finitely presented left \( R \)-modules \( N \) with \( \text{pd}_R N \leq n \). A left \( R \)-module \( A \) is said to be \( n \)-FP-injective if \( \text{Ext}_1^R(N, A) = 0 \) holds for all finitely presented left \( R \)-modules \( N \) of projective dimension \( \leq n \).

Given a class \( \mathcal{C} \) of \( R \)-modules, let \( \perp \mathcal{C} \) be the class of \( R \)-modules \( F \) such that \( \text{Ext}_1^R(F, C) = 0 \) for every \( C \in \mathcal{C} \) and let \( \mathcal{C} \perp \) be the class of \( R \)-modules \( F \) such that \( \text{Ext}_1^R(C, F) = 0 \) for every \( C \in \mathcal{C} \). A pair of classes of \( R \)-modules \((\mathcal{F}, \mathcal{C})\) is called a cotorsion theory if \( \mathcal{F} \perp = \mathcal{C} \) and \( \perp \mathcal{C} = \mathcal{F} \). A cotorsion theory is said to be complete if for every \( R \)-module \( M \) there is an exact sequence \( 0 \rightarrow C \rightarrow F \rightarrow M \rightarrow 0 \) such that \( C \in \mathcal{C} \) and \( F \in \mathcal{F} \). A cotorsion theory is said to be perfect if every \( R \)-module has an \( \mathcal{F} \)-cover and a \( \mathcal{C} \)-envelope. A cotorsion theory is said to be hereditary if \( 0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0 \) is exact with \( F, F'' \in \mathcal{F} \), then \( F' \in \mathcal{F} \).

We recall that given a class of \( R \)-modules \( T \), a morphism \( \phi: T \rightarrow M \) where \( T \in T \) is called a \( T \)-cover of \( M \) if the following conditions hold:

1. For any linear map \( \phi': T' \rightarrow M \) with \( T' \in T \), there exists a linear map \( f: T' \rightarrow T \) with \( \phi' = \phi f \), or equivalently, \( \text{Hom}_R(T', T) \rightarrow \text{Hom}_R(T', M) \rightarrow 0 \) is exact for any \( T' \in T \).
2. If \( f \) is an endomorphism of \( T \) with \( \phi = \phi f \), then \( f \) must be an automorphism.

If (1) holds (and perhaps not (2)), \( \phi: T \rightarrow M \) is called a \( T \)-precover. A \( T \)-envelope and \( T \)-preenvelope are defined dually.

2. \( n \)-FLAT AND \( n \)-FP-INJECTIVE MODULES

Let \( n \) be a non-negative integer. In what follows, \( \mathcal{F}_n \) stands for the class of all \( n \)-flat right \( R \)-modules and \( \mathcal{F}\mathcal{I}_n \) denotes the class of all \( n \)-FP-injective left \( R \)-modules.

**Proposition 2.1.** \( \mathcal{F}_n \) and \( \mathcal{F}\mathcal{I}_n \) are closed under pure submodules.

**Proof.** Let \( B \in \mathcal{F}_n \) and let \( A \subseteq B \) be a pure submodule. Then \( 0 \rightarrow (B/A)^+ \rightarrow B^+ \rightarrow A^+ \rightarrow 0 \) is split and \( B^+ \) is \( n \)-FP-injective by [6, Lemma 5], and so \( A \) is \( n \)-flat by [6, Lemma 5].

Let \( M \in \mathcal{F}\mathcal{I}_n \), let \( S \) be a pure submodule of \( M \) and let \( N \) be any finitely presented left \( R \)-module with \( \text{pd}_R N \leq n \). Then we can get an induced exact sequence

\[
0 \rightarrow \text{Hom}_R(N, S) \rightarrow \text{Hom}_R(N, M) \rightarrow \text{Hom}_R(N, M/S) \rightarrow 0,
\]

and so \( \text{Ext}_1^R(N, S) = 0 \) since \( \text{Ext}_1^R(N, M) = 0 \). It follows that \( S \in \mathcal{F}\mathcal{I}_n \). \( \square \)

**Lemma 2.1.** The following conditions are equivalent:

1. \( M \) is \( n \)-FP-injective if and only if \( \text{Ext}_1^R(R/I, M) = 0 \) for any finitely generated left ideal \( I \) with \( \text{pd}_R I \leq n - 1 \);
(2) \( N \) is \( n \)-flat if and only if \( \text{Tor}_1^R(N, R/I) = 0 \) for any finitely generated left ideal \( I \) with \( \text{pd}_R I \leq n - 1 \).

**Proof.** (1) “\( \Rightarrow \)” is trivial.

“\( \Leftarrow \)” Let \( L \) be any finitely presented left \( R \)-module with \( \text{pd}_R L \leq n \). Then there is an exact sequence \( 0 \rightarrow A \rightarrow R^n \rightarrow L \rightarrow 0 \) for some \( n \geq 0 \) and \( A \subseteq R^n \) finitely generated with \( \text{pd}_R A \leq n - 1 \). Consider the following pullback of \( A \rightarrow R^n \) and \( R \rightarrow R^n \):

\[
\begin{array}{ccccccccc}
0 & 0 & & & & & & & \\
0 & B & A & R^{n-1} & 0 & & & & \\
& f & & & & & & & \\
0 & R & R^n & R^{n-1} & 0 & & & & \\
& & L & L & & & & & \\
& 0 & 0 & & & & & & \\
\end{array}
\]

Then \( L \cong R/\text{Im} f \) and \( \text{Im} f \cong B \) is finitely generated with \( \text{pd}_R \text{Im} f \leq n - 1 \). Thus \( \text{Ext}_1^R(L, M) \cong \text{Ext}_1^R(R/\text{Im} f, M) = 0 \), which gives that \( M \) is \( n \)-FP-injective.

(2) By analogy with the proof of (1). \( \square \)

**Theorem 2.1.** Let \( n \) be a non-negative integer and \( R \) a ring. Then

1. \( (\mathcal{F}_n, \mathcal{F}_n^\perp) \) is a perfect cotorsion theory;
2. \( (\perp \mathcal{F}_n, \mathcal{F}_n) \) is a complete cotorsion theory.

**Proof.** (1) Let \( \text{Card}(R) \leq \aleph_\beta \) and \( F \in \mathcal{F}_n \). Then we can write \( F \) as a union of a continuous chain \( (F_\alpha)_{\alpha<\lambda} \) of pure submodules of \( F \) such that \( \text{Card}(F_0) \leq \aleph_\beta \) and \( \text{Card}(F_{\alpha+1}/F_\alpha) \leq \aleph_\beta \) whenever \( \alpha + 1 < \lambda \). If \( N \) is a right \( R \)-module such that \( \text{Ext}_1^R(F_0, N) = 0 \) and \( \text{Ext}_1^R(F_{\alpha+1}/F_\alpha, N) = 0 \) whenever \( \alpha + 1 < \lambda \), then \( \text{Ext}_1^R(F, N) = 0 \) by [5, Theorem 7.3.4]. Since \( F_\alpha \) is a pure submodule of \( F \) for any \( \alpha < \lambda \), we have \( F_\alpha \in \mathcal{F}_n \) by Proposition 2.1. On the other hand, \( F_\alpha \) is a pure submodule of \( F_{\alpha+1} \) whenever \( \alpha + 1 < \lambda \), hence \( F_{\alpha+1}/F_\alpha \in \mathcal{F}_n \) by Proposition 2.1. Let \( X \) be a set of representatives of all modules \( G \in \mathcal{F}_n \) with \( \text{Card}(G) \leq \aleph_\beta \). Then \( \mathcal{F}_n^\perp = X^\perp \). So \( (\mathcal{F}_n, \mathcal{F}_n^\perp) \) is a cotorsion theory by [1, Corollary 2.13]. Since \( (\mathcal{F}_n, \mathcal{F}_n^\perp) \) is cogenerated by the set \( X \), \( (\mathcal{F}_n, \mathcal{F}_n^\perp) \) is a complete cotorsion theory by [5, Theorem 7.4.1]. Moreover, \( (\mathcal{F}_n, \mathcal{F}_n^\perp) \) is a perfect cotorsion theory by [5, Theorem 7.2.6] since \( \mathcal{F}_n \) is closed under direct limits.
(2) Let $X \in (\perp \mathcal{F}\mathcal{I}_n)^\perp$ and let $N$ be finitely presented with $\text{pd}_R N \leq n$. Then $N \in \perp \mathcal{F}\mathcal{I}_n$. So $\text{Ext}_R^1(N, X) = 0$, which gives that $X \in \mathcal{F}\mathcal{I}_n$ and $(\perp \mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n)$ is a cotorsion theory. By Lemma 2.1, $M$ is $n$-FP-injective if and only if $\text{Ext}_R^1(R/A, M) = 0$ for any finitely generated $A \subseteq R$ with $\text{pd}_R A \leq n - 1$. Set $X = \oplus R/A$, where the sum is over all finitely generated left ideals $A$ of $R$ with $\text{pd}_R A \leq n - 1$. Then $\mathcal{F}\mathcal{I}_n = X^\perp$. So $(\perp \mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n)$ is a complete cotorsion theory by [5, Theorem 7.4.1]. □

3. $n$-coherent rings

In this section we characterize $n$-coherent rings in terms of the $n$-FP-injective and $n$-flat modules. We obtain some characterizations of the situation when every $R$-module has a monic $\mathcal{F}_n$-preenvelope and an epic $\mathcal{F}_n$-preenvelope.

**Theorem 3.1.** For a ring $R$ and any $n$ ($0 < n \leq \infty$), the following conditions are equivalent:

1. $R$ is left $n$-coherent;
2. every right $R$-module has an $\mathcal{F}_n$-preenvelope;
3. any direct limit of $n$-FP-injective left $R$-modules is $n$-FP-injective;
4. $\text{Ext}_R^1(N, \lim_{\to} M_i) \to \lim_{\to} \text{Ext}_R^1(N, M_i)$ is an isomorphism for any finitely presented left $R$-module $N$ with $\text{pd}_R N \leq n$ and any direct system $(M_i)_{i \in I}$ of left $R$-modules;
5. $\mathcal{F}\mathcal{I}_n$ is a coresolving subcategory;
6. $(\perp \mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n)$ is a hereditary cotorsion theory.

**Proof.** (1) ⇒ (4) By [3, Lemma 2.9(2)]; (4) ⇒ (3) and (5) ⇒ (6) are obvious.

(1) ⇒ (2) Let $N$ be any right $R$-module. Then there is a cardinal number $\aleph_\alpha$ such that for any homomorphism $f: N \to L$ with $L$ $n$-flat, there is a pure submodule $Q$ of $L$ such that $\text{Card}(Q) \leq \aleph_\alpha$ and $f(N) \subseteq Q$. Note that $Q$ is $n$-flat by Proposition 2.1 and $\mathcal{F}_n$ is closed under products by [6, Theorem 5], and so $N$ has an $\mathcal{F}_n$-preenvelope by [5, Proposition 6.2.1].

(2) ⇒ (1) Let $(F_i)_{i \in I}$ be a family of $n$-flat right $R$-modules and let $\prod_{i \in I} F_i \to F$ be an $\mathcal{F}_n$-preenvelope. Then there are factorizations $\prod_{i \in I} F_i \to F \to F_j$, where $\prod_{i \in I} F_i \to F_j$ is the canonical projection for each $j$. This gives rise to a map $F \to \prod_{i \in I} F_i$ with the composition $\prod_{i \in I} F_i \to F \to \prod_{i \in I} F_i$ being the identity. Hence $\prod_{i \in I} F_i$ is isomorphic to a summand of $F$, and so $\prod_{i \in I} F_i$ is $n$-flat, which implies that $R$ is left $n$-coherent.

(3) ⇒ (1) Let $K$ be a finitely generated submodule of a free left $R$-module $F$ whose projective dimension is $\leq n - 1$. Consider the exact sequence $0 \to K \to F \to$
$F/K \to 0$. Then $F/K$ is finitely presented and $\text{pd}_R F/K \leq n$. So we have the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
\text{Hom}_R(F/K, \lim M_i) & \to & \text{Hom}_R(F, \lim M_i) & \to & \text{Hom}_R(K, \lim M_i) & \to & 0 \\
\alpha & & & & \beta & & \\
\lim \text{Hom}_R(F/K, M_i) & \to & \lim \text{Hom}_R(F, M_i) & \to & \lim \text{Hom}_R(K, M_i) & \to & 0 \\
\gamma & & & & & & \\
\end{array}
$$

Since $\alpha$ and $\beta$ are isomorphisms, $\gamma$ is an isomorphism by Five lemma. Thus $K$ is finitely presented.

$(1) \Rightarrow (5)$ Let $N$ be a finitely presented left $R$-module with $\text{pd}_R N \leq n$ and let $0 \to A \to B \to C \to 0$ be exact in $R$-Mod with $A, B \in \mathcal{F} \mathcal{I}_n$. Then

$$0 = \text{Ext}_R^1(N, B) \to \text{Ext}_R^1(N, C) \to \text{Ext}_R^2(N, A) = 0$$

by [6, Theorem 1], and so $C \in \mathcal{F} \mathcal{I}_n$. Thus $\mathcal{F} \mathcal{I}_n$ is a coresolving subcategory.

$(6) \Rightarrow (1)$ Let $S$ be a finitely generated submodule of a free left $R$-module $F$ whose projective dimension is $\leq n - 1$. We need to prove that $S$ is finitely presented. Let $M$ be FP-injective and let $0 \to M \to E \to C \to 0$ be exact with $E$ injective. Then $M \in \mathcal{F} \mathcal{I}_n$ and $C \in \mathcal{F} \mathcal{I}_n$, and so

$$\text{Ext}_R^1(S, M) \cong \text{Ext}_R^2(F/S, M) \cong \text{Ext}_R^1(F/S, C) = 0.$$

Thus $S$ is finitely presented, which means that $R$ is left $n$-coherent.

**Proposition 3.1.** The following conditions are equivalent:

1. $R$ is a left $n$-coherent ring;
2. $\text{Ext}_R^1(I, N) = 0$ for any FP-injective left $R$-module $N$ and any finitely generated left ideal $I$ with $\text{pd}_R I \leq n - 1$;
3. $\text{Ext}_R^2(R/I, N) = 0$ for any FP-injective left $R$-module $N$ and any finitely generated left ideal $I$ with $\text{pd}_R I \leq n - 1$;
4. if $0 \to N \to M \to L \to 0$ is an exact sequence of left $R$-modules with $N$ FP-injective and $M$ $n$-FP-injective, then $L$ is $n$-FP-injective.

**Proof.** $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$ Let $N$ be an FP-injective left $R$-module and $I$ a finitely generated left ideal with $\text{pd}_R I \leq n - 1$. Then the exact sequence $0 \to I \to R \to R/I \to 0$ gives rise to the exact sequence

$$0 = \text{Ext}_R^1(I, N) \to \text{Ext}_R^2(R/I, N) \to \text{Ext}_R^2(R, N) = 0$$

by (2). Thus $\text{Ext}_R^2(R/I, N) = 0$. 363
(3) ⇒ (4) Let \( I \) be a finitely generated left ideal of \( R \) with \( \text{pd}_R I \leq n - 1 \). The exact sequence \( 0 \to N \to M \to L \to 0 \) induces the exactness of

\[
0 = \text{Ext}^1_R(R/I, M) \to \text{Ext}^1_R(R/I, L) \to \text{Ext}^2_R(R/I, N) = 0
\]

by (3), and hence \( \text{Ext}^1_R(R/I, L) = 0 \). That is, \( L \) is \( n \)-FP-injective by Lemma 2.1.

(4) ⇒ (1) Let \( I \) be a finitely generated left ideal with \( \text{pd}_R I \leq n - 1 \). For any \( \text{FP-injective} \) left \( R \)-module \( N \), there is an exact sequence

\[
0 \to N \to E \to E/N \to 0
\]

with \( E \) injective. Note that \( E/N \) is \( n \)-FP-injective by (4). Hence we get the exact sequence

\[
0 = \text{Ext}^1_R(R/I, E/N) \to \text{Ext}^2_R(R/I, N) \to \text{Ext}^2_R(R/I, E) = 0,
\]

and so \( \text{Ext}^1_R(I, N) \cong \text{Ext}^2_R(R/I, N) = 0 \). It follows that \( I \) is finitely presented. Therefore \( R \) is left \( n \)-coherent.

\[\square\]

**Lemma 3.1.** Let \( R \) be a left \( n \)-coherent ring and let \( |M| = \lambda \) for a left \( R \)-module \( M \). Let \( k \) be as in El Bashir’s result. Then any map \( A \to M \) with \( A \) \( n \)-FP-injective can be factored through an \( n \)-FP-injective left \( R \)-module \( B \) with \( |B| < k \).

\[\text{Proof.} \quad \text{Consider any homomorphism} \ A \to M \text{ with} \ A \text{ \( n \)-FP-injective. If} \ |A| < k, \text{ let} \ B = A. \text{ So suppose} \ |A| \geq k. \text{ Consider a submodule} \ S \subseteq A \text{ maximal with respect to the two properties that} \ S \text{ is pure in} \ A \text{ and that} \ S \subseteq \text{Ker}(A \to M). \text{ Let} \ B = A/S. \text{ Then} \ B \text{ is} \ n \text{-FP-injective by Theorem 3.1. We wish to argue that} \ |B| < k. \text{ Let} \ K \text{ be the kernel of} \ B \to M. \text{ Then} \ |B/K| \leq |M| = \lambda. \text{ So if} \ |B| \geq k, \text{ there is a nonzero pure submodule} \ T/S \text{ of} \ B \text{ contained in} \ K. \text{ But then} \ T \text{ is pure in} \ A \text{ and is contained in the kernel of} \ A \to M. \text{ This contradicts the choice of} \ S. \quad \square\]

**Theorem 3.2.** Let \( R \) be a left \( n \)-coherent ring. Then every left \( R \)-module has an \( \mathcal{F} \mathcal{I}_n \)-cover.

\[\text{Proof.} \quad \text{By Lemma 3.1 and [5, Proposition 5.2.2 and Corollary 5.2.7].} \quad \square\]

**Proposition 3.2.** Let \( R \) be left \( n \)-coherent. Then the following conditions are equivalent:

1. every left \( R \)-module has an \( n \)-FP-injective cover with the unique mapping property (see [4]);

2. for every left \( R \)-modules exact sequence \( A \to B \to C \to 0 \) with \( A \) and \( B \) \( n \)-FP-injective, \( C \) is \( n \)-FP-injective.
Proof. (1) \(\Rightarrow\) (2) Let \(A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0\) be exact in \(R\)-Mod with \(A, B\) \(n\)-FP-injective and \(\theta: H \rightarrow C\) an \(n\)-FP-injective cover with the unique mapping property. Then there exists a map \(\delta: B \rightarrow H\) such that \(g = \theta \delta\). Thus \(\delta f = gf = 0 = \theta 0\), and hence \(\delta f = 0\), which implies that \(\text{Ker } g = \text{Im } f \subseteq \text{Ker } \delta\). Therefore there is a morphism \(\gamma: C \rightarrow H\) such that \(\gamma g = \delta\), and so \(\theta \gamma g = \theta \delta = g\), which gives that \(\theta \gamma = 1_H\). Thus \(C\) is isomorphic to a direct summand of \(H\), and so \(C\) is \(n\)-FP-injective.

(2) \(\Rightarrow\) (1) Let \(M\) be any left \(R\)-module. Then \(M\) has an \(n\)-FP-injective cover \(f: L \rightarrow M\) by Theorem 3.2. It is enough to show that for any \(n\)-FP-injective left \(R\)-module \(G\) and any homomorphism \(g: G \rightarrow L\) such that \(fg = 0\), we have \(g = 0\). In fact, there is a homomorphism \(\beta: L/\text{Im } g \rightarrow M\) such that \(\beta \pi = f\), where \(\pi: L \rightarrow L/\text{Im } g\) is the natural map. Since \(L/\text{Im } g\) is \(n\)-FP-injective, there is a map \(\alpha: L/\text{Im } g \rightarrow L\) such that \(\beta = f \alpha\), and so \(f \alpha \pi = f\). Hence \(\alpha \pi\) is an isomorphism. Therefore \(\pi\) is monic and \(g = 0\).

Proposition 3.3. The following conditions are equivalent:

1. \((\mathcal{F}_n, \mathcal{F}_n)\) is a hereditary cotorsion theory;
2. \(R\) is left \(n\)-coherent and \((\mathcal{F}_n, \mathcal{F}_n)\) is a hereditary cotorsion theory;
3. \(\text{Ext}_R^2(R/I, M) = 0\) for any finitely generated left ideal \(I\) with \(\text{pd}_R I \leq n - 1\) and any \(n\)-FP-injective left \(R\)-module \(M\);
4. \(R\) is left \(n\)-coherent and \(\text{Tor}_R^2(N, R/I) = 0\) for any finitely generated left ideal \(I\) with \(\text{pd}_R I \leq n - 1\) and any \(n\)-flat right \(R\)-module \(N\).

Proof. (1) \(\Rightarrow\) (2) Since \((\mathcal{F}_n, \mathcal{F}_n)\) is hereditary, \(R\) is left \(n\)-coherent by Theorem 3.1. On the other hand, let \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\) be exact with \(B, C \in \mathcal{F}_n\). Then \(0 \rightarrow C^+ \rightarrow B^+ \rightarrow A^+ \rightarrow 0\) is exact with \(B^+, C^+ \in \mathcal{F}_n\) by [6, Theorem 3], and so \(A^+ \in \mathcal{F}_n\) by (1), which implies that \(A \in \mathcal{F}_n\). That is, \((\mathcal{F}_n, \mathcal{F}_n)\) is hereditary.

(2) \(\Rightarrow\) (3) \(\Rightarrow\) (1) By [6, Theorem 1], (4) \(\Rightarrow\) (2). It is easy.

(2) \(\Rightarrow\) (4) Let \(N \in \mathcal{F}_n\) and let \(0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0\) be exact with \(P\) projective. Then \(K \in \mathcal{F}_n\) by (2), and hence \(\text{Tor}_R^2(N, R/I) \cong \text{Tor}_1^R(K, R/I) = 0\) for any finitely generated left ideal \(I\) with \(\text{pd}_R I \leq n - 1\).

Proposition 3.4. The following conditions are equivalent for a left \(n\)-coherent ring \(R\):

1. every \(n\)-flat right \(R\)-module is flat;
2. every cotorsion right \(R\)-module belongs to \(\mathcal{F}_n\);
3. every \(n\)-FP-injective left \(R\)-module is FP-injective;
4. every finitely presented left \(R\)-module belongs to \(\mathcal{F}_n\).
Proof. (1) \iff (2) and (3) \iff (4) follow from Theorem 2.1.

(1) \Rightarrow (3) Let \( M \) be any \( n \)-FP-injective left \( R \)-module. Then \( M^+ \) is \( n \)-flat, and so \( M^+ \) is flat by (1). On the other hand, for any finitely presented left \( R \)-module \( N \), there is an exact sequence

\[
\text{Tor}_1^R(M^+, N) \longrightarrow \text{Ext}_R^1(N, M)^+ \longrightarrow 0
\]

by [3, Lemma 2.7(1)]. Thus \( \text{Ext}_R^1(N, M) = 0 \), and so \( M \) is FP-injective.

(3) \Rightarrow (1) Let \( M \) be an \( n \)-flat right \( R \)-module. Then \( M^+ \) is \( n \)-FP-injective, and so \( M^+ \) is FP-injective by (3). Hence \( M \) is flat. \hfill \Box

Now we study when every right \( R \)-module has a monic \( \mathcal{F}_n \)-preenvelope and an epic \( \mathcal{F}_n \)-preenvelope.

**Proposition 3.5.** The following conditions are equivalent:

1. every right \( R \)-module has a monic \( \mathcal{F}_n \)-preenvelope;
2. \( R \) is left \( n \)-coherent and every flat left \( R \)-module is \( n \)-FP-injective;
3. \( R \) is left \( n \)-coherent and \( R R \) is \( n \)-FP-injective;
4. \( R \) is left \( n \)-coherent and \((\mathcal{F}_n, \mathcal{F}_n^\perp)\) is a perfect cotorsion theory;
5. \( R \) is left \( n \)-coherent and every left \( R \)-module has an epic \( \mathcal{F}_n \)-cover.

Proof. (2) \Rightarrow (3) and (4) \Rightarrow (5) are obvious.

(1) \Rightarrow (2) Let \( M \) be a flat left \( R \)-module. Then \( M^+ \) is injective and \( M^+ \) has a monic \( \mathcal{F}_n \)-preenvelope \( \varphi: M^+ \to F \). Set \( C = \text{Coker} \varphi \). Then \( 0 \to M^+ \to F \to C \to 0 \) is split, and so \( C \in \mathcal{F}_n \), which gives that \( M^+ \in \mathcal{F}_n \) since \( R \) is left \( n \)-coherent. Thus \( M \) is \( n \)-FP-injective.

(3) \Rightarrow (4) By analogy with the proof of Theorem 2.1.

(5) \Rightarrow (1) Let \( M \) be any right \( R \)-module. Then \( M \) has an epic \( \mathcal{F}_n \)-cover \( E \to M^+ \) by (5), and so there is a monomorphism \( M \to E^+ \). Thus every right \( R \)-module has a monic \( \mathcal{F}_n \)-preenvelope by Theorem 3.1. \hfill \Box

**Proposition 3.6.** The following conditions are equivalent:

1. every right \( R \)-module has an epic \( \mathcal{F}_n \)-preenvelope;
2. \( R \) is left \( n \)-coherent and every submodule of any \( n \)-flat right \( R \)-module is \( n \)-flat;
3. every quotient module of any \( n \)-FP-injective left \( R \)-module is \( n \)-FP-injective;
4. every left \( R \)-module has a monic \( \mathcal{F}_n \)-cover.

Proof. (1) \Rightarrow (2) \( R \) is left \( n \)-coherent by Theorem 3.1. Now suppose that \( N \) is a submodule of an \( n \)-flat right \( R \)-module \( L \) and \( \iota: N \to L \) is the inclusion.
By (1), $N$ has an epic $\mathcal{F}_n$-preenvelope $f: N \to F$. Then there is a homomorphism $g: F \to L$ such that the following diagram is commutative:

\[
\begin{array}{c}
N \xrightarrow{f} F \\
\downarrow \quad \downarrow \quad \downarrow \\
\quad \quad \quad \quad \quad \quad L \\
\end{array}
\]

So $gf = \iota$ is monic, and hence $f$ is monic, which gives that $f$ is an isomorphism and $N \cong F$ is $n$-flat.

(2) $\Rightarrow$ (3) Let $M$ be any $n$-FP-injective left $R$-module and let $M \to N \to 0$ be exact. Then $0 \to N^+ \to M^+$ is exact and $M^+$ is $n$-flat, and so $N^+$ is $n$-flat by (2). Thus $N$ is $n$-FP-injective by [6, Theorem 3].

(3) $\Rightarrow$ (4) By [6, Theorem 2], $R$ is left $n$-coherent, and hence every left $R$-module $M$ has an $\mathcal{F}_n$-precover $\varphi: C \to M$. Note that $\text{Im} \varphi$ is $n$-FP-injective by (3), so $\text{Im} \varphi \to M$ is a monic $\mathcal{F}_n$-cover.

(4) $\Rightarrow$ (1) Let $E$ be an injective left $R$-module and $S \subseteq E$ a pure submodule. Then $E/S$ has a monic $\mathcal{F}_n$-cover $f: C \to E/S$. By analogy with the proof (1) $\Rightarrow$ (2), $f$ is an isomorphism and $E/S$ is $n$-FP-injective, and hence $R$ is left $n$-coherent by [6, Theorem 2], which means that every right $R$-module has an $\mathcal{F}_n$-preenvelope by Theorem 3.1. Let $M$ be any right $R$-module. Then $M^+$ has a monic $\mathcal{F}_n$-cover $E \to M^+$, and hence $M^{++} \to E^+ \to 0$ is exact. Set $K = \text{Ker}(M^{++} \to E^+)$. Consider the following pullback of $M \to M^{++}$ and $K \to M^{++}$:

\[
\begin{array}{c}
0 \\
\downarrow & \downarrow & \downarrow \\
0 \quad X \quad K \quad M^{++}/M \quad 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 \quad M \quad M^{++} \quad M^{++}/M \quad 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
E^+ \quad E^+ \quad E^+ \quad E^+ \quad 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \\
\end{array}
\]

Since $E^+$ is $n$-flat, $M$ has an epic $\mathcal{F}_n$-preenvelope. \qed
Proposition 3.7. The following conditions are equivalent:

(1) every left $R$-module is $n$-FP-injective;
(2) every cotorsion left $R$-module is $n$-FP-injective;
(3) every right $R$-module is $n$-flat;
(4) every cotorsion right $R$-module is $n$-flat;
(5) every right $R$-module in $\mathcal{F}_n^\perp$ is injective;
(6) every left $R$-module in $\mathcal{F}_n^\perp$ is projective;
(7) every nonzero right $R$-module contains a nonzero $n$-flat submodule;
(8) $(\mathcal{F}_n^\perp, \mathcal{F}_n^\perp)$ is a hereditary cotorsion theory and every left $R$-module in $\mathcal{F}_n^\perp$ is $n$-FP-injective.

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), (3) $\Rightarrow$ (7) and (1) $\Rightarrow$ (8) are obvious.

(2) $\Rightarrow$ (3) Let $M$ be any right $R$-module. Then $M^+$ is $n$-FP-injective by (2), and so $M$ is $n$-flat.

(4) $\Rightarrow$ (1) Let $M$ be any left $R$-module. Then $M^+$ is $n$-flat by (4), and so $M^{++}$ is $n$-FP-injective. Note that $M$ is a pure submodule of $M^{++}$. So $M$ is $n$-FP-injective.

(3) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (6) follow from Theorem 2.1.

(7) $\Rightarrow$ (5) Assume that $0 \to A \to B \to C \to 0$ is any exact sequence. To simplify the notation, we think of $A$ as a submodule of $B$. Let $M \in \mathcal{F}_n^\perp$ and let $f: A \to M$ be any homomorphism. By a simple application of Zorn’s Lemma, we can find $g: D \to M$, where $A \subseteq D \subseteq B$ and $g|_A = f$, such that $g$ cannot be extended to any submodule of $B$ properly containing $D$. We claim that $D = B$. Indeed, if $D \neq B$, then $B/D \neq 0$. By (7), there is a nonzero submodule $N/D$ of $B/D$ such that $N/D$ is $n$-flat. Since $M \in \mathcal{F}_n^\perp$, there exists $h: N \to M$ such that $h|_D = g$. It is obvious that $h$ extends $g$, thus we get the desired contradiction, and so $M$ is injective.

(8) $\Rightarrow$ (1) Let $M$ be a left $R$-module. By Theorem 2.1, there is an exact sequence $0 \to K \to F \to M \to 0$ with $F \in \mathcal{F}_n^\perp$, $K \in \mathcal{F}_n^\perp$. Then $F \in \mathcal{F}_n^\perp$, and hence $M \in \mathcal{F}_n^\perp$ by (8).

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References


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