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L^∞ ESTIMATES OF SOLUTION FOR m -LAPLACIAN PARABOLIC
EQUATION WITH A NONLOCAL TERM

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Abstract. In this paper, we consider the global existence, uniqueness and L^∞ estimates of weak solutions to quasilinear parabolic equation of m -Laplacian type $u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) = u|u|^{\beta-1} \int_\Omega |u|^\alpha dx$ in $\Omega \times (0, \infty)$ with zero Dirichlet boundary condition in $\partial\Omega$. Further, we obtain the L^∞ estimate of the solution $u(t)$ and $\nabla u(t)$ for $t > 0$ with the initial data $u_0 \in L^q(\Omega)$ ($q > 1$), and the case $\alpha + \beta < m - 1$.

Keywords: m -Laplacian parabolic equations, global existence, uniqueness, L^∞ estimates

MSC 2010: 35K65, 35K92

1. INTRODUCTION

In this paper we study the global existence, uniqueness, and L^∞ estimates of the solution for the initial boundary value problem for the parabolic equation of m -Laplacian type with a nonlocal term

$$(1.1) \quad \begin{cases} u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) = u|u|^{\beta-1} \int_\Omega |u|^\alpha dx, & x \in \Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \\ u(x, t) = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where $2 < m < N$, $\alpha \geq 0$, $\beta \geq 1$ and Ω is a bounded domain in \mathbb{R}^N ($N \geq 3$) with the smooth boundary $\partial\Omega$. If $\alpha + \beta \geq m - 1$ and $|\Omega|$ or $u_0(x)$ is properly large, we know the problem (1.1) need not have a global solution, see [9]. So we mainly consider the problem (1.1) with $\alpha + \beta < m - 1$. Many results concerning global

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existence, uniqueness, blow-up and asymptotic behavior of the solution for (1.1) have been established. In particular, it is well known that (1.1) admits a unique global solution if $\alpha = 0$ and $u_0 \in W_0^{1,m}(\Omega)$.

Many physical phenomena were formulated as non-local mathematical models and studied by many authors (cf. [1],[5], [9]). Li and Xie in [9] considered the problem

$$(1.2) \quad \begin{cases} u_t - \operatorname{div}(|\nabla u|^{m-2}\nabla u) = \int_{\Omega} |u|^{\alpha} dx, & x \in \Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \Omega, \\ u(x, t) = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

by making use of super-subsolution techniques with $2 < m < N, \alpha \geq 1, u_0 \in L^{\infty}(\Omega) \cap W_0^{1,m}(\Omega)$ and $\partial u_0/\partial\nu < 0$ on $\partial\Omega$, where ν denotes the unit outer normal vector on the boundary $\partial\Omega$. Under the appropriate hypotheses, they developed local theory of the solution and obtained that the solution either exists globally or blows up in finite time.

Rouchon in [14] proved the existence of a universal bound for all nonnegative global solutions of (1.2) with $m = 2$, where $\alpha > 1$ and $u_0 \in L^{\infty}(\Omega)$.

In [3] Chen considered the nonlocal problem (1.1) with $\alpha = 0$ and $u_0 \in L^q$ ($1 < q < 2$), proved the global existence of $u(t)$ and gave an L^{∞} estimates of $u(t)$ and $\nabla u(t)$ for $t \in (0, T]$. However, as far as we know, there are few results concerning the L^{∞} estimates of $u(t)$ and $\nabla u(t)$ for $u_0 \in L^q(\Omega)$ ($q > 1$) for the problem (1.1).

In this paper we are interested in the global existence and the uniqueness of solution for (1.1) with $u_0 \in L^q(\Omega)$ ($q > 1$), $\alpha + \beta < m - 1$, and give L^{∞} estimates for $u(t)$ and $\nabla u(t)$ with $t > 0$. For L^{∞} estimates, we use Moser's technique as in [2]–[4], [11]–[13]. To obtain an estimate of $\|\nabla u(t)\|_{\infty}$, we also make the assumption that the mean curvature $H(x)$ of $\partial\Omega$ at x is non-positive with respect to the outward normal; such assumption is made also in [2], [7]. We know that $H(x) \leq 0$ if Ω is convex.

This paper is organized as follows. In Section 2, we state the main results and present some lemmas which will be used below. In Sections 3 and 4, we use these lemmas to derive L^{∞} estimates for $u(t)$ and $\nabla u(t)$, respectively. The proof of the main results will be given in Sections 3 and 4.

2. PRELIMINARIES AND RESULTS

Let $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ denote the $L^p(\Omega)$ and $W^{1,p}(\Omega)$ ($1 \leq p \leq \infty$) norms respectively.

Definition 1. A measurable function $u(x, t)$ on $\Omega \times \mathbb{R}^+$ is said to be a weak solution of the problem (1.1) if $u = u(x, t) \in L^{\infty}((0, \infty), W_0^{1,m}) \cap L^{m-1}(\mathbb{R}^+, W_0^{1,m-1})$

and the equality

$$(2.1) \quad \int_0^t \int_{\Omega} \{-u\varphi_s + |\nabla u|^{m-2} \nabla u \nabla \varphi - \|u(s)\|_{\alpha}^{\alpha} |u|^{\beta-1} u \varphi\} dx ds \\ = \int_{\Omega} \{u_0(x)\varphi(x, 0) - u(x, t)\varphi(x, t)\} dx$$

is valid for any $t > 0$ and $\varphi = \varphi(x, t) \in C^1(\mathbb{R}^+, C_0^1(\Omega))$, where $\mathbb{R}^+ = [0, +\infty)$.

We make the following assumptions.

(H₁) $u_0 \in L^q(\Omega)$, $q > 1$;

(H₂) $N > m > 2$, $\alpha \geq 0$, $\beta \geq 1$, and $\alpha + \beta < m - 1$;

(H₃) the mean curvature $H(x)$ of $\partial\Omega$ at x is non-positive with respect to the outward normal.

Remark 1. Since Ω is a bounded domain, we have $L^p(\Omega) \subset L^q(\Omega)$ for $p > q \geq 1$.

Our main results read as follows.

Theorem 1. Assume (H₁)–(H₂) hold. Then (1.1) admits a unique global solution $u(t)$ which satisfies

$$(2.2) \quad u(t) \in L^{\infty}(\mathbb{R}^+, L^q) \cap L_{\text{loc}}^{\infty}((0, \infty), W_0^{1,m}) \cap L_{\text{loc}}^{m-1}(\mathbb{R}^+, W_0^{1,m-1}), \\ u_t \in L_{\text{loc}}^2((0, \infty), L^2),$$

and the estimates

$$(2.3) \quad \|u(t)\|_p \leq C_p(1 + t^{-1/(m-2)}), \quad t > 0, \quad \forall p > q,$$

$$(2.4) \quad \|u(t)\|_{\infty} \leq C_0 t^{-\lambda}, \quad 0 < t \leq T,$$

$$(2.5) \quad \|\nabla u(t)\|_m \leq C_0 t^{-(1+2\lambda(\alpha+\beta))/m}, \quad 0 < t \leq T,$$

where $\lambda = N/((m-2)N + mq)$, $C_0 = C_0(|\Omega|, T, \|u_0\|_q) > 0$ and C_p depends on p .

Theorem 2. Assume that (H₁)–(H₃) hold. Then the solution $u(t)$ of (1.1) has the gradient estimate

$$(2.6) \quad \|\nabla u(t)\|_{\infty} \leq C_0 t^{-\mu}, \quad 0 < t \leq T.$$

Further, if $u_0 \in W_0^{1,m}(\Omega)$ and $2\beta < m^* = Nm/(N-m)$, we have

$$(2.7) \quad \|\nabla u(t)\|_m \leq \|\nabla u_0\|_m e^{-\lambda_1 t} + C_0, \quad t \geq 0$$

with some $\lambda_1 > 0$ and $\mu = (2(1 + 2\lambda(\alpha + \beta)) + N^2)/(2m + (m-2)N^2)$.

To obtain the above results, we will use the following lemmas.

Lemma 1 ([11], [16]). Let $\beta \geq 0$, $N > m \geq 1$, $\beta + 1 \leq q$ and $1 \leq r \leq p \leq (\beta + 1)Nm/(N - m)$. Then for $|u|^\beta u \in W^{1,m}(\Omega)$ we have

$$\|u\|_q \leq C_1^{1/(\beta+1)} \|u\|_r^{1-\theta} \| |u|^\beta u \|_{1,m}^{\theta/(\beta+1)}$$

with $\theta = (\beta + 1)(r^{-1} - p^{-1})/(N^{-1} - m^{-1} + (\beta + 1)r^{-1})$, where C_1 is a constant independent of p, r, β and θ .

Lemma 2 ([13]). Let $y(t)$ be a nonnegative differentiable function on $(0, T]$ satisfying

$$y'(t) + At^{\lambda\theta-1}y^{1+\theta} \leq Bt^{-k}y(t) + Ct^{-\delta}$$

with $A, \theta > 0$, $\lambda\theta \geq 1$, $B, C \geq 0$, $k \leq 1$. Then we have

$$y(t) \leq A^{-1/\theta}(2\lambda + 2BT^{1-k})^{1/\theta}t^{-\lambda} + 2C(\lambda + BT^{1-k})^{-1}t^{1-\delta}, \quad 0 < t \leq T.$$

Lemma 3 ([15]). Let $y(t)$ be a nonnegative differentiable function on $(0, \infty)$ satisfying

$$y'(t) + Ay^{1+\mu}(t) \leq B, \quad t > 0,$$

with $A, \mu > 0$, $B \geq 0$. Then

$$y(t) \leq (BA^{-1})^{1/(1+\mu)} + (A\mu t)^{-1/\mu}, \quad t > 0.$$

Further, if $y(t)$ is continuous on $[0, +\infty)$ then

$$y(t) \leq (BA^{-1})^{1/(1+\mu)} + (y(0))^{-\mu} + A\mu t)^{-1/\mu}, \quad t > 0.$$

3. L^∞ ESTIMATE FOR $u(t)$

Let $u_{0,i} \in C_0^2(\Omega) \rightarrow u_0$ in $L^q(\Omega)$ ($q > 1$) as $i \rightarrow \infty$. For $i = 1, 2, \dots$, we consider the approximate problem of (1.1):

$$(3.1) \quad \begin{cases} u_t - \operatorname{div}((|\nabla u|^2 + i^{-1})^{(m-2)/2} \nabla u) = u|u|^{\beta-1} \int_\Omega |u|^\alpha dx, & x \in \Omega, t > 0, \\ u(x, 0) = u_{0,i}(x), & x \in \Omega, \\ u(x, t) = 0, & x \in \partial\Omega, t > 0. \end{cases}$$

Then the problem (3.1) has a unique smooth solution $u_i(x, t)$ (see [8]). For simplicity of notation, we write u instead of u_i and u^p for $|u|^{p-1}u$ when $p > 0$. Also, let C, C_j, μ_j be generic constants independent of i and p , and changeable from line to line.

Proposition 1. Assume (H₁)–(H₂) hold and $u(t) = u(x, t)$ is the solution of (3.1). Then $u(t) \in L^\infty(\mathbb{R}^+, L^q)$.

The proof of Proposition 1 is similar to that of Proposition 1 in [3] and is omitted here.

Proposition 2. Under the assumptions of Proposition 1 and $p \geq q > 1$, the solution $u(t)$ of (3.1) also satisfies

$$(3.2) \quad \|u(t)\|_p \leq C_p(1 + t^{-1/(m-2)}), \quad t > 0, \quad \forall p > q,$$

and for any $T > 0$,

$$(3.3) \quad \|u(t)\|_\infty \leq C_1 t^{-\lambda}, \quad 0 < t \leq T,$$

$$(3.4) \quad \|\nabla u(t)\|_m \leq C_1 t^{-(1+2\lambda(\alpha+\beta))/m}, \quad 0 < t \leq T,$$

$$(3.5) \quad \int_0^T s^{1+\gamma} \|u_s(s)\|_2^2 ds \leq C_1,$$

where C_p depends on p , $\lambda = N/((m-2)N + mq)$, $C_1 = C_1(|\Omega|, T, \|u_0\|_q)$, $\gamma > \lambda(2-q)^+$, and $(2-q)^+ = \max\{0, 2-q\}$.

Proof. Multiplying (3.1) by u^{p-1} ($p \geq q > 1$), we have

$$(3.6) \quad \frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p + \mu_0 p^{1-m} \|\nabla u^{(p+m-2)/m}(t)\|_m^m \leq \|u(t)\|_\alpha^\alpha \int_\Omega |u(t)|^{\beta+p-1} dx \equiv A.$$

By the Sobolev inequality, we get

$$\|\nabla u^{(p+m-2)/m}\|_m^m \geq \mu_1 \|u\|_{p+m-2}^{p+m-2}$$

where $\mu_0, \mu_1 > 0$ are independent of p .

Since $\alpha + \beta < m - 1$, we use Young's inequality and get

$$(3.7) \quad \|u(t)\|_\alpha^\alpha \int_\Omega |u(t)|^{\beta+p-1} dx = \|u(t)\|_\alpha^\alpha \|u(t)\|_{p+\beta-1}^{p+\beta-1} \leq \frac{\mu_0 \mu_1}{2} \|u(t)\|_{p+m-2}^{p+m-2} + C_p.$$

Then (3.6) becomes

$$(3.8) \quad \frac{d}{dt} \|u(t)\|_p^p + \frac{\mu_0 \mu_1}{2} p^{2-m} \|u(t)\|_{p+m-2}^{p+m-2} \leq p C_p.$$

Note that

$$(3.9) \quad \|u(t)\|_{p+m-2}^{p+m-2} \geq C_0 \|u(t)\|_p^{p+m-2}$$

with $C_0 = C_0(|\Omega|) > 0$. Then, the application of Lemma 3 to (3.8) gives

$$(3.10) \quad \|u(t)\|_p \leq C_p(1 + t^{-1/(m-2)}), \quad t > 0.$$

In order to derive (3.3), we must treat carefully the differential inequality (3.6). Since $0 \leq \alpha < m$, it follows from the Sobolev's inequality that

$$\|u(t)\|_\alpha \leq C_0 \|u^{(p+m-2)/m}(t)\|_m^{m/(p+m-2)} \leq C_0 \|\nabla u^{(p+m-2)/m}(t)\|_m^{m/(p+m-2)}$$

Further, by Lemma 1 and Proposition 1, we get

$$\begin{aligned} A &= \|u(t)\|_\alpha^\alpha \|u(t)\|_{p+\beta-1}^{p+\beta-1} \leq \|u(t)\|_\alpha^\alpha \|u(t)\|_p^{\theta_1} \|u(t)\|_q^{\theta_2} \|u(t)\|_{p^*}^{\theta_3} \\ &\leq C_0 \|\nabla u^{(p+m-2)/m}(t)\|_m^{m\alpha/(p+m-2)} \|u(t)\|_p^{\theta_1} \|u(t)\|_{p^*}^{\theta_3} \\ &\leq C_0 \|u\|_p^{\theta_1} \|\nabla u^{(p+m-2)/m}(t)\|_m^{m(\alpha+\theta_3)/(p+m-2)} \\ &\leq \frac{1}{2} \mu_0 p^{1-m} \|\nabla u^{(p+m-2)/m}(t)\|_m^m + Cp^\sigma \|u(t)\|_p^p, \end{aligned}$$

in which $\sigma = \lambda\alpha$, $p^* = N(p+m-2)/(N-m)$ and $(\theta_1, \theta_2, \theta_3)$ is the positive solution of the following system

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 = p + \beta - 1, \\ \frac{\theta_1}{p} + \frac{\theta_2}{q} + \frac{\theta_3(N-m)}{N(p+m-2)} = 1, \\ \frac{\theta_1}{p} + \frac{\theta_3 + \alpha}{p+m-2} = 1. \end{cases}$$

It is easy to obtain

$$\begin{aligned} \theta_1 &= \frac{p(p+m-2-\alpha)[N(m-2)+mq] - pN(p+m-2)(\beta-1) - pN\alpha(p-q)}{(p+m-2)[N(m-2)+mq]}, \\ \theta_2 &= q\alpha(p+m-2)^{-1} + mq[\beta-1 + \alpha(p-q)(p+m-2)^{-1}][N(m-2)+mq]^{-1}, \\ \theta_3 &= [N(p+m-2)(\beta-1) + N\alpha(p-q)][N(m-2)+mq]^{-1}. \end{aligned}$$

Then (3.6) becomes

$$(3.11) \quad \frac{d}{dt} \|u(t)\|_p^p + \frac{1}{2} C_0 p^{2-m} \|\nabla u^{(p+m-2)/m}(t)\|_m^m \leq C_2 p^{1+\sigma} \|u(t)\|_p^p.$$

Let $r > mq^{-1}$, $p_1 = q$, $p_n = rp_{n-1} - m + 2$, $\theta_n = rN(1 - p_n^{-1}p_{n-1})(m + Nr - N)^{-1}$, $\beta_n = (p_n + m - 2)\theta_n^{-1} - p_n$, $n = 2, 3, \dots$. By Lemma 1 we know that

$$\|u(t)\|_{p_n} \leq C^{m/(p_n+m-2)} \|u(t)\|_{p_{n-1}}^{1-\theta_n} \|\nabla u^{(p_n+m-2)/m}(t)\|_m^{m\theta_n/(p_n+m-2)}.$$

Putting this into (3.11) ($p = p_n$) we find that for $0 < t \leq T$,

$$(3.12) \quad \frac{d}{dt} \|u(t)\|_{p_n} + C_0 C^{-m/\theta_n} p_n^{2-m} \|u(t)\|_{p_{n-1}}^{m-2-\beta_n} \|u(t)\|_{p_n}^{1+\beta_n} \leq C_2 p_n^{1+\sigma} \|u(t)\|_{p_n}.$$

We claim that there exist a bounded sequence $\{\xi_n\}$ and a convergent sequence $\{\lambda_n\}$ such that

$$(3.13) \quad \|u(t)\|_{p_n} \leq \xi_n t^{-\lambda_n}, \quad 0 < t \leq T.$$

In fact, by Proposition 1, this holds for $n = 1$ if we take $\lambda_1 = 0$, $\xi_1 = \sup_{t \geq 0} \{\|u(t)\|_q\}$.

If (3.13) is true for $n - 1$, then we have from (3.12) that

$$(3.14) \quad \begin{aligned} \frac{d}{dt} \|u(t)\|_{p_n} + C_0 C^{-m/\theta_n} p_n^{2-m} \xi_{n-1}^{m-2-\beta_n} t^{\lambda_{n-1}(\beta_n-m+2)} \|u(t)\|_{p_n}^{1+\beta_n} \\ \leq C_2 p_n^{1+\sigma} \|u(t)\|_{p_n}. \end{aligned}$$

Applying Lemma 2 to (3.14), we conclude that (3.13) also holds for n with $\lambda_n = (1 + \lambda_{n-1}(\beta_n - m + 2))\beta_n^{-1}$ and $\xi_n = (C_0^{-1} C^{m/\theta_n} p_n^{2-m} \xi_{n-1}^{m-2})^{-1/\beta_n} (2\lambda_n + 2C_2 p_n^{1+\sigma})^{1/\beta_n} \xi_{n-1}$, $n = 2, 3, \dots$

It is not difficult to show that $\lambda_n \rightarrow \lambda = N/((m-2)N + mq)$ as $n \rightarrow \infty$ and $\{\xi_n\}$ is bounded (cf. [11]). Then (3.3) follows from (3.13) as $n \rightarrow \infty$.

In order to obtain (3.4), we first choose $\gamma > \lambda(2-q)^+$. Without loss of generality, we suppose $q \in (1, 2)$. Further, let $\eta(t) \in C[0, \infty) \cap C^1(0, \infty)$ such that $\eta(t) = t^\gamma$ if $t \in [0, 1]$; $\eta(t) = 2$ if $t \geq 2$ and $\eta(t), \eta'(t) \geq 0$ in $[0, \infty)$.

Then, multiplying (3.1) by $\eta(t)u$ we arrive at

$$(3.15) \quad \begin{aligned} \int_0^t \int_\Omega \eta(s) |\nabla u(s)|^m dx ds + \frac{1}{2} \eta(t) \|u(t)\|_2^2 \\ \leq \frac{1}{2} \int_0^t \eta'(s) \|u(s)\|_2^2 ds + \int_0^t \int_\Omega \eta(s) |u(s)|^{\beta+1} \|u(s)\|_\alpha^\alpha dx ds. \end{aligned}$$

Noticing that for $t \in (0, T]$,

$$(3.16) \quad \int_0^t \eta'(s) \|u(s)\|_2^2 ds \leq C_0 \int_0^t s^{\gamma-1} \|u(s)\|_q^q \|u(s)\|_\infty^{2-q} ds \leq C t^{\gamma-\lambda(2-q)}$$

and

$$(3.17) \quad \|u(s)\|_\alpha^\alpha \int_\Omega |u(s)|^{\beta+1} dx \leq C \|u(s)\|_m^{\alpha+\beta+1} \leq \frac{1}{2} \|\nabla u(s)\|_m^m + C,$$

where the fact $\alpha + \beta < m - 1$ has been used.

Therefore, we obtain from (3.15)–(3.17) that

$$(3.18) \quad \int_0^t \int_{\Omega} \eta(s) |\nabla u(s)|^m dx ds \leq Ct^{\gamma-\lambda(2-q)}, \quad 0 < t \leq T.$$

Next, let $\varrho(t) = \int_0^t \eta(s) ds$, $t \geq 0$. Similarly, multiplying (3.1) by $\varrho(t)u_t$, we get

$$(3.19) \quad \begin{aligned} \frac{1}{m} \frac{d}{dt} \int_{\Omega} \varrho(t) (|\nabla u(t)|^2 + i^{-1})^{m/2} dx + \varrho(t) \|u_t(t)\|_2^2 \\ \leq \frac{\varrho'(t)}{m} \int_{\Omega} (|\nabla u(t)|^2 + i^{-1})^{m/2} dx + \varrho(t) \|u(t)\|_{\alpha}^{\alpha} \int_{\Omega} |u(t)|^{\beta} |u_t(t)| dx. \end{aligned}$$

Moreover, for $t \in (0, T]$ we have

$$(3.20) \quad \begin{aligned} \varrho(t) \|u(t)\|_{\alpha}^{\alpha} \int_{\Omega} |u(t)|^{\beta} |u_t(t)| dx &\leq \varrho(t) \|u_t(t)\|_2 \|u(t)\|_{2\beta}^{\beta} \|u(t)\|_{\alpha}^{\alpha} \\ &\leq \frac{1}{2} \varrho(t) \|u_t(t)\|_2^2 + C \varrho(t) \|u(t)\|_{2\beta}^{2\beta} \|u(t)\|_{\alpha}^{2\alpha} \\ &\leq \frac{1}{2} \varrho(t) \|u_t(t)\|_2^2 + Ct^{\gamma+1-2(\alpha+\beta)\lambda}. \end{aligned}$$

Now, the application of (3.18)–(3.20) and the integration of (3.19) on $[0, t]$ yield

$$(3.21) \quad \frac{1}{m} \varrho(t) \|\nabla u(t)\|_m^m + \frac{1}{2} \int_0^t \varrho(s) \|u_s(s)\|_2^2 ds \leq C(t^{\gamma-\lambda(2-q)} + t^{\gamma+2-2(\alpha+\beta)\lambda}).$$

Thus (3.21) gives

$$(3.22) \quad \frac{\varrho(t)}{m} \|\nabla u(t)\|_m^m \leq Ct^{\gamma-2\lambda(\alpha+\beta)}, \quad 0 < t \leq T.$$

This implies (3.4). Similarly, we have the estimate (3.5) from (3.21) and (3.22). Then the proof of Proposition 2 is completed. \square

Proof of Theorem 1. Noticing that the estimate constants C_1, C_p in (3.2)–(3.5) are independent of i , we can obtain the desired solution $u(t)$ as the limit of $\{u_i\}$ (or a subsequence) by the standard compactness argument in [10, 11]. The solution $u(t)$ of (1.1) also satisfies (3.2)–(3.5) and (2.3)–(2.5).

It remains to prove the uniqueness. First, for $n = 1, 2, \dots$ we define $a_n^+(s) = 1$ if $ns \geq 1$, and $a_n^+(s) = n^2 s^2 e^{1-n^2 s^2}$ if $0 \leq ns < 1$. Let $A_n(t) = \int_0^t a_n(s) ds$, $t \in \mathbb{R}^1$, where $a_n(s)$ is an odd extension of $a_n^+(s)$ in \mathbb{R}^1 .

Let $u_1(t), u_2(t)$ be two solutions of (1.1) which satisfy (2.2) and (2.4). Denote $u(t) = u_1(t) - u_2(t)$. Then by Proposition 1 and Lemma 4.4 in [6, chap 1] we obtain

$$(3.23) \quad \frac{d}{dt} \int_{\Omega} A_n(u(t)) dx + \gamma_0 \int_{\Omega} |\nabla u|^m a'_n(u) dx \leq I$$

for some $\gamma_0 > 0$, where

$$\begin{aligned}
 (3.24) \quad I &= \int_{\Omega} (\|u_1\|_{\alpha}^{\alpha} |u_1|^{\beta-1} u_1 - \|u_2\|_{\alpha}^{\alpha} |u_2|^{\beta-1} u_2) a_n(u) \, dx \\
 &\leq \|u_1\|_{\alpha}^{\alpha} \int_{\Omega} |u_1|^{\beta-1} u_1 - |u_2|^{\beta-1} u_2 \, dx + \|\|u_1\|_{\alpha}^{\alpha} - \|u_2\|_{\alpha}^{\alpha}\| \int_{\Omega} |u_2|^{\beta-1} u_2 \, dx \\
 &\leq Ct^{-\lambda(\alpha+\beta-1)} \|u(t)\|_1.
 \end{aligned}$$

Then combining (3.23) with (3.24), we obtain for $t \in (0, T]$

$$(3.25) \quad \frac{d}{dt} \int_{\Omega} A_n(u(t)) \, dx \leq Ct^{-\lambda(\alpha+\beta-1)} \|u(t)\|_1,$$

where $C > 0$ is independent of i and n . Integrating (3.25) on $[r, t]$ and letting $n \rightarrow \infty$, we have

$$(3.26) \quad \|u(t)\|_1 \leq \|u(r)\|_1 + \int_r^t s^{-\lambda(\alpha+\beta-1)} \|u(s)\|_1 \, ds, \quad 0 < r < t \leq T.$$

Since $u(t) \in L^q([0, T], L^q(\Omega))$ ($q > 1$) and $u(0) = 0$, we let $r \rightarrow 0^+$ and find that

$$\|u(t)\|_1 \leq \int_0^t s^{-\lambda(\alpha+\beta-1)} \|u(s)\|_1 \, ds, \quad 0 < t \leq T.$$

Since $0 < \lambda(\alpha+\beta-1) < 1$, the application of the Gronwall's Lemma brings $\|u(t)\|_1 = 0$ on $[0, T]$. Thus $u_1(t) = u_2(t)$ on $[0, T]$. This completes the proof of Theorem 1. \square

4. L^∞ ESTIMATE FOR $\nabla u(t)$

In this section we give the proof of Theorem 2. We also use an argument similar to that in [2], [7], [13], but we must treat carefully the nonlinear nonlocal term in the L^∞ estimate of $\nabla u(t)$. As above, we only consider the estimate of $\|\nabla u(t)\|_\infty$ for the smooth solution $u(t)$ of (3.1). As above, let C, C_j be generic constants independent of p and i changeable from line to line. Denote

$$|D^2 u|^2 = \sum_{i,j=1}^N u_{i,j}^2, \quad u_{i,j} = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Multiplying (3.1) by $-\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \geq 2$ and integrating on Ω by parts, we have

$$\begin{aligned}
 (4.1) \quad & \frac{1}{p} \frac{d}{dt} \|\nabla u(t)\|_p^p + \int_{\Omega} |\nabla u(t)|^{p+m-4} |D^2 u(t)|^2 dx \\
 & + C_p \int_{\Omega} |\nabla u(t)|^{p+m-6} |\nabla(|\nabla u(t)|^2)|^2 dx \\
 & - (N-1) \int_{\partial\Omega} H(x) |\nabla u(t)|^{p+m-2} dS \\
 & \leq -\|u(t)\|_{\alpha}^{\alpha} \int_{\Omega} |u(t)|^{\beta-1} u(t) \operatorname{div}(|\nabla u(t)|^{p-2}\nabla u(t)) dx \equiv B
 \end{aligned}$$

with $C_p = \frac{1}{4}(p-2)$. It follows from (2.4) that

$$\begin{aligned}
 (4.2) \quad & B = \beta \|u(t)\|_{\alpha}^{\alpha} \int_{\Omega} |u(t)|^{\beta-1} |\nabla u(t)|^p dx \\
 & \leq C \|u(t)\|_{\infty}^{\alpha+\beta-1} \|\nabla u(t)\|_p^p \leq C t^{-\lambda(\alpha+\beta-1)} \|\nabla u(t)\|_p^p.
 \end{aligned}$$

If $H(x) \leq 0$ on $\partial\Omega$ and $N > 1$, then by the argument for the elliptic eigenvalue problem (cf. [7]) there exists $\lambda_0 > 0$ such that

$$(4.3) \quad \|\nabla v\|_2^2 - (N-1) \int_{\partial\Omega} v^2 H(x) dS \geq \lambda_0 \|v\|_{1,2}^2, \quad \forall v \in W^{1,2}(\Omega).$$

From (4.1)–(4.3) we see that there exist C_1 and C_2 , independent of p , such that

$$(4.4) \quad \frac{d}{dt} \|\nabla u(t)\|_p^p + C_1 \|\nabla u(t)\|_{1,2}^{(p+m-2)/2} \|\nabla u(t)\|_p^2 \leq C_2 p t^{-\lambda(\alpha+\beta-1)} \|\nabla u(t)\|_p^p.$$

Let $p_1 = m$, $p_n = N p_{n-1} - m + 2$, $\theta_n = N(1 - p_{n-1} p_n^{-1})(N - 1 + 2/N)^{-1}$, $n = 2, 3, \dots$. Then it follows from Lemma 1 that

$$(4.5) \quad \|\nabla u(t)\|_{p_n} \leq C^{2/(p_n+m-2)} \|\nabla u(t)\|_{p_{n-1}}^{1-\theta_n} \|\nabla u(t)\|_{1,2}^{2\theta_n/(p_n+m-2)}.$$

Putting this into (4.4) ($p = p_n$), we obtain

$$\begin{aligned}
 & \frac{d}{dt} \|\nabla u(t)\|_{p_n}^{p_n} + C_1 C^{-2/\theta_n} \|\nabla u(t)\|_{p_{n-1}}^{(p_n+m-2)(1-1/\theta_n)} \|\nabla u(t)\|_{p_n}^{(p_n+m-2)/\theta_n} \\
 & \leq C_2 p_n t^{-\lambda(\alpha+\beta-1)} \|\nabla u(t)\|_{p_n}^{p_n}.
 \end{aligned}$$

Then we take $y_1 = \max\{1, C_0\}$, $z_1 = (1 + 2\lambda(\alpha + \beta))/m$, where C_0 is the constant in the estimate (2.5).

As the proof of Proposition 2, we can show that there exist a bounded sequence $\{y_n\}$ and a convergent sequence $\{z_n\}$ such that

$$(4.6) \quad \|\nabla u(t)\|_{p_n} \leq y_n t^{-z_n}, \quad 0 < t \leq T,$$

for which $z_n \rightarrow \mu = (2(1 + 2\lambda(\alpha + \beta)) + N^2)/(2m + (m - 2)N^2)$, see [11]. Then the estimate (2.6) is obtained from (4.6) as $n \rightarrow \infty$.

Now we consider the estimate (2.7). Let

$$F(t) = \frac{1}{m} \int_{\Omega} |\nabla u(t)|^m dx.$$

Multiplying (1.1) by u_t and integrating on Ω by parts, we obtain

$$\|u_t(t)\|_2^2 + F'(t) = \|u(t)\|_{\alpha}^{\alpha} \int_{\Omega} u_t(t) |u(t)|^{\beta-1} u(t) dx \leq \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \|u(t)\|_{\alpha}^{2\alpha} \|u(t)\|_{2\beta}^{2\beta}.$$

Hence,

$$(4.7) \quad \frac{1}{2} \|u(t)\|_{\alpha}^{2\alpha} \|u(t)\|_{2\beta}^{2\beta} - F'(t) \geq \frac{1}{2} \|u_t(t)\|_2^2.$$

Similarly, multiplying (1.1) by $u(t)$ gives

$$(4.8) \quad \|\nabla u(t)\|_m^m = \|u(t)\|_{\beta+1}^{\beta+1} \|u(t)\|_{\alpha}^{\alpha} - \int_{\Omega} u(t) u_t(t) dx \leq \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \|\nabla u(t)\|_m^m + C_1.$$

This shows that

$$(4.9) \quad \frac{1}{2} \|u(t)\|_{\alpha}^{2\alpha} \|u(t)\|_{2\beta}^{2\beta} - F'(t) \geq \lambda_1 F(t) - C_1$$

for some $\lambda_1 \in (0, 1)$ and $C_1 > 0$. We now estimate the first term of (4.9).

By the assumption $u_0 \in W_0^{1,m}(\Omega)$, we obtain $u_0 \in L^{m^*}(\Omega)$ by the Sobolev embedding theorem, where $m^* = mN/(N - m)$. Then, by Proposition 1, the solution satisfies $u(t) \in L^{\infty}(\mathbb{R}^+, L^{m^*}(\Omega))$. Since $\alpha + \beta < m - 1$ and $2\beta \leq m^*$, we get

$$(4.10) \quad \|u(t)\|_{\alpha}^{2\alpha} \leq C_0, \quad \|u(t)\|_{2\beta}^{2\beta} \leq C_0, \quad \forall t \geq 0,$$

where C_0 depends only on the initial data u_0 . Then we have from (4.9) that

$$(4.11) \quad F'(t) + \lambda_1 F(t) \leq C_0.$$

This implies that

$$(4.12) \quad F(t) \leq F(0)e^{-\lambda_1 t} + C_0, \quad t \geq 0,$$

or

$$(4.13) \quad \|\nabla u(t)\|_m \leq \|\nabla u_0\|_m e^{-\lambda_1 t} + C_0, \quad t \geq 0.$$

This is the estimate (2.7). We have completed the proof of Theorem 2. □

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