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TRANSFERRAL OF ENTAILMENT IN DUALITY THEORY II: STRONG DUALISABILITY

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Abstract. Results saying how to transfer the entailment in certain minimal and maximal ways and how to transfer strong dualisability between two different finite generators of a quasi-variety of algebras are presented. A new proof for a well-known result in the theory of natural dualities which says that strong dualisability of a quasi-variety is independent of the generating algebra is derived.

Keywords: natural duality, (strong) dualisability, entailment

MSC 2010: 08A35, 08C15

1. Introduction

For the basic facts about natural duality theory we recommend [1] and [9]. A very brief introduction into the theory and the summary of the basic concepts can also be found in our companion paper [7].

Throughout this paper we assume that $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are finite algebras of the same type such that $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$. Consequently, for the quasi-varieties $\mathscr{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ and $\mathscr{M} := \mathbb{ISP}(\underline{\mathbf{M}})$ we have $\mathscr{D} \subseteq \mathscr{M}$. We assume that there are homomorphisms $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, for some k, and $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ such that β and $\alpha \circ \beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{D}}^k$ are one-to-one.

In Section 3 we introduce minimal and maximal extensions of algebraic relations and (partial) operations from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$. While in [7] we concentrated on the transferral of the entailment and dualisability "up" from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$ via the 'minimal extensions', here we equally focus on the transferral via the 'maximal extensions'. Then in

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Section 4, where we assume that $\mathscr{D} = \mathscr{M}$ and that α is one-to-one, we present a natural way of transferring a strong duality "up" from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$ by generalizing the Strong Duality Transfer Theorem of Davey and Haviar [3]. In Section 5 we give a new proof for a well-known result due to Hyndman [8] saying that strong dualisability of a quasi-variety is independent of the generating algebra. We use the additional assumption that the strongly dualising structures are finite, which, however, is the case in all known strong dualities to date. Finally, as an application of the results of Sections 4 and 5, in Section 6 we show how to transfer a strong duality "down" from $\underline{\mathbf{M}}$ to $\underline{\mathbf{D}}$.

2. Preliminaries

Let $\mathscr{A} := \mathbb{ISP}(\underline{\mathbf{M}})$ be a quasi-variety generated by a finite algebra $\underline{\mathbf{M}}$. Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ be a discrete topological structure, in which the set M is endowed with the discrete topology $\mathcal{T}$ and with families G, H and R of finitary operations, partial operations and relations, respectively. The structure $\underset{\sim}{\mathbf{M}}$ is said to be *algebraic over* $\underline{\mathbf{M}}$ if the relations in R and the graphs of operations and partial operations in $G \cup H$ are subalgebras of appropriate powers of $\underline{\mathbf{M}}$; we shall always be assuming that $\underset{\sim}{\mathbf{M}} = (M; G, H, R, \mathcal{T})$ is algebraic over $\underline{\mathbf{M}}$. The structure $\underset{\sim}{\mathbf{M}}$ is called the *alter ego* of $\underline{\mathbf{M}}$ and finding $\underset{\sim}{\mathbf{M}}$ with desirable properties to a given $\underline{\mathbf{M}}$ is one of the central tasks in natural duality theory.

Given a closed substructure $\mathbf{X}$ of a non-zero power of $\underset{\sim}{\mathbf{M}}$, we define a morphism from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$ to be a map $\alpha\colon X \to \underset{\sim}{\mathbf{M}}$ that preserves the relations in R and the graphs of operations and partial operations in $G \cup H$ and is continuous. Note that when $\mathbf{X}$ is finite, the morphisms are all the structure preserving maps from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$.

Now we say that $G \cup H \cup R$, or simply $\underset{\sim}{\mathbf{M}}$, *entails the relation* s (*the partial operation* h) *on the structure* $\mathbf{X}$ if each morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ preserves s (the graph(h)). The dual category to $\mathscr{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ is the class $\mathscr{X} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$ of all isomorphic copies of closed substructures of non-zero powers of $\underset{\sim}{\mathbf{M}}$. The morphisms of the category $\mathscr{X}$ are the continuous structure preserving maps.

We use a pair of contravariant functors $\mathrm{D}\colon \mathscr{A} \to \mathscr{X}$ and $\mathrm{E}\colon \mathscr{X} \to \mathscr{A}$ defined as follows. For every $\mathbf{A} \in \mathscr{A}$, $\mathrm{D}(\mathbf{A})$ is the homset $\mathscr{A}(\mathbf{A}, \underline{\mathbf{M}})$ regarded as a closed substructure of $\underset{\sim}{\mathbf{M}}^{A}$; the structure $\mathrm{D}(\mathbf{A}) \in \mathscr{X}$ is called the *dual of* $\mathbf{A}$. Similarly, for every $\mathbf{X} \in \mathscr{X}$, its *dual* $\mathrm{E}(\mathbf{X}) \in \mathscr{A}$ is defined to be the homset $\mathscr{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ regarded as a subalgebra of $\underline{\mathbf{M}}^{X}$.

Let $\mathbf{A} \in \mathscr{A}$, $\mathbf{X} \in \mathscr{X}$ and let $e_A\colon \mathbf{A} \to ED(\mathbf{A})$ and $\varepsilon_X\colon \mathbf{X} \to DE(\mathbf{X})$ be maps given by evaluation:

$$\begin{aligned} e_A(a)(h) = h(a) \quad &\text{for every } \ a \in A \ \text{ and } \ h \in D(\mathbf{A}), \\ \varepsilon_X(y)(\varphi) = \varphi(y) \quad &\text{for every } \ y \in X \ \text{ and } \ \varphi \in E(\mathbf{X}). \end{aligned}$$

It is said that the structure $\underset{\sim}{\mathbf{M}}$ (or just $G \cup H \cup R$) *yields a (natural) duality on* $\mathscr{A}$ *based on* $\underline{\mathbf{M}}$, or $\underset{\sim}{\mathbf{M}}$ *dualises* $\underline{\mathbf{M}}$, if for every $\mathbf{A} \in \mathscr{A}$, the embedding e_A is an isomorphism.

Let $\mathscr{A}_{\text{fin}}$ be the category of all finite algebras in $\mathscr{A}$. If for every $\mathbf{A}$ in $\mathscr{A}_{\text{fin}}$, $e_{\mathbf{A}}$ is an isomorphism, then $\underset{\sim}{\mathbf{M}}$ *dualises* $\underline{\mathbf{M}}$ *at the finite level.*

The following Duality Compactness Theorem is due independently to Willard [12] and Zádori [13].

Theorem 2.1 ([1], Theorem 2.2.11)**.** *If $\underset{\sim}{\mathbf{M}}$ is of finite type and yields a duality on $\mathscr{A}_{\text{fin}}$, then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathscr{A}$.*

If $e_{\mathbf{A}}$ and $\varepsilon_{\mathbf{X}}$ are isomorphisms for all $\mathbf{A} \in \mathscr{A}$ and $\mathbf{X} \in \mathscr{X}$, then $\underset{\sim}{\mathbf{M}}$ is said to yield a *full duality on* $\mathscr{A}$ or one says that $\underset{\sim}{\mathbf{M}}$ *fully dualises* $\underline{\mathbf{M}}$. In this case, the categories $\mathscr{A}$ and $\mathscr{X}$ are dually equivalent.

If $\underset{\sim}{\mathbf{M}}$ *fully dualises* $\underline{\mathbf{M}}$ and moreover, $\underset{\sim}{\mathbf{M}}$ is injective in $\mathscr{X}$, then we say that $\underset{\sim}{\mathbf{M}}$ *strongly dualises* $\underline{\mathbf{M}}$. However, the usual definition of a strong duality is the following one (see Chapter 3 of [1] for a proof that this is equivalent to the former definition given above): $\underset{\sim}{\mathbf{M}}$ *strongly dualises* $\underline{\mathbf{M}}$ if $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$ and, for every non-empty set S, every closed substructure $\underset{\sim}{\mathbf{M}}^S$ of $\underset{\sim}{\mathbf{M}}$ is term-closed. We recall that given a non-empty set S, a closed substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^S$ is *term-closed* if whenever $y \in M^S \backslash X$ there exist S-ary term functions t_1 and t_2 of the algebra $\underline{\mathbf{M}}$ such that $t_1 \restriction X = t_2 \restriction X$ but $t_1(y) \neq t_2(y)$.

As far as duality is concerned we are interested only in the entailment on the structures $\mathbf{X}$ of the form $\mathrm{D}(\mathbf{A})$ for $\mathbf{A} \in \mathscr{A}$. Thus we say (cf. [1], p. 55) that $\underset{\sim}{\mathbf{M}}$ *entails* s if it entails s on every structure of the form $\mathrm{D}(\mathbf{A})$, for $\mathbf{A} \in \mathscr{A}$. Let $\mathscr{B}_M$ be the class of all finitary algebraic relations on $\underline{\mathbf{M}}$ and let $\Omega \subseteq \mathscr{B}_M$. If a set R of relations in Ω is such that R entails s for every $s \in \Omega$, then we say that R is *entailment-dense in* Ω.

Let $\mathbf{s}$ be the subalgebra of $\underline{\mathbf{M}}^n$ corresponding to the n-ary algebraic relation s on $\underline{\mathbf{M}}$, $n \geqslant 1$. For each $i \in \{1, \ldots, n\}$, let $\varrho_i^s := \pi_i \restriction s\colon\ s \to M$, where $\pi_i\colon\ M^n \to M$ is the natural projection.

The following result is fundamental to the study of entailment (for its proof see [4], 2.3 or [1], 8.1.3, 9.1.2; cf. also [5], 1.4).

Theorem 2.2 (The Test Algebra Theorem)**.** *Let $\underline{\mathbf{M}}$ be a finite algebra and let $\underset{\sim}{\mathbf{M}}$ be its alter ego. Let s be an n-ary algebraic relation on $\underline{\mathbf{M}}$ for some $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) *$\underset{\sim}{\mathbf{M}}$ entails s;*
(2) *$\underset{\sim}{\mathbf{M}}$ entails s on $\mathrm{D}(\mathbf{s})$;*
(3) *every morphism $\alpha\colon \mathrm{D}(\mathbf{s}) \to \underset{\sim}{\mathbf{M}}$ satisfies $(\alpha(\varrho_1^s), \dots, \alpha(\varrho_n^s)) \in \mathbf{s}$;*
(4) *$s = \{(\alpha(\varrho_1^s), \dots, \alpha(\varrho_n^s))\colon\ \alpha \in \mathrm{ED}(\mathbf{s})\}$;*
(5) *s may be obtained from $G \cup H \cup R$ via a* primitive positive construct, *that is, for some primitive positive formula $\Phi(x_1, \dots, x_n)$ in the language of $\underset{\sim}{\mathbf{M}}$, $s = \{(c_1, \dots, c_n) \in M^n\colon\ \underset{\sim}{\mathbf{M}} \models \Phi(c_1, \dots, c_n)\}$ and $\mathrm{D}(\mathbf{s})$ satisfies $\Phi(\varrho_1^s, \dots \varrho_n^s)$.*

As an immediate consequence we obtain that if $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$, then $G \cup H \cup R$ is entailment-dense in $\mathscr{B}_M$.

The Brute Force Duality Theorem (cf. [1], Theorem 2.3.1) says that the set $\mathscr{B}_M$ of all finitary algebraic relations on $\underline{\mathbf{M}}$ (the brute force) yields a duality on $\mathscr{A}_{\text{fin}}$. The following result is called a *Density Lemma* and it explains the central role of entailment in duality theory. It also holds at the finite level.

Lemma 2.3 ([1], Lemma 8.2.2)**.** *Let $\underline{\mathbf{M}}$ be a finite algebra. Assume that $\Omega \subseteq \mathscr{B}_M$ yields a duality on $\mathscr{A}$ and let $R \subseteq \Omega$. Then the following conditions are equivalent:*

(1) *R yields a duality on $\mathscr{A}$;*
(2) *R is entailment-dense in Ω;*
(3) *R entails s for each $s \in \Omega \setminus R$;*
(4) *R entails s on $D(\mathbf{s})$ for each $s \in \Omega \setminus R$.*

Hence if a finite set $R \subseteq \mathscr{B}_M$ entails the brute force $\mathscr{B}_M$, then R yields a duality on $\mathscr{A}_{\text{fin}}$, and by the Duality Compactness Theorem 2.1, R yields a duality on $\mathscr{A}$. Thus we conclude:

Lemma 2.4. *Let $\underline{\mathbf{M}}$ be a finite algebra. A finite set $R \subseteq \mathscr{B}_M$ yields a duality on $\mathscr{A}$ if and only if R entails $\mathscr{B}_M$.*

It follows from this lemma that studying the dualisability of $\underline{\mathbf{M}}$ via a finite set of relations $R \subseteq \mathscr{B}_M$ is equivalent to studying when R entails $\mathscr{B}_M$.

Let us recall that in our setting $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are finite algebras of the same type such that $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$ and we assume the existence of a homomorphism $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, for some k, and a one-to-one homomorphism $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$. Let now, for every $i \in \{1, \dots, k\}$, $\omega_i := \beta \circ \varrho_i^{\alpha(M)} \circ \alpha$. We see that ω_i is an endomorphism of $\underline{\mathbf{M}}$. Let us further denote $\Gamma_{\beta\alpha} := \{\omega_1, \dots, \omega_k\}$.

We define a homomorphism $\omega := \omega_1 \sqcap \ldots \sqcap \omega_k \colon \underline{\mathbf{M}} \to \underline{\mathbf{M}}^k$ by $\omega(a) := (\omega_1(a), \ldots, \omega_k(a))$ for all $a \in M$. When the maps $\omega_1, \ldots, \omega_k$ separate the points of M, ω is an embedding. In such a case we define $\sigma \colon \omega(M) \to M$ to be the inverse of ω regarded as a k-ary algebraic partial operation on $\underline{\mathbf{M}}$. It follows that for all $a \in M$, $\sigma(\omega_1(a), \ldots, \omega_k(a)) = a$. The partial operation σ on $\underline{\mathbf{M}}$ was introduced in Davey and Haviar [3] for the particular case when $\underline{\mathbf{D}}$ is a subalgebra of $\underline{\mathbf{M}}$ and $\beta \colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ is the inclusion map. Our setting where $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$ is a more general one. Hence we shall call the partial operation σ on $\underline{\mathbf{M}}$ in this more general setting the *generalized schizophrenic operation* corresponding to $\omega_1, \ldots, \omega_k$.

3. Transferring entailment up via minimal and maximal extensions

Throughout this section we again assume that $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are finite algebras of the same type such that $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$, so for $\mathscr{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ and $\mathscr{M} := \mathbb{ISP}(\underline{\mathbf{M}})$ we have $\mathscr{D} \subseteq \mathscr{M}$. We also assume that there exist homomorphisms $\alpha \colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, for some k, and $\beta \colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ such that β and $\alpha \circ \beta \colon \underline{\mathbf{D}} \to \underline{\mathbf{D}}^k$ are one-to-one. In our companion paper [7] we concentrated on the transferral of the entailment and dualisability "up" from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$ via certain 'minimal extensions'. In this section we equally focus on the transferral "up" via 'maximal extensions'.

Let $(\alpha \circ \beta)^{-1}$ be the inverse of the isomorphism $\underline{\mathbf{D}} \to (\alpha \circ \beta)(\underline{\mathbf{D}}) \leqslant \underline{\mathbf{D}}^k$ given by $\alpha \circ \beta$ and, for any $X \subseteq D^{mk}$, let us use $\alpha^{-1}(X)$ to denote the subset of $\underline{\mathbf{M}}^m$ of all elements $(c_1, \ldots, c_m)$ of M^m that satisfy

$$((\pi_1 \circ \alpha)(c_1), \ldots, (\pi_k \circ \alpha)(c_1), \ldots, (\pi_1 \circ \alpha)(c_m), \ldots, (\pi_k \circ \alpha)(c_m)) \in X.$$

For every n-ary algebraic relation r on $\underline{\mathbf{D}}$, we consider the *minimal extension* $\min r$ of r to $\underline{\mathbf{M}}$ to be the algebraic relation

$$\min r := \{(\beta(a_1), \ldots, \beta(a_n)) \colon (a_1, \ldots, a_n) \in r\}$$

already defined in [10] or in [7], where it was denoted by r_β. We now introduce the *maximal extension* $\max r$ of r to $\underline{\mathbf{M}}$ to be the algebraic relation $\alpha^{-1}((\alpha \circ \beta)(r))$. We note that taking $\alpha^{-1}((\alpha \circ \beta)(r))$ instead of $\alpha^{-1}(r)$ guarantees, for example, that $\min r \subseteq \max r$. Every n-ary algebraic relation r_M on $\underline{\mathbf{M}}$ satisfying $\min r \leqslant r_M \leqslant \max r$ will be called an *extension* of r to $\underline{\mathbf{M}}$.

For every n-ary (partial) homomorphism $h \colon \operatorname{dom} h \subseteq \underline{\mathbf{D}}^n \to \underline{\mathbf{D}}$, we use $\min h$ to denote its *minimal extension* to a partial homomorphism of $\underline{\mathbf{M}}$ that assigns to each $(a_1, \ldots, a_n) \in \min(\operatorname{dom} h)$ the element $\beta(h(\beta^{-1}(a_1), \ldots, \beta^{-1}(a_n)))$ of M; we note that this has already been considered in [10] and in [7] where it was denoted by h_β. We introduce the *maximal extension* $\max h$ to be the partial homomorphism of $\underline{\mathbf{M}}$ with the domain $\max(\operatorname{dom} h)$ that assigns to each element $(a_1,\dots,a_n)$ of $\max(\operatorname{dom} h)$ the element $\beta(h(((\alpha\circ\beta)^{-1}(\alpha(a_1)),\dots,(\alpha\circ\beta)^{-1}(\alpha(a_n)))))$ of M.

For every set S of algebraic relations or (partial) operations on $\underline{\mathbf{D}}$, we define $\min S := \{\min s\colon\ s\in S\}$ and $\max S := \{\max s\colon\ s\in S\}$.

Definition 3.1. Let r be an n-ary algebraic relation on $\underline{\mathbf{D}}$ and r_M one of its extensions to $\underline{\mathbf{M}}$. For every map $u\colon \mathscr{M}(r_M,\underline{\mathbf{M}})\to M$ that preserves the unary algebraic relation $\beta(D)$ on $\underline{\mathbf{M}}$, we define the *associated map* $u_D\colon \mathscr{D}(r,\underline{\mathbf{D}})\to D$ by

$$u_D(x)=\beta^{-1}(u(\beta\circ x\circ(\alpha\circ\beta)^{-1}\circ\alpha_{|r_M})).$$

Lemma 3.2. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras for which there exist homomorphisms $\alpha\colon \underline{\mathbf{M}}\to\underline{\mathbf{D}}^k$, for some k, and $\beta\colon \underline{\mathbf{D}}\to\underline{\mathbf{M}}$ such that β and $\alpha\circ\beta$ are one-to-one. Let r and s be two finitary algebraic relations on $\underline{\mathbf{D}}$ and let r_M and s_M be their extensions to $\underline{\mathbf{M}}$.*

If $u\colon \mathscr{M}(r_M,\underline{\mathbf{M}})\to M$ is a map that preserves s_M and $\beta(D)$, then the map $u_D\colon \mathscr{D}(r,\underline{\mathbf{D}})\to D$ preserves s.

Proof. Let $\mathbf{s}\leqslant\underline{\mathbf{D}}^n$ and let $(x_1,\dots,x_n)\in s$ for $x_i\in\mathscr{D}(r,\underline{\mathbf{D}})$, with $i\in\{1,\dots,n\}$. Then $(x_1,\dots,x_n)\in s$ yields the following sequence of assertions of which each implies the subsequent one:

$$\begin{aligned}
&(\beta\circ x_1\circ(\alpha\circ\beta)^{-1}\circ\alpha_{|r_M},\dots,\beta\circ x_n\circ(\alpha\circ\beta)^{-1}\circ\alpha_{|r_M})\in\beta(s)\subseteq s_M,\\
&(u(\beta\circ x_1\circ(\alpha\circ\beta)^{-1}\circ\alpha_{|r_M}),\dots,u(\beta\circ x_n\circ(\alpha\circ\beta)^{-1}\circ\alpha_{|r_M}))\in s_M\cap\beta(D)^n,\\
&(\beta(u_D(x_1)),\dots,\beta(u_D(x_n)))\in s_M\subseteq\alpha^{-1}((\alpha\circ\beta)(s)),\\
&((\alpha\circ\beta)(u_D(x_1)),\dots,(\alpha\circ\beta)(u_D(x_1)))\in(\alpha\circ\beta)(s),\\
&(u_D(x_1),\dots,u_D(x_n))\in s,
\end{aligned}$$

the last step using the fact that $\alpha\circ\beta$ is one-to-one. $\square$

Proposition 3.3. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras for which there exist homomorphisms $\alpha\colon \underline{\mathbf{M}}\to\underline{\mathbf{D}}^k$, for some k, and $\beta\colon \underline{\mathbf{D}}\to\underline{\mathbf{M}}$ such that β and $\alpha\circ\beta$ are one-to-one. Let S be a subset of $\mathscr{B}_D$ and r an n-ary algebraic relation on $\underline{\mathbf{D}}$. For each $s\in S$ take s_M to be an arbitrary (but fixed) extension of s to $\underline{\mathbf{M}}$ and define $S_M=\{s_M\colon\ s\in S\}$. If S entails r on $\underline{\mathbf{D}}$, then $S_M\cup\{\beta(D)\}$ entails $\min r$ on $\underline{\mathbf{M}}$.*

Proof. Let $u\colon \mathscr{M}(\min r,\underline{\mathbf{M}})\to M$ be a map that preserves S_M and $\beta(D)$. By the previous lemma, the map u_D preserves every relation in S. We now take

$(x_1, \ldots, x_n) \in \min r$ where every x_i is in $\mathscr{M}(\min r, \underline{\mathbf{M}})$. Since $(\alpha(x_1), \ldots, \alpha(x_n)) \in (\alpha \circ \beta)(r)$, we have

$$((\alpha \circ \beta)^{-1} \circ \alpha(x_1), \ldots, (\alpha \circ \beta)^{-1} \circ \alpha(x_n)) \in r.$$

Hence

$$((\alpha \circ \beta)^{-1} \circ \alpha \circ x_1 \circ \beta_{|r}, \ldots, (\alpha \circ \beta)^{-1} \circ \alpha \circ x_n \circ \beta_{|r}) \in r$$

with

$$(\alpha \circ \beta)^{-1} \circ \alpha \circ x_i \circ \beta_{|r} \in \mathscr{D}(\mathbf{r}, \underline{\mathbf{D}}).$$

By hypothesis S entails r and consequently u_D preserves r. Hence

$$(u_D((\alpha \circ \beta)^{-1} \circ \alpha \circ x_1 \circ \beta_{|r}), \ldots, u_D((\alpha \circ \beta)^{-1} \circ \alpha \circ x_n \circ \beta_{|r})) \in r.$$

We note that

$$\begin{aligned} &u_D((\alpha \circ \beta)^{-1} \circ \alpha \circ x_i \circ \beta_{|r}) \\ &\qquad = \beta^{-1}(u(\beta \circ (\alpha \circ \beta)^{-1} \circ \alpha \circ x_i \circ \beta_{|r} \circ (\alpha \circ \beta)^{-1} \circ \alpha_{|\min r})) = \beta^{-1}(u(x_i)) \end{aligned}$$

because

$$\beta_{|r} \circ (\alpha \circ \beta)^{-1} \circ \alpha_{|\min r} = \mathrm{id}_{|\min r}$$

and

$$\beta \circ (\alpha \circ \beta)^{-1} \circ \alpha \circ x_i = x_i$$

once $x_i \in \beta(D)$ for every $i \in \{1, \ldots, n\}$. But then we get

$$(\beta^{-1}(u(x_1)), \ldots, \beta^{-1}(u(x_n))) \in r,$$

which implies $(u(x_1), \ldots, u(x_n)) \in \min r$. □

Corollary 3.4. *Let* $\underline{\mathbf{D}}$ *and* $\underline{\mathbf{M}}$ *be finite algebras for which there exist homomorphisms* $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$*, for some* k*, and* $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ *such that* β *and* $\alpha \circ \beta$ *are one-to-one. Let* r *be a finitary algebraic relation on* $\underline{\mathbf{D}}$*. Then* $\{\max r, \beta(D)\}$ *entails* $\min r$ *on* $\underline{\mathbf{M}}$*.*

As an immediate consequence we obtain the following result which is a restricted version of Theorem 3.3 of our companion paper [7].

Corollary 3.5. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras such that $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$. Let $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ be an embedding.*

If $G \cup H \cup R$ entails an algebraic relation $s \subseteq D^n$ (the graph of a partial algebraic operation h) on $\underline{\mathbf{D}}$, then $\min G \cup \min H \cup \min R \cup \{\beta(D)\}$ entails the relation $\min s$ (the graph of the partial operation $\min h$) on $\underline{\mathbf{M}}$.

One can feel that a result in the same spirit on the transferral of dualisability up from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$ via the minimal extensions might be true as well. This is indeed the case and Corollary 3.5 has served us the springboard for proving this in our companion paper [7]. The proof required some essential work to be done (see [7], pp. 48–51) and here we just present the final result. We note that this was one of the main results of [10]; comparing it with the present result, here we use the additional assumption that the dualising structure $G \cup H \cup R$ is finite which, however, is the case in all known dualities to date.

Corollary 3.6 ([7], Corollary 3.9; see also [10], Proposition 2.1)**.** *Let $\underline{\mathbf{M}}$ be a finite algebra in $\mathscr{D} = \mathbb{ISP}(\underline{\mathbf{D}})$ and assume that $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$. If $\underline{\mathbf{D}}$ is dualisable via a finite set of relations, then $\underline{\mathbf{M}}$ is dualisable.*

More specifically, assume that $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ yields a duality on $\mathscr{D}$ such that $G \cup H \cup R$ is a finite set and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$ and $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ are one-to-one homomorphisms. Then the structure

$$\underset{\sim}{\mathbf{M}} = \langle M; \Gamma_{\beta\alpha}, \min G, \min H, \min R, \mathcal{T}\rangle$$

yields a duality on $\mathbb{ISP}(\underline{\mathbf{M}}) = \mathscr{D}$.

In the particular setting of Corollary 3.6, that is, when α is one-to one and so $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are finite generators of the same quasivariety, the maximal and minimal extensions of any relation on $\underline{\mathbf{D}}$ coincide. So it is formally correct to replace $\min G, \min H$ and $\min R$ in Corollary 3.6 by $\max G, \max H$ and $\max R$, respectively and to say that "dualisability can be transferred from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$ via the maximal extensions".

In the more general setting of having $\mathscr{D}$ as a subquasivariety of $\mathscr{M}$ with $\underline{\mathbf{D}}$ being a subalgebra of $\underline{\mathbf{M}}$, the maximal and minimal extensions do not coincide but are somehow identifiable in the entailment sense. As we will see, in the presence of a specific set of endomorphisms of $\underline{\mathbf{M}}$ plus a unary relation on $\underline{\mathbf{M}}$, the minimal extension of a relation entails and is entailed by the maximal one.

Proposition 3.7. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras for which there exist homomorphisms $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, for some k, and $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ such that β and $\alpha \circ \beta$ are one-to-one. Let S be a subset of $\mathscr{B}_D$ and r be an n-ary algebraic relation on $\underline{\mathbf{D}}$. For each $s \in S$ take s_M to be an arbitrary (but fixed) extension of s to $\underline{\mathbf{M}}$ and define $S_M = \{s_M\colon s \in S\}$. If S entails r on $\underline{\mathbf{D}}$, then $S_M \cup \{\beta(D)\} \cup \Gamma_{\beta\alpha}$ entails* $\max r$ *on $\underline{\mathbf{M}}$.*

Proof. Suppose that r is an n-ary relation. By Test Algebra Theorem 2.2, we only need to prove that $(u(\varrho_1^{\max r}), \dots, u(\varrho_n^{\max r})) \in \max r$ whenever $u\colon \mathscr{M}(\max r, \underline{\mathbf{M}}) \to M$ is a map preserving $S_M \cup \{\beta(D)\} \cup \Gamma_{\beta\alpha}$. Let $u\colon \mathscr{M}(\max r, \underline{\mathbf{M}}) \to M$ be a map that preserves S_M and $\beta(D)$ and $\{\beta \circ \pi_j \circ \alpha\colon j \in \{1, \dots, k\}\}$. By Lemma 3.2, the map u_D preserves every relation in S and so it preserves r. Since we have that (we are shortening $\varrho_i^{\max r}$ to ϱ_i now) $((\alpha\circ\beta)^{-1} \circ \alpha \circ \varrho_1, \dots, (\alpha\circ\beta)^{-1} \circ \alpha \circ \varrho_n) \in r$ we also have that $(u_D((\alpha\circ\beta)^{-1} \circ \alpha \circ \varrho_1 \circ \beta_{|r}), \dots, u_D((\alpha\circ\beta)^{-1} \circ \alpha \circ \varrho_n \circ \beta_{|r})) \in r$. We note that for every $i \in \{1, \dots, n\}$,

$$\beta \circ (\alpha \circ \beta)^{-1} \circ \alpha \circ \varrho_i \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha_{|\max r} = \varrho_i \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha_{|\max r}$$

once $\varrho_i \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha_{|\max r} \in \beta(D)$. Then

$$(u(\varrho_1 \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha_{|\max r}), \dots, u(\varrho_n \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha_{|\max r})) \in \beta(r),$$

which implies that

$$(\alpha(u(\varrho_1 \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha_{|\max r})), \dots, \alpha(u(\varrho_n \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha_{|\max r}))) \in \alpha \circ \beta(r).$$

For every $i \in \{1, \dots, n\}$, let $v_i := \varrho_i \circ \beta \circ (\alpha \circ \beta)^{-1} \circ \alpha_{|\max r}$. We have

$$\begin{aligned}
\alpha(u(v_i)) &= (\beta^{-1} \circ \beta \circ \pi_1 \circ \alpha(u(v_i)), \dots, \beta^{-1} \circ \beta \circ \pi_k \circ \alpha(u(v_i))) \\
&= (\beta^{-1}(u(\beta \circ \pi_1 \circ \alpha \circ v_i)), \dots, \beta^{-1}(u(\beta \circ \pi_k \circ \alpha \circ v_i))) \\
&= (\beta^{-1}(u(\beta \circ \pi_1 \circ \alpha \circ \varrho_i)), \dots, \beta^{-1}(u(\beta \circ \pi_k \circ \alpha \circ \varrho_i))) \\
&= (\beta^{-1} \circ \beta \circ \pi_1 \circ \alpha(u(\varrho_i)), \dots, \beta^{-1} \circ \beta \circ \pi_k \circ \alpha(u(\varrho_i))) \\
&= \alpha(u(\varrho_i))
\end{aligned}$$

and so $(u(\varrho_1), \dots, u(\varrho_n)) \in \max r$ as required. □

Corollary 3.8. *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite algebras for which there exist homomorphisms $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, for some k, and $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ such that β and $\alpha \circ \beta$ are one-to-one. Let r be an n-ary algebraic relation on $\underline{\mathbf{D}}$. Then* $\min r \cup \{\beta(D)\} \cup \Gamma_{\beta\alpha}$ *entails* $\max r$ *on $\underline{\mathbf{M}}$.*

If we also apply Corollary 3.4 then we get the following result.

Corollary 3.9. *Let* $\underline{\mathbf{D}}$ *and* $\underline{\mathbf{M}}$ *be finite algebras for which there exist homomorphisms* $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$*, for some* k*, and* $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ *such that* β *and* $\alpha\circ\beta$ *are one-to-one. Let* S *be any set of* n*-ary algebraic relations on* $\underline{\mathbf{D}}$*. Then* $\min S\cup\{\beta(D)\}\cup\Gamma_{\beta\alpha}$ *entails and is entailed by* $\max S\cup\{\beta(D)\}\cup\Gamma_{\beta\alpha}$ *on* $\underline{\mathbf{M}}$*.*

Remark 3.10. We note that from the last corollary one can conclude that every globally minimal failset (for the concept see [11] or [1], Chapter 8) not containing partial endomorphisms and $\beta(D)$, contains $\min r$ if and only if it contains $\max r$. If $(\alpha\circ\beta)(D) = \alpha(M)$ then every globally minimal failset without partial endomorphisms contains $\min r$ if and only if it contains $\max r$ (see Lemma 3.4 of [7]).

4. Transferring strong dualities up

In this section we assume that $\mathscr{D} = \mathscr{M}$ and that α is one-to-one. Recall that this means that the maximal and the minimal extensions coincide. We will present what seems to be the only natural way of transferring the strong duality "up" from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$ where $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$.

We start with two lemmas considered as part of the folklore.

Lemma 4.1. *Let* X *and* Y *be topological spaces. For every continuous map* $\varphi\colon X \to Y$*, the map* $\varphi\colon X^S \to Y^S$ *defined by*

$$\varphi(\langle x_s\rangle_{s\in S}) = \langle\varphi(x_s)\rangle_{s\in S}$$

is continuous for the product topologies.

Lemma 4.2 ([1], Lemma B.1, p. 337)**.** *If* $\gamma\colon X \to Y$ *is a continuous map from a compact topological space* X *to a Hausdorff topological space* Y *and* $U \subseteq X$ *is closed, then* $\gamma(U) \subseteq Y$ *is closed, too.*

For any map $\gamma\colon A \to B$ we denote by γ^S the map from A to B^S defined by $\gamma^S(a) = \langle b_s\rangle_{s\in S}$ where $b_s = \gamma(a)$ for every $s \in S$.

The first main result of this section shows how the strong dualisability can be transferred via a one-to-one homomorphism.

Theorem 4.3. *Let $\underline{\mathbf{D}}$ be a finite algebra and $\underline{\mathbf{M}} = \beta(\underline{\mathbf{D}})$, where β is a one-to-one homomorphism. If $\underset{\sim}{\mathbf{D}}$ yields a strong duality via a finite set $G \cup H \cup R$ of (partial) operations and relations on $\underline{\mathbf{D}}$, then $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\underline{\mathbf{M}}$ via the set $\Gamma_{\beta\alpha} \cup \min G \cup \min H \cup \min R \cup \{\beta(D)\}$.*

Proof. By Corollary 3.6, $\underset{\sim}{\mathbf{M}}$ yields a duality on $\underline{\mathbf{M}}$. It remains to prove that every closed substructure of $\underset{\sim}{\mathbf{M}}^S$ is term-closed. So let X be a closed substructure of $\underset{\sim}{\mathbf{M}}^S$. We take the corresponding subset $\beta^{-1}(X) = \{\langle \beta^{-1}(x_s)\rangle_{s\in S}\colon \langle x_s\rangle_{s\in S} \in X\}$ of D^S. Since the topological space $\underset{\sim}{\mathbf{D}}^S$ is Hausdorff, by Lemma 4.1 and Lemma 4.2 we have that $\beta^{-1}(X)$ is a closed subset of $\underset{\sim}{\mathbf{D}}^S$. For every $h \in G \cup H$, let $(\overline{x}_1, \ldots, \overline{x}_n) \in \operatorname{dom} h$ with $\overline{x}_1, \ldots, \overline{x}_n \in \beta^{-1}(X)$. Then there exist $\overline{y}_1, \ldots, \overline{y}_n \in X$ such that for all $s \in S$ we have $\overline{x}_i(s) = \beta^{-1}(\overline{y}_i(s))$ for $i \in \{1, \ldots, n\}$. But then $(\overline{y}_1, \ldots, \overline{y}_n) \in \min(\operatorname{dom} h)$. We are assuming that X is a substructure of $\underset{\sim}{\mathbf{M}}^S$ whence $\min h(\overline{y}_1, \ldots, \overline{y}_n) \in X$, and so $\beta(h(\overline{x}_1, \ldots, \overline{x}_n)) \in X$ and finally $h(\overline{x}_1, \ldots, \overline{x}_n) \in \beta^{-1}(X)$. Thus $\beta^{-1}(X)$ is a closed substructure of $\underset{\sim}{\mathbf{D}}^S$. Now we take an element $\overline{z}$ of $M^S \setminus X$. Then $\beta^{-1}(\overline{z}) \in D^S \setminus \beta^{-1}(X)$. As $\underset{\sim}{\mathbf{D}}$ is strongly dualisable, $\beta^{-1}(X)$ is term-closed, whence there exist term functions t_1, t_2 such that t_1, t_2 agree on $\beta^{-1}(X)$ but $t_1(\beta^{-1}(\overline{z})) \neq t_2(\beta^{-1}(\overline{z}))$. Since h is a one-to-one homomorphism, we get that t_1, t_2 agree on X but $t_1(\overline{z}) \neq t_2(\overline{z})$. Hence X is term-closed and the proof is complete. $\square$

We continue with the Strong Duality Transfer Theorem of Davey and Haviar [3]. We note that in the case $\underline{\mathbf{D}}$ is a subalgebra of $\underline{\mathbf{M}}$, the embedding $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ is just the inclusion map, the set of the endomorphisms $\Gamma_{\beta\alpha}$ of $\underline{\mathbf{M}}$ has the specific form $\Omega := \{\varrho_i^{\alpha(M)} \circ \alpha\colon 1 \leqslant i \leqslant k\}$ and the structure $\min G \cup \min H \cup \min R$ is denoted by $G_D \cup H_D \cup R_D$.

Theorem 4.4 ([3], Strong Duality Transfer Theorem)**.** *Let $\underline{\mathbf{D}}$ be a finite algebra, let $\underline{\mathbf{M}}$ be a finite algebra in $\mathscr{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ which has $\underline{\mathbf{D}}$ as a subalgebra. If the structure $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ yields a strong duality on $\mathscr{D}$ based on $\underline{\mathbf{D}}$, then the structure*

$$\underset{\sim}{\mathbf{M}} := \langle M; \Omega, G_D, H_D \cup \{\sigma\}, R_D \cup \{D\}, \mathcal{T}\rangle$$

yields a strong duality on $\mathscr{D}$ based on $\underline{\mathbf{M}}$.

Our aim here is to present this result in our more general setting where $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$. Let $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ be a one-to-one homomorphism and assume that $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ yields a strong duality on $\mathscr{D}$ via a finite set $G \cup H \cup R$. We derive the transferral of the strong duality from $\underline{\mathbf{D}}$ into $\underline{\mathbf{M}}$ in two steps. In the first step we apply Theorem 4.3 to transfer the strong duality from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}' := \beta(\underline{\mathbf{D}})$. We obtain that $\underline{\mathbf{M}}'$ is strongly dualisable via $\Gamma_{\beta\alpha} \cup \min G \cup \min H \cup \min R \cup \{\beta(D)\}$. In the second step, having now $\underline{\mathbf{M}}'$ as a subalgebra of $\underline{\mathbf{M}}$, we apply the Strong Duality Transfer Theorem 4.4 above. We obtain a new result:

Theorem 4.5 (Generalized Strong Duality Transfer Theorem)**.** *Let $\underline{\mathbf{M}}$ be a finite algebra in $\mathscr{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ and assume that $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$. Let $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, for some k, be one-to-one homomorphisms. If $\underline{\mathbf{D}}$ is strongly dualisable then also $\underline{\mathbf{M}}$ is strongly dualisable.*

More specifically, if $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ yields a strong duality on $\mathscr{D}$ via a finite set $G \cup H \cup R$, then the structure

$$\underset{\sim}{\mathbf{M}} = \langle M; \Gamma_{\beta\alpha}, \min G, \min H \cup \{\sigma\}, \min R \cup \{\beta(D)\}, \mathcal{T}\rangle$$

yields a strong duality on $\mathscr{M} = \mathbb{ISP}(\underline{\mathbf{M}})$, where σ is the generalized schizophrenic operation and $\Gamma_{\beta\alpha} := \{\beta \circ \varrho_i^{\alpha(M)} \circ \alpha\colon\ i \in \{1, \dots, k\}\}$.

Remark 4.6. When transferring the strong duality from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$, where $\underline{\mathbf{D}}$ is (isomorphic to) a subalgebra of $\underline{\mathbf{M}}$, it turns out that we cannot avoid the presence of the (generalized) schizophrenic operation σ in the structure $\underset{\sim}{\mathbf{M}}$. In [5], Theorem 4.3., using the concept of *structural entailment* introduced there, the authors proved that provided $\underline{\mathbf{D}}$ is actually a subalgebra of $\underline{\mathbf{M}}$, if (i) $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ is an alter ego that strongly dualises $\underline{\mathbf{D}}$, (ii) $\underset{\sim}{\mathbf{M}} := \langle M; G', H', R', \mathcal{T}\rangle$ is an alter ego of $\underline{\mathbf{M}}$ that strongly dualises $\underline{\mathbf{M}}$, and (iii) for every non-empty set T and every closed substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^T$, the set $X \cap D^T$ is a substructure of $\underset{\sim}{\mathbf{D}}^T$ (this is shown in [3], p. 219), then σ has an extension in the enriched partial clone of $\underset{\sim}{\mathbf{M}}$, which means that σ is essentially present in $\underset{\sim}{\mathbf{M}}$.

5. Strong dualisability of a quasi-variety is independent of the generating algebra

The result in the title of this section was proved by J. Hyndman in [8]. In her 'categorical approach' she uses Lemma 3.8 of [2] and the necessary and sufficient conditions in it needed for a dualising set to yield a strong duality. After presenting her proof, she mentions the construction used in [3] (which is Theorem 4.4 presented in the previous section) and the use of "one further partial operation", meaning the schizophrenic operation. But the efficient use of the schizophrenic operation had obviously not been employed in Hyndman's approach. It will be employed in our approach in this section.

Here we present a new and, we hope, a little bit more transparent proof of Hyndman's result in three steps and under the assumption that the strongly dualising structures are finite. Having a strongly dualisable finite generator (let us call it the first generator) of a quasi-variety embedded in some finite power of another finite generator (let us call it the second generator), in the first step one can apply our Theorem 4.3 to transfer the strong dualisability of the first generator via this embedding into a subalgebra of the power of the second generator. Then by the Strong Duality Transfer Theorem 4.4 and the use of the schizophrenic operation, the strong dualisability can be transferred from this subalgebra to the actual power of the second generator. Finally, in the third step, one needs to show how the strong dualisability can be transferred from that power to the second generator itself. We only need to present the third step of our method.

In the next two results the relations of $\underline{\mathbf{D}}^k$ are regarded as relations on $\underline{\mathbf{D}}$ and each (partial) operation $h\colon \operatorname{dom} h \subseteq \underline{\mathbf{D}}^{kn} \to \underline{\mathbf{D}}^k$ of $\underline{\mathbf{D}}^k$ in $G \cup H$ is then replaced by the homomorphisms $\pi_j \circ h\colon \operatorname{dom} h \to \underline{\mathbf{D}}$, where $\pi_j\colon \underline{\mathbf{D}}^k \to \underline{\mathbf{D}}$ is the j-th projection, $j \in \{1, \ldots, k\}$. We will call such structure *the associated structure on* $\underline{\mathbf{D}}$.

Proposition 5.1. *If a set of finitary algebraic relations $G \cup H \cup R$ on $\underline{\mathbf{D}}^k$ entails $\mathscr{B}_{D^k}$, then the associated structure on $\underline{\mathbf{D}}$ entails $\mathscr{B}_D$.*

Proof. Let $r \in \mathscr{B}_D$ be an n-ary relation and let $u\colon \mathscr{D}(\underline{\mathbf{r}}, \underline{\mathbf{D}}) \to D$ be a map that preserves the structure on $\underline{\mathbf{D}}$ associated with $G \cup H \cup R$. We want to prove that $(u(\varrho_1^r), \ldots, u(\varrho_n^r)) \in r$. We define $v\colon \mathscr{D}(\mathrm{id}^k(\underline{\mathbf{r}}), \underline{\mathbf{D}}^k) \to D^k$ by

$$v(x) := (u(\pi_1 \circ x \circ \mathrm{id}^k), \ldots, u(\pi_k \circ x \circ \mathrm{id}^k)).$$

We note that here id^k denotes the homomorphism from $\underline{\mathbf{r}} \leqslant \underline{\mathbf{D}}^n$ to $(\underline{\mathbf{D}}^k)^n$ given by $\mathrm{id}^k(a_1, \ldots, a_n) = ((a_1, \ldots, a_1), \ldots, (a_n, \ldots, a_n))$. We claim that v preserves $G \cup H \cup R$. We take a relation s in R (a graph of a partial operation $h \in G \cup H$) of arity m and we assume that $(x_1, \ldots, x_m) \in s$ with $x_i = (\pi_1 \circ x_i, \ldots, \pi_k \circ x_i) \in \mathscr{D}(\mathrm{id}^k(\underline{\mathbf{r}}), \underline{\mathbf{D}}^k)$. Then $((\pi_1 \circ x_1 \circ \mathrm{id}^k, \ldots, \pi_k \circ x_1 \circ \mathrm{id}^k), \ldots, (\pi_1 \circ x_m \circ \mathrm{id}^k \ldots, \pi_k \circ x_m \circ \mathrm{id}^k)) \in s$ with $\pi_j \circ x_i \circ \mathrm{id}^k \in \mathscr{D}(\underline{\mathbf{r}}, \underline{\mathbf{D}})$. Since u preserves s we have that

$$((u(\pi_1 \circ x_1 \circ \mathrm{id}^k), \ldots, u(\pi_k \circ x_1 \circ \mathrm{id}^k)), \ldots, (u(\pi_1 \circ x_m \circ \mathrm{id}^k) \ldots, u(\pi_k \circ x_m \circ \mathrm{id}^k)))$$

belongs to s, that is, $(v(x_1), \ldots, v(x_m)) \in s$, thus v preserves s. Since v preserves $G \cup H \cup R$, we conclude that v preserves $\mathrm{id}^k(r)$. We denote by p_i the i-th projection from $(\underline{\mathbf{D}}^k)^n$ onto $\underline{\mathbf{D}}^k$. For every $i \in \{1, \ldots, n\}$ and every $j \in \{1, \ldots, k\}$, the composite $p_i \circ \mathrm{id}^k \restriction_r$ is a homomorphism from $\underline{\mathbf{r}}$ into $\underline{\mathbf{D}}^k$ that maps every element $(a_1, \ldots, a_n)$

to $(a_i, \dots, a_i)$. As $(p_1 \restriction_{\mathrm{id}^k(r)}, \dots, p_n \restriction_{\mathrm{id}^k(r)}) \in \mathrm{id}^k(r)$ with $p_i \restriction_{\mathrm{id}^k(r)} \in \mathscr{D}(\mathrm{id}^k(\underline{\mathbf{r}}), \underline{\mathbf{D}}^k)$, we have $(v(p_1 \restriction_{\mathrm{id}^k(r)}), \dots, v(p_n \restriction_{\mathrm{id}^k(r)})) \in \mathrm{id}^k(r)$, that is,

$$((u(\pi_1 \circ p_1 \restriction_{\mathrm{id}^k(r)} \circ \mathrm{id}^k), \dots, u(\pi_k \circ p_1 \restriction_{\mathrm{id}^k(r)} \circ \mathrm{id}^k)), \dots$$
$$\dots, (u(\pi_1 \circ p_n \restriction_{\mathrm{id}^k(r)} \circ \mathrm{id}^k), \dots, u(\pi_k \circ p_n \restriction_{\mathrm{id}^k(r)} \circ \mathrm{id}^k)))$$

belongs to $id^k(r)$. We note that

$$(u(\pi_1 \circ p_i \restriction_{\mathrm{id}^k(r)} \circ \mathrm{id}^k), \dots, u(\pi_k \circ p_i \restriction_{\mathrm{id}^k(r)} \circ \mathrm{id}^k)) = (u(\varrho_i^r), \dots, u(\varrho_i^r)),$$

and so $((u(\varrho_1^r), \dots, u(\varrho_1^r)), \dots, (u(\varrho_n^r), \dots, u(\varrho_n^r))) \in \mathrm{id}^k(r)$. We finally obtain $(u(\varrho_1^r), \dots, u(\varrho_n^r)) \in r$ as required. □

Proposition 5.2. *If $\underset{\sim}{\mathbf{D}}^k$ strongly dualises $\underline{\mathbf{D}}^k$ via a finite set of (partial) operations and relations $G \cup H \cup R$ on $\underline{\mathbf{D}}^k$, then the associated structure $\underset{\sim}{\mathbf{D}}$ on $\underline{\mathbf{D}}$ strongly dualises $\underline{\mathbf{D}}$.*

Proof. By applying Proposition 5.1 we get that the structure $\underset{\sim}{\mathbf{D}}$ on $\underline{\mathbf{D}}$ associated with $G \cup H \cup R$ entails $\mathscr{B}_D$. By Lemma 2.4, $\underset{\sim}{\mathbf{D}}$ dualises $\underline{\mathbf{D}}$. We claim that this duality is strong. Let X be a closed substructure of $\underset{\sim}{\mathbf{D}}^S$. Consider the closed subset X^k of $(\underset{\sim}{\mathbf{D}}^k)^S$. We claim that X^k is a substructure of $(\underset{\sim}{\mathbf{D}}^k)^S$. Let $h \in G \cup H$ and assume $\operatorname{dom} h \subseteq (\underline{\mathbf{D}}^k)^n$ and $(\overline{x}_1, \dots, \overline{x}_n) \in \operatorname{dom} h$ with $\overline{x}_i \in X^k$. Then for all $j \in \{1, \dots, k\}$, $(\pi_j(\overline{x}_1), \dots, \pi_j(\overline{x}_n)) \in \operatorname{dom} h \subseteq X^n$. Since X is a closed substructure of $\underset{\sim}{\mathbf{D}}^S$, we have that $h(\pi_j(\overline{x}_1), \dots, \pi_j(\overline{x}_n)) \in X$. But then $(h(\pi_1(\overline{x}_1), \dots, \pi_1(\overline{x}_n)), \dots, h(\pi_k(\overline{x}_1), \dots, \pi_k(\overline{x}_n))) \in X^k$, or equivalently, $h(\overline{x}_1, \dots, \overline{x}_n) \in X^k$. So X^k is a substructure of $(\underset{\sim}{\mathbf{D}}^k)^S$.

Now we take an element y of $D^S \setminus X$. Then $\mathrm{id}^k(y) \in (D^k)^S \setminus X^k$. As X^k is a closed substructure of $(\underset{\sim}{\mathbf{D}}^k)^S$ and, by assumption, $\underset{\sim}{\mathbf{D}}^k$ strongly dualises $\underline{\mathbf{D}}^k$, there exist term functions t_1, t_2 such that t_1, t_2 agree on X^k but $t_1(\mathrm{id}^k(y)) \neq t_2(\mathrm{id}^k(y))$. Since $\mathrm{id}^k(X) \subseteq X^k$ we have that t_1, t_2 agree on $\mathrm{id}^k(X)$. But id^k is a one-to-one homomorphism, which implies that t_1, t_2 agree on X but $t_1(y) \neq t_2(y)$. Hence X is term-closed. □

Under the additional assumption that the strongly dualising structures are finite, which, as mentioned previously, is the case in all known strong dualities to date, we now give a new proof for the result due to Hyndman [8], which is interpreted so that strong dualisability of a quasi-variety is independent of the generating algebra.

Theorem 5.3. *Let $\underline{\mathbf{D}}$ be a finite algebra and let $\underset{\sim}{\mathrm{D}}$ strongly dualise $\underline{\mathbf{D}}$ via a finite dualising structure. Then every finite generator $\underline{\mathbf{M}}$ of $\mathbb{ISP}(\underline{\mathbf{D}})$ is strongly dualisable via a finite structure, too.*

P r o o f. Let $\underline{\mathbf{M}}$ be a finite generator of $\mathbb{ISP}(\underline{\mathbf{D}})$. Note that $\underline{\mathbf{D}}$ is embedded in some finite power $\underline{\mathbf{M}}^k$ of $\underline{\mathbf{M}}$ via a one-to-one homomorphism β. If $\underset{\sim}{\mathrm{D}}$ strongly dualises $\underline{\mathbf{D}}$ via a finite structure, then by Theorem 4.3, the copy $\beta(\underline{\mathbf{D}})$ of $\underline{\mathbf{D}}$ in $\underline{\mathbf{M}}^k$ is strongly dualisable via a finite structure, too. By the Strong Duality Transfer Theorem 4.4, $\underline{\mathbf{M}}^k$ is then also strongly dualisable via a finite structure. Finally, by Proposition 5.2, $\underline{\mathbf{M}}$ is strongly dualisable via a finite structure. $\square$

6. Transferring strong dualities down

In Section 4 we have presented a natural way of transferring a strong duality "up" from a finite algebra $\underline{\mathbf{D}}$ to a finite algebra $\underline{\mathbf{M}}$, where $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$ and $\underline{\mathbf{M}} \in \mathbb{ISP}(\underline{\mathbf{D}})$. In this section we apply the results of Sections 4 and 5 to show how the transferral of strong dualisability can also be made "down" from the algebra $\underline{\mathbf{M}}$ to the algebra $\underline{\mathbf{D}}$. We use a structure introduced in [10] (and also used in [7]). We start with recalling the basic setting and the shape of the transferred structure.

Let the finite algebras $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be such that $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$ and $\underline{\mathbf{M}} \in \mathbb{ISP}(\underline{\mathbf{D}})$, and therefore $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ generate the same quasi-variety. Let $\beta\colon \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, where $k \geqslant 1$, be one-to-one homomorphisms.

For every n-ary partial operation $h\colon \operatorname{dom} h \subseteq M^n \to M$, let $\operatorname{dom} h_\alpha \subseteq D^{nk}$ be the set of all elements $(\varrho \circ \alpha)((a_1, \ldots, a_n))$ of the form

$$((\varrho_1 \circ \alpha)(a_1), \ldots, (\varrho_k \circ \alpha)(a_1), \ldots, (\varrho_1 \circ \alpha)(a_n), \ldots, (\varrho_k \circ \alpha)(a_n)),$$

where $(a_1, \ldots, a_n) \in \operatorname{dom} h$. Let $h_\alpha\colon \operatorname{dom} h_\alpha \to D^k$ be the map defined so that for all $(a_1, \ldots, a_n) \in \operatorname{dom} h$,

$$h_\alpha((\varrho \circ \alpha)((a_1, \ldots, a_n))) = \alpha(h(a_1, \ldots, a_n)).$$

We observe that h_α is a homomorphism from the subalgebra $\operatorname{dom} h_\alpha$ of $\underline{\mathbf{D}}^{nk}$ into $\underline{\mathbf{D}}^k$ whenever h is algebraic over $\underline{\mathbf{M}}$ (cf. [10], p. 201).

For every m-ary relation r on M, we define the mk-ary relation r_α on D as

$$r_\alpha := \{(\varrho \circ \alpha)((a_1, \ldots, a_m))\colon (a_1, \ldots, a_m) \in r\}.$$

We again observe that r_α is algebraic over $\underline{\mathbf{D}}$ whenever r is algebraic over $\underline{\mathbf{M}}$ (cf. [10], p. 202). Let us denote a structure on $\underline{\mathbf{D}}$ defined from a given structure $G \cup H \cup R$ on $\underline{\mathbf{M}}$ as follows:

(a) $G_\alpha := \{\varrho_1 \circ g_\alpha, \dots, \varrho_k \circ g_\alpha \colon\ g \in G\}$;
(b) $H_\alpha := \{\varrho_1 \circ h_\alpha, \dots, \varrho_k \circ h_\alpha \colon\ h \in H\}$;
(c) $R_\alpha := \{r_\alpha \colon\ r \in R\}$.

This structure was used in the following result from [7].

Lemma 6.1 ([7], Corollary 4.7). *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite generators of the same quasi-variety and let $\beta\colon\ \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon\ \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, for some $k \geqslant 1$, be one-to-one homomorphisms. If $\underline{\mathbf{M}}$ is dualisable via a finite set $\{\omega_1, \dots, \omega_k\} \cup G \cup H \cup R$, then $\underline{\mathbf{D}}$ is dualisable via the set $\mathrm{End}(\underline{\mathbf{D}}) \cup G_\alpha \cup H_\alpha \cup R_\alpha \cup \{\alpha(M)\}$.*

Now using the results of Sections 4 and 5, we can present the final result of this paper.

Theorem 6.2. *Let $\underline{\mathbf{D}}$ be a finite algebra, let $\underline{\mathbf{M}}$ be a finite algebra in $\mathscr{D} := \mathbb{ISP}(\underline{\mathbf{D}})$ and assume that $\underline{\mathbf{D}} \in \mathbb{IS}(\underline{\mathbf{M}})$, so that $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ generate the same quasi-variety. Let $\beta\colon\ \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ and $\alpha\colon\ \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$, for some k, be one-to-one homomorphisms. Let us assume that*

$$\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$$

yields a strong duality on $\mathbb{ISP}(\underline{\mathbf{M}})$ and the set $G \cup H \cup R$ is finite.

Then the structure $\underset{\sim}{\mathbf{D}}_\alpha$ defined to be

$$\langle D; \mathrm{End}(\underline{\mathbf{D}}), G_\alpha, H_\alpha \cup \{(\alpha\circ\beta)^{-1} \circ \varrho_j^{((\alpha\circ\beta)(D))^k} \colon\ 1 \leqslant j \leqslant k\}, R_\alpha \cup \{\alpha(M)\}, \mathcal{T}\rangle$$

yields a strong duality on $\mathbb{ISP}(\underline{\mathbf{D}})$.

P r o o f. First we introduce the following notation: given homomorphisms $\gamma_i\colon \mathbf{A}_i \to \mathbf{B}_i$ with $i \in \{1, \dots, k\}$, we take $\gamma_1 \times \dots \times \gamma_k$ to denote the homomorphism from $\mathbf{A}_1 \times \dots \times \mathbf{A}_k$ to $\mathbf{B}_1 \times \dots \times \mathbf{B}_k$ that maps each $(a_1, \dots, a_k)$ to $(\gamma_1(a_1), \dots, \gamma_k(a_k))$. We apply the Generalized Strong Duality Transfer Theorem 4.5 to transfer the strong duality from $\underline{\mathbf{M}}$ via the embedding $\beta' := \alpha\colon\ \underline{\mathbf{M}} \to \underline{\mathbf{D}}^k$. We obtain that $\Gamma_{\beta'\alpha'} \cup \min G \cup \min H \cup \{\sigma'\} \cup \min R \cup \{\beta'(M)\}$ strongly dualises $\underline{\mathbf{D}}^k$, where

$$\begin{aligned}
\alpha' &= \beta \times \dots \times \beta\colon\ \underline{\mathbf{D}}^k \to \underline{\mathbf{M}}^k,\\
\Gamma_{\beta'\alpha'} &= \{\beta' \circ \pi_j \circ \alpha'\colon\ 1 \leqslant j \leqslant k\} = \{\alpha \circ \beta \circ \pi_j\colon\ \underline{\mathbf{D}}^k \to \underline{\mathbf{D}}^k\colon\ 1 \leqslant j \leqslant k\},\\
\sigma' &= (\alpha\circ\beta)^{-1} \times \dots \times (\alpha\circ\beta)^{-1}\colon\ ((\alpha\circ\beta)(D))^k \to \underline{\mathbf{D}}^k,
\end{aligned}$$

and $\min G = \{g_\alpha\colon\ g \in G\}$, $\min H = \{h_\alpha\colon\ h \in H\}$, $\min R = \{r_\alpha\colon\ r \in R\}$.

Now the result follows from Proposition 5.2. $\square$

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