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HALL EXPONENTS OF MATRICES, TOURNAMENTS AND THEIR LINE DIGRAPHS

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Abstract. Let A be a square $(0,1)$-matrix. Then A is a Hall matrix provided it has a nonzero permanent. The Hall exponent of A is the smallest positive integer k, if such exists, such that A^k is a Hall matrix. The Hall exponent has received considerable attention, and we both review and expand on some of its properties. Viewing A as the adjacency matrix of a digraph, we prove several properties of the Hall exponents of line digraphs with some emphasis on line digraphs of tournament (matrices).

Keywords: Hall matrix, Hall exponent, irreducible, primitive, tournament (matrix), line digraph

MSC 2010: 15A15, 15B34, 05C20

1. Introduction

A nonnegative matrix $A = [a_{ij}]$ of order n has been called a *Hall matrix* [11] provided there is a permutation $i_1, i_2, \ldots, i_n$ of $\{1, 2, \ldots, n\}$ such that

$$a_{1i_1} a_{2i_2} \ldots a_{ni_n} \neq 0;$$

equivalently, the permanent of A is nonzero. With A we can associate a *bipartite graph* $G(A)$ and a *digraph* $D(A)$ as follows: $G(A)$ has $2n$ vertices corresponding to the n rows and n columns of A with an edge between a row vertex i and a column vertex j if and only if $a_{ij} \neq 0$ $(1 \leqslant i, j \leqslant n)$. $D(A)$ has n vertices $1, 2, \ldots, n$, corresponding simultaneously to the rows and columns of A with an edge from a vertex i to a vertex j if and only if $a_{ij} \neq 0$ $(1 \leqslant i, j \leqslant n)$. The matrix A is the adjacency matrix of this digraph.

We have that A is a Hall matrix if and only if $G(A)$ has a perfect matching. Also, A is a Hall matrix if and only if the vertices of $D(A)$ can be partitioned into sets such that each is the set of vertices of a directed cycle of $D(A)$ (this corresponds to the factorization of a permutation into permutation cycles). The well-known Frobenius-König theorem [5] is equivalent to the fact that A is a Hall matrix if and only if A does not have a p by q zero submatrix for some p and q with $p+q \geqslant n+1$. We assume that the reader is familiar with the relationship between matrices and digraphs or bipartite graphs, and refer the reader to [5] for undefined terms and basic results. Thus A is a Hall matrix if and only if given any nonempty set U of vertices of $D(A)$, the set W of terminal vertices of the edges with their initial vertex in U satisfies $|W| \geqslant |U|$. Note that W and U need not be disjoint.

In investigating Hall matrices, there is no loss of generality in assuming, as we do, that our matrices are $(0,1)$-matrices with Boolean arithmetic used so that $1+1=1$. We let $\mathscr{B}_n$ denote the set of all $(0,1)$-matrices of order n. Via the adjacency matrix, there is a one-to-one correspondence between matrices in $\mathscr{B}_n$ and digraphs with vertices labeled as $1,2,\ldots,n$. As a result, we identify $\mathscr{B}_n$ also with the set of digraphs with vertices labeled $1,2,\ldots,n$ and freely interchange a matrix and a digraph. For $X \subseteq \{1,2,\ldots,n\}$ and $t \geqslant 0$, let $R_t(X)$ be the set of all vertices that can be reached by a walk of length t starting from a vertex in X. Then by the Frobenius-König theorem, $h(A)$ (or $h(D(A))$) equals the smallest t such that

$$|R_t(X)| \geqslant |X| \quad (X \subseteq \{1,2,\ldots,n\}).$$

Let $\mathscr{H}_n$ denote the set of all Hall matrices of order n (or their digraphs), and let $\widetilde{\mathscr{H}_n}$ denote the *Hall class* of all $(0,1)$-matrices of order n (or their digraphs) some positive power of which is a Hall matrix:

$$\widetilde{\mathscr{H}_n} = \{A \in \mathscr{B}_n\colon\ A^k \in H_n \text{ for some } k \geqslant 1\}.$$

In the digraph interpretation, $A^k \in \mathscr{H}_n$ means that there is a permutation σ of $\{1,2,\ldots,n\}$ such that there is a walk of length k from a vertex i to a vertex $\sigma(i)$ for $i=1,2,\ldots,n$. If $A \in \widetilde{H}_n$, then the *Hall exponent* $h(A)$ of A is the smallest positive integer k such that $A^k \in \mathscr{H}_n$. Clearly, $\mathscr{H}_n \subseteq \widetilde{\mathscr{H}_n}$ and indeed for $A \in \widetilde{\mathscr{H}_n}$, $h(A)=1$ if and only if $A \in \mathscr{H}_n$. In view of our identification of matrices in $\mathscr{B}_n$ with digraphs, we can also refer to the Hall exponent and, later, the strict Hall exponent of a digraph.

Recall that a matrix A is irreducible if and only is its digraph $D(A)$ is strongly connected. Thus A is irreducible if and only if for each set U of vertices with $\emptyset \subset U \subset \{1,2,\ldots,n\}$ there is at least one edge with its initial vertex in U and its terminal vertex in its complement $\overline{U}$, that is, $R_1(U) \cap \overline{U} \neq \emptyset$. In investigating the Hall class $\widetilde{\mathscr{H}_n}$, it is natural to consider only irreducible matrices (strongly connected digraphs), since according to Theorem 1 of [4], $A \in \widetilde{H}_n$ if and only if A does not have an irreducible component equal to the zero matrix of order 1. For the remainder of this paper, we assume that $n \geqslant 2$ and confine our attention to the class $\mathscr{I}_n$ of irreducible matrices in $\mathscr{B}_n$, that is, the irreducible matrices in $\widetilde{\mathscr{H}_n}$.

In [4] the *strict Hall class* $\mathscr{H}_n^*$ is defined to be the class of all $(0,1)$-matrices of order n such that $A^p \in \mathscr{H}_n$ for all sufficiently large positive integers p. The *strict Hall exponent* $h^*(A)$ of a matrix $A \in \mathscr{H}_n^*$ is the smallest integer k such that $A^p \in \mathscr{H}_n$ for all $p \geqslant k$. Clearly, $h(A) \leqslant h^*(A)$. Let d be the *index of imprimitivity* of an irreducible matrix A, that is, the greatest common divisor of the lengths of the cycles in the digraph $D(A)$. Then [5] there is a partition of $\{1,2,\ldots,n\}$ into sets $U_1, U_2, \ldots, U_d$ such that all edges of D go from a vertex in U_i to a vertex in U_{i+1} (where $U_{d+1} = U_1$). The sets $U_1, U_2, \ldots, U_d$ are called the *sets of imprimitivity* of A and of $D(A)$, and their cardinalities are the *imprimitivity parameters* of A and of $D(A)$. In [4] it is shown that for an irreducible matrix $A \in \mathscr{B}_n$, $A \in \mathscr{H}_n^*$ if and only if its imprimitivity parameters are all equal. Clearly, $\mathscr{H}_n \subseteq \mathscr{H}_n^* \subseteq \widetilde{\mathscr{H}_n}$. Finally, recall that a matrix $A \in \mathscr{B}_n$ is *primitive* if $d = 1$ and this is equivalent to some positive power of A having no 0s; the smallest such power $e(A)$ is the (*primitive*) *exponent* of A. It follows easily that if A is primitive, then A^k has no 0s for all $k \geqslant e(A)$. Let $\mathscr{P}_n$ denote the class of all primitive matrices of order n. Then $\mathscr{P}_n \subseteq \mathscr{H}_n^*$, and if $A \in \mathscr{P}_n$, then $h^*(A) \leqslant e(A)$.

We now briefly summarize some known facts about the exponents $h(A)$ and $h^*(A)$ for irreducible matrices $A \in \mathscr{I}_n$ that are relevant to our investigations. First we note that both the Hall and strict Hall exponents are invariant under simultaneous row and column permutations.

Fact (i) [4] If A has s 1s on its main diagonal ($D(A)$ has s loops) where $1 \leqslant s \leqslant n-1$, then $h^*(A) \leqslant n - s$. (If $s = n$, that is, A has all 1s on its main diagonal, then $A \in \mathscr{H}_n^*$ and $h^*(A) = 1$.) The maximum (strict) Hall exponent of an irreducible matrix in $\mathscr{B}_n$ with nonzero trace is $n - 1$.

Fact (ii) [4] If the digraph $D(A)$ of A has a cycle of length r and there are s vertices in the union of all cycles of length r in $D(A)$, then

$$h(A) \leqslant \begin{cases} r & \text{if } s = n, \\ r(n-s) & \text{if } 1 \leqslant s \leqslant n-1. \end{cases}$$

(In fact, there is a small error in the proof in [4]. We fix this error in the next section, and indeed prove a somewhat better inequality.)

Fact (iii) [4] $h(A) \leqslant \left\lfloor \frac{1}{4}(n^2 - 1) \right\rfloor$ if $n \geqslant 3$. It was *conjectured* that in fact $h(A) \leqslant 2n - 8$ for $n \geqslant 8$, which was the largest Hall exponent of a matrix constructed in [4].

Fact (iv) [8] If $A \in \mathscr{P}_n$ has at least one pair of symmetrically opposite 1s ($D(A)$ has a cycle of length 2), then $h^*(A) \leqslant 2n - 3$. More generally, if there are s rows that contain a 1 whose symmetrically opposite 1 is also in A, then $h^*(A) \leqslant 2n - s - 1$.

Fact (v) [8] The maximum strict Hall exponent of a symmetric matrix in $\mathscr{H}_n^*$ is $n-2$ if n is even and $n - 1$ if n is odd.

Fact (vi) [9] The maximum Hall exponent of a matrix in $\mathscr{P}_n$ with at least one pair of symmetrically opposite 1s is $2n - 8$ for $n \geqslant 8$, thus verifying the conjecture in (iii) for this restricted class of matrices. In [7] the conjecture is also verified for primitive matrices having a set of cycles with at least one common vertex the lengths of which have the greatest common divisor equal to 1.

Fact (vii) [9] The maximum Hall exponent of a symmetric matrix in $\mathscr{P}_n$ is 2 if $n \geqslant 2$.

There are other more specialized results in the papers [4], [8], [9] that we do not quote here.

We now briefly summarize the contents of this paper. First, we improve the bound in Fact (ii) for the Hall exponent We also find some bounds for the Hall exponent of a matrix with additional assumptions on the matrix; in particular, we obtain the bound $2n - 8$ for another class of matrices. Our main emphasis is on the Hall exponent of tournament digraphs (matrices) and their line digraphs, and we obtain several bounds on their Hall exponents.

2. Hall exponents of matrices

We begin with an improvement (and corrected proof) of Corollary 3 in [4] (Fact (ii)).

Theorem 2.1. *Let A be a matrix in $\mathscr{I}_n$ with sets of imprimitivity $U_1, U_2, \ldots, U_d$ and imprimitivity parameters $n_1 = |U_1|$, $n_2 = |U_2|$, ..., $n_d = |U_d|$. Assume that $D(A)$ has a cycle of length r and that for $i = 1, 2, \ldots, d$, there are u_i vertices of U_i that belong to cycles of length r. Then*

$$\begin{aligned} h(A) &\leqslant r \max\{1, \max\{n_1 - u_1, n_2 - u_2, \ldots, n_d - u_d\}\} \\ &\leqslant r \max\{1, n - s\}, \end{aligned}$$

where $s = u_1 + u_2 + \ldots + u_d$ is the number of vertices of $D(A)$ belonging to cycles of length r.

Proof. Note that if all vertices of $D(A)$ belong to a cycle of length r, then A^r has all 1s on its main diagonal.

There exists a permutation matrix such that PAP^t has the form

$$\begin{bmatrix} O_{n_1} & A_1 & O & \dots & O \\ O & O_{n_2} & A_2 & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & A_{d-1} \\ A_d & O & O & \dots & O_{n_d} \end{bmatrix}$$

where the sets $U_1, U_2, \dots, U_d$ correspond to the rows of $A_1, A_2, \dots, A_d$ and the columns of $A_d, A_1, \dots, A_{d-1}$, respectively (see [5]). Each cycle of the digraph $D(A)$ is a multiple of d and contains the same number of vertices from each U_i. Hence $d|r$ and each cycle of length r contains r/d vertices of each U_i. Without loss of generality we may assume that A itself has the form given in (2.1). Then

$$A^d = \operatorname{diag}(B_1, B_2, \dots, B_d)$$

where $B_1, B_2, \dots, B_d$ are primitive matrices (see [5]), and

$$A^r = (A^d)^{r/d} = \operatorname{diag}(C_1, C_2, \dots, C_d),$$

where $C_i = B_i^{r/d}$ for $i = 1, 2, \dots, r$.[1] The matrices C_i being powers of a primitive matrix are also primitive and have u_i 1s on their main diagonals. By Fact (i), $h^*(C_i) \leqslant \max\{1, n_i - u_i\}$ for $i = 1, 2, \dots, d$. Hence

$$h^*(A) \leqslant r \max\{1, \max\{n_1 - u_1, n_2 - u_2, \dots, n_d - u_d\}\} \leqslant r \max\{1, n - s\}.$$

□

Corollary 2.2. *If in Theorem* 2.1 *the imprimitivity parameters are all equal and* $D(A)$ *contains a cycle of length* d, *then* $A \in \mathscr{H}_n^*$ *and*

$$h(A) \leqslant h^*(A) \leqslant \max\{1, n - d\}.$$

P r o o f. Equality of the imprimitivity parameters implies that $A \in \mathscr{H}_n^*$. If $d = n$, then $h(A) = h^*(A) = 1$. Suppose that $d < n$. Then

$$A^d = \operatorname{diag}(B_1, B_2, \dots, B_d),$$

where each B_i is a primitive matrix of order n/d and has at least one 1 on its main diagonal. Hence

$$h(A) \leqslant h^*(A) \leqslant d\left(\frac{n}{d} - 1\right) = n - d.$$

□

[1] In [4], it is asserted that A^r is irreducible, which is clearly false.

By Fact (vi), if A is a primitive matrix (i.e. $d = 1$) and $D(A)$ has a cycle of length 2, then $h(A) \leqslant 2n - 8$ for $n \geqslant 8$. We show that the assumption of primitivity can be removed and replaced by the assumption that the index of imprimitivity is 2.

Theorem 2.3. *Let A be a matrix in $\mathscr{I}_n$ with index of imprimitivity $d = 2$. Assume that the digraph $D(A)$ contains a cycle of length 2. Then*

$$h(A) \leqslant 2n - 8 \quad (n \geqslant 6).$$

P r o o f. Let the imprimitivity parameters of A be n_1 and n_2 where $n_1 \geqslant n_2$. Suppose that $n_2 = 1$. Then the only length of cycles in $D(A)$ is 2, and $D(A)$ is uniquely determined with $h(A) = 2$ ($D(A)$ consists of $n - 1$ cycles of length 2 each containing one specified vertex). We have $2 \leqslant 2n - 8$ if and only if $n \geqslant 5$. Now suppose that $n_2 = 2$. Then the only lengths of cycles are 2 and possibly 4, and $h(A) \leqslant 4$. We have $4 \leqslant 2n - 8$ if and only if $n \geqslant 6$. Finally suppose that $n_2 \geqslant 3$. Then $n_1 \leqslant n - 3$ and by Theorem 2.1,

$$h(A) \leqslant 2\max\{1, \max\{n_1 - u_1, n_2 - u_2, \ldots, n_d - u_d\} \leqslant 2(n_1 - 1) \leqslant 2(n - 4).$$

$\square$

If in $D(A)$ there is not a cycle of length 2 but there is a cycle of length 4, then we have the following result.

Theorem 2.4. *Let A be a matrix in $\mathscr{I}_n$ with index of imprimitivity $d = 2$. Assume that the digraph $D(A)$ does not contain a cycle of length 2 but does contain a cycle of length 4. Then*

$$h(A) \leqslant 4n - 10.$$

If the imprimitivity indices of A are equal, then

$$h(A) \leqslant 2n - 6.$$

P r o o f. The matrix A can be assumed to have the form

$$A = \begin{bmatrix} O_{n_1} & A_1 \\ A_2 & O_{n_2} \end{bmatrix}.$$

We have

$$A^2 = \operatorname{diag}(A_1 A_2, A_2 A_1),$$

where A_1A_2 and A_2A_1 are primitive matrices of orders n_1 and n_2, respectively, whose digraphs contain a cycle of length 2. Hence by Fact (iv), $h^*(A_1A_2) \leqslant 2n_1 - 3$ and $h^*(A_2A_1) \leqslant 2n_2 - 3$. Thus

$$h(A) \leqslant 2\max\{2n_1 - 3, 2n_2 - 3\} \leqslant 4n - 10.$$

If $n_1 = n_2 = \frac{1}{2}n$, then we obtain

$$h(A) \leqslant 2\max\{2n_1 - 3, 2n_2 - 3\} = 2\max\{n - 3, n - 3\} = 2n - 6.$$

□

One way to construct a digraph D^* with index of imprimitivity 2 without a cycle of length 2 is to take a digraph D with index of imprimitivity 2 and subdivide each edge by replacing it with a path of length 2. Let A be an adjacency matrix of D, and let A^* be an adjacency matrix of D^* where the vertices of D are listed first. Then

$$(A^*)^2 = A \oplus B,$$

where B is an adjacency matrix of the line digraph $LD(A)$ of $D(A)$ (see the next section).

More bounds on the Hall exponent of a matrix in $\mathscr{I}_n$ can be obtained if additional information on the matrix is available.

Proposition 2.5. *Let A be a matrix in $\mathscr{H}_n^*$ with index of imprimitivity d and suppose that $D(A)$ contains a cycle of length d. Then $h(A) \leqslant n - d$.*

P r o o f. Since $A \in \mathscr{H}_n^*$, its imprimitivity parameters are all equal. Hence we have $A^d = \mathrm{diag}(B_1, B_2, \ldots, B_d)$ where the B_i are primitive matrices of order n/d. Since $D(A)$ has a cycle of length d, each B_i has at least one 1 on its diagonal. Hence by Fact (i), $h^*(B_i) \leqslant n/d - 1$, and hence $B_i^{n/d-1} \in \mathscr{H}_n$. Thus $A^{d(n/d-1)} \in \mathscr{H}_n$ and $h(A) \leqslant d(n/d - 1) = n - d$. □

We note that a similar argument shows that if in the hypotheses we assume that $D(A)$ has a cycle of length $2d$, rather than of length d, then $h(A) \leqslant 2(n - d)$.

Other information on the index of imprimitivity d can also be used to bound the Hall exponent.

Proposition 2.6. *Let A be a matrix in $\mathscr{I}_n$ with index of imprimitivity equal to d.*

(i) *If $d > \frac{1}{2}n$, then $h(A) \leqslant d$.*
(ii) *If $\frac{1}{3}n < d \leqslant \frac{1}{2}n$, then $h(A) < n$.*
(iii) *If $\frac{1}{4}n < d \leqslant \frac{1}{3}n$, then $h(A) \leqslant 2n$.*

P r o o f. Since the index of imprimitivity of A is d, the lengths of all cycles in $D(A)$ are multiples of d. First consider (i). In this case, all cycles of $D(A)$ have length equal to d and hence every vertex is on a cycle of length d. Hence A^d has all 1s on its diagonal and thus $h(A) \leqslant d$. In case (ii), each cycle of $D(A)$ has length d or $2d$, and hence $h(A) \leqslant 2d \leqslant n$. Now case (iii). Each cycle of $D(A)$ has length d, $2d$, or $3d$, and hence the least common multiple of the cycle lengths is at most $6d$. Hence A^l has all 1s on its main diagonal for some $l \leqslant 6d$ and hence $h(A) \leqslant 6d \leqslant 2n$. □

3. Hall exponents of line digraphs

In this section we initiate a discussion of the Hall properties of adjacency matrices of line digraphs.

Let $A \in \mathscr{B}_n$ have a total of m 1s, that is, $D(A)$ has m edges. The *line digraph* [6] of $D(A)$ is the digraph $LD(A)$ whose vertices are the m edges $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $D(A)$ with an edge from α_i to α_j if and only if the terminal vertex of α_i is the same as the initial vertex of α_j. Thus the edges of $LD(A)$ correspond to walks of length 2 in $D(A)$. If $n \geqslant 3$ and $D(A)$ has no isolated vertices, then $D(A)$ is strongly connected if and only if $LD(A)$ is strongly connected [1]. Moreover, A is primitive if and only if $LD(A)$ is primitive, in which case

$$e(LD(A)) = e(A) + 1. \tag{3.1}$$

This is a consequence of the observation that if there is a walk of length t from a vertex i to a vertex j, then there is a walk of length $t+1$ in $LD(A)$ from any edge with the terminal vertex i to any edge with the initial vertex j. It follows from (3.1) that

$$h(LD(A)) \leqslant h^*(LD(A)) \leqslant e(A) + 1.$$

We denote the adjacency matrix of the line digraph of a digraph by M. Thus, assuming that A is irreducible, as we have done, both A and M have a Hall exponent but there is no simple relationship between these two Hall exponents.

Examples. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the adjacency matrix of $LD(A)$ is

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $h(A) = h(M) = 2$. Now let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then the adjacency matrix of $LD(A)$ is

$$M = P \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} Q$$

for some permutation matrices P and Q. Then $h(A) = 1$, and it is easy to check that $h(M) = 3$. Finally, let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then the adjacency matrix M of $LD(A)$ is

$$M = P \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} Q$$

for some permutation matrices P and Q. Thus $h(A) = 2$ and $h(M) = 1$. In general, if $A \in \mathscr{I}_n$ with $h(A) = 1$, then $h(M)$ can be $n - 1$. One needs only to take a full cycle permutation matrix of order n (its digraph is a cycle of length n) and put in a chord cutting off one vertex. The resulting matrix A and the matrix of its line digraph satisfy $h(A) = 1$ and $h(M) = n - 1$.

The adjacency matrix M of the line digraph of the digraph $D(A)$ is a Hall matrix if and only if the edges of $D(A)$ can be partitioned into closed walks. If the indegree of each vertex of $D(A)$ equals its outdegree and $D(A)$ is strongly connected, it contains a closed walk using each edge exactly once, and hence the adjacency matrix M of $LD(A)$ is a Hall matrix.

The adjacency matrix M of $LD(A)$ has order m and has a special structure, as illustrated in the previous examples. Consider a vertex v of $D(A)$ and the set E_v of edges of which v is the terminal vertex and the set F_v of which v is the initial vertex. Let $k_v = |E_v|$ and $l_v = |F_v|$. Then this pair of sets of edges corresponds in M to a k_v by l_v submatrix J_{k_v,l_v} matrix of all 1s. These 1s account for all the 1s in M and it follows that the adjacency matrix of $LD(A)$ is given by

$$M = P\left(\oplus_v J_{k_v,l_v}\right)Q$$

for some permutation matrices P and Q. Here the direct sum is over all vertices v of $D(A)$. Whether or not M is a Hall matrix, and if it is, the value of its Hall exponent, depends on P and Q.

Proposition 3.1. *Let $A \in \mathscr{B}_n$. Then the adjacency matrix of $LD(A)$ is a Hall matrix if and only if the indegree of each vertex of $D(A)$ equals its outdegree. In particular, this holds when $D(A)$ is a regular digraph.*

Proof. Using the above notation, the adjacency matrix of $LD(A)$ is a Hall matrix if and only if $\bigoplus_v J_{k_v,l_v}$ is a Hall matrix and, by the Frobenius-König theorem, the latter is so if and only if $k_v = l_v$ for all vertices v. $\square$

It follows from Proposition 3.1 that one can tell immediately if the adjacency matrix of the line digraph of $D(A)$ is a Hall matrix by checking if a row i contains the same number of 1s as the column i for each i. It also follows that A can be a Hall matrix while the adjacency matrix M of $LD(A)$ is not a Hall matrix. One needs only to construct a Hall matrix for which the number of 1s in some row does not equal the number of 1s in the corresponding column.

We next turn to the main focus of this paper, namely tournaments.

4. Hall exponents of tournaments and their line digraphs

A *tournament of order* n is a digraph obtained from the complete graph K_n of order n by giving a direction to each of its edges. A tournament T of order n has $\binom{n}{2}$ (directed) edges. The sequence $(r_1, r_2, \ldots, r_n)$ of outdegrees of the vertices $\{1, 2, \ldots, n\}$ of T, ordered so that $r_1 \leqslant r_2 \leqslant \ldots \leqslant r_n$, is called the *score sequence* of T. By Landau's theorem (see [2] for a recent proof), a nondecreasing sequence $R = (r_1, r_2, \ldots, r_n)$ of nonnegative integers is the outdegree sequence of a tournament if and only if

$$\sum_{i=1}^{k} r_i \geqslant \binom{k}{2} \quad (k = 1, 2, \ldots, n)$$

with equality for $k = n$. The sequence of indegrees of the vertices of T is given by

$$(s_1 = n - 1 - r_1,\ s_2 = n - 1 - r_2, \ldots,\ s_n = n - 1 - r_n)$$

and satisfies $s_1 \geqslant s_2 \geqslant \ldots \geqslant s_n$. We have $r_i + s_i = n - 1$ for $i = 1, 2, \ldots, n$. In the tournament T' obtained from T by reversing the direction of each edge, the indegree sequence and outdegree sequence are interchanged; the score vector of T' equals $(s_1, s_2, \ldots, s_n)$ with the s_i in nonincreasing order. In fact, the adjacency matrix of T and that of T' are transposes of each other. A tournament of order n is *regular* provided each vertex has its outdegree and indegree equal to $\frac{1}{2}(n-1)$. Regularity requires that n is odd. If n is even, a tournament of order n is *nearly regular* provided its outdegree sequence is $(\frac{1}{2}n, \ldots, \frac{1}{2}n, \frac{1}{2}n - 1, \ldots, \frac{1}{2}n - 1)$ with an equal number of occurrences of the two values of the outdegree. Standard examples of the adjacency matrices of regular and nearly regular tournaments are

$$P + P^2 + \ldots + P^{(n-1)/2} \quad (n \text{ odd}),$$

where P is the full cycle permutation matrix with 1s in positions $(1,2)$, $(2,3)$, $\ldots$, $(n-1,n)$, $(n,1)$, and

$$\left[\begin{array}{c|c} U_{n/2} & I_{n/2} U^t_{n/2} \\ \hline U^t_{n/2} & U_{n/2} \end{array}\right] \quad (n \text{ even})$$

where $U_{n/2}$ is the upper triangular matrix with 1s everywhere above the main diagonal. For example, when $n = 7$ and 8 we get

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Another special tournament of which we shall make use is the tournament T_n^* with the score vector $(1, 1, 2, 3, \ldots, n-3, n-2, n-2)$ obtained by taking the lower triangular matrix U_n^t and deleting the 1s on the subdiagonal and replacing the 0s on the superdiagonal with 1s. For example,

$$T_6^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Recall that if $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are two n-tuples of real numbers in nonincreasing order, then $x \preceq y$ (read x is *majorized* by y or x is *dominated* by y) provided

$$\sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} y_i \quad (k = 1, 2, \ldots, n),$$

with equality for $k = n$; equivalently, provided

$$\sum_{i=k+1}^{n} x_i \geqslant \sum_{i=k+1}^{n} y_i \quad (k = 0, 1, \ldots, n-1),$$

with equaliy for $k = 0$. If x and y are not assumed to be in nonincreasing order, then $x \preceq y$ provided their nonincreasing rearrangements x' and y' satisfy $x' \preceq y'$. The score sequence of T_n^* is the largest in the majorization order, and the score sequence of the regular (n odd), and nearly regular (n even) tournaments are the smallest [3].

As is often done, we shall use T to denote both a tournament and its adjacency matrix, called a *tournament matrix*. We usually just say tournament in referring to both.

A regular tournament T of order $n \geqslant 3$ is a Hall matrix and hence $h(T) = h^*(T) = 1$. The regular tournament T is strongly connected and the indegree and outdegree of each vertex equal $\frac{1}{2}(n-1)$. Hence the adjacency matrix M of the line digraph of a regular tournament is a matrix each of whose row and column sums equals $\frac{1}{2}(n-1)$ and so is a Hall matrix, that is, $h(M) = h^*(M) = 1$.

A nearly regular tournament of order $n \geqslant 4$ is also a Hall matrix since such a tournament must be strongly connected and strongly connected tournaments have Hamilton cycles. This latter fact is also a consequence of the next proposition.

Proposition 4.1. *A zero submatrix of a nearly regular tournament T_n of order n that does not meet the main diagonal has size r by s for some integers r and s with*

$$r + s \leqslant \frac{2}{3}(n+1)$$

where we have strict inequality if either $r > \frac{1}{2}n$ or $s > \frac{1}{2}n$.

P r o o f. Suppose that T_n has an r by s zero submatrix O_{rs} not meeting the main diagonal. Without loss of generality, T_n has the form

$$\left[\begin{array}{c|c|c} T_r & & O_{rs} \\ \hline & & \\ \hline J_{sr} & & T_s \end{array}\right],$$

where T_r and T_s are tournaments of orders r and s, respectively. Considering the first r columns we see that

$$\binom{r}{2} + sr \leqslant r\frac{n}{2},$$

with strict inequality if $r > \frac{1}{2}n$, since then one of these columns must sum to $\frac{1}{2}n - 1$. Simplifying, we get

$$r + 2s \leqslant n + 1. \tag{4.1}$$

Similarly, considering the last s rows, we get

$$s + 2r \leqslant n + 1. \tag{4.2}$$

Adding (4.1) and (4.2) we get

$$r + s \leqslant \frac{2}{3}(n+1)$$

with strict inequality if either $r > \frac{1}{2}n$ or $s > \frac{1}{2}n$. $\square$

Equality can occur in the inequality in Proposition 4.1. For example, if $n = 8$, the nearly regular tournament

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

has a 3 by 3 zero submatrix with $3 + 3 = \frac{2}{3}(8 + 1)$.

For a nearly regular tournament we have the following result.

Theorem 4.2. *Let T be a nearly regular tournament of order n. Let M be the adjacency matrix of $LD(T)$. Then $h(M) = 2$ for $n \geqslant 6$. If $n = 4$, then $h(M) = 3$.*

Proof. By Proposition 3.1, $h(M) \geqslant 2$. We first show that $h(M) \leqslant 2$ if $n \geqslant 10$. Let E be a set of edges of T, and let U be the set of terminal vertices of the edges in E. For a given U, the set E is largest when E is the set of all edges whose terminal vertices are in U, and it is enough to consider E to be this extremal set. Let F be the set of all edges with initial vertices in U, and let V be the set of terminal vertices of edges in F. Finally, let G be the set of edges whose initial vertices are in V. Pictorially we have

$$\xrightarrow{E} U \xrightarrow{F} V \xrightarrow{G}.$$

In order that M^2 be a Hall matrix, we must have $|G| \geqslant |E|$. Hence we may assume that $|U| \leqslant n - 1$. If $|U| \geqslant n - 3$, then since T is nearly regular, and $n \geqslant 10$, $V = \{1, 2, \ldots, n\}$ and hence $|G| = \binom{n}{2} \geqslant |E|$.

Now assume that $|U| \leqslant n - 4$. Since T is a tournament, its induced subdigraph on the set of vertices in U is also a tournament T_U, and thus T_U has at most one vertex of outdegree equal to 0. Hence $|U \cap V| \geqslant |U| - 1$. After simultaneous permutations of rows and columns we may take T in the form

$$\left[\begin{array}{c|c|c} T_U & X & O_{rk} \\ \hline & & \end{array}\right],$$

where k is a nonnegative integer and where $|U| = r$ and each column of X contains at least one 1 (thus the vertices of T corresponding to the columns of X all belong to V as do all but at most one vertex in U). The tournament T_U has at most one zero row and at most one zero column. In fact, since T is nearly regular, if $|U| > \frac{1}{2}(n+2)$, then T_U cannot have a zero row or zero column.

First suppose that $k = 0$. Then $|V| \geqslant n-1$. If $|V| = n$, then clearly $|G| \geqslant |E|$. Assume that $|V| = n-1$, so that $|G| \geqslant \binom{n}{2} - \frac{1}{2}n = \frac{1}{2}n(n-2)$. Then T_U has one zero column and, since T is nearly regular, $|U| \leqslant \frac{1}{2}n+1$. Hence $|E| \leqslant \frac{1}{2}n \cdot \frac{1}{2}n + \frac{1}{2}n - 1 = \frac{1}{2}n(\frac{1}{2}n+1) - 1$, and it follows that $|G| \geqslant |E|$. Now suppose that $k \geqslant 1$ so that the zero matrix O_{rk} is not vacuous. By Proposition 4.1, $r + k \leqslant \frac{2}{3}(n+1)$, that is, $k \leqslant \frac{2}{3}(n+1) - |U|$. Hence

$$|V| \geqslant (|U|-1)+(n-k-|U|) = n-k-1 \geqslant n-\left(\frac{2}{3}(n+1)-|U|\right)-1 \geqslant \frac{n-5}{3}+|U|.$$

Now $|E| \leqslant \frac{1}{2}n|U|$ and $|G| \geqslant \frac{1}{2}(n-2)|V|$ where

$$\frac{n-2}{2}|V| \geqslant \frac{n-2}{2}\left(\frac{n-5}{3}+|U|\right) = \frac{n-2}{2} \cdot \frac{n-5}{2} + \frac{n-2}{2}|U|.$$

Thus $|G| \geqslant |E|$ provided that

$$\frac{n-2}{2} \cdot \frac{n-5}{2} + \frac{n-2}{2}|U| \geqslant \frac{n}{2}|U|,$$

that is, provided that

$$\frac{n-2}{2} \cdot \frac{n-5}{3} \geqslant |U|.$$

Since $|U| \leqslant n-4$, this holds if $n \geqslant 10$.

It is not difficult to check by hand that $h(M) = 2$ if $n = 6$ or 8. Up to isomorphism there is only one nearly regular tournament of order 4, namely,

$$T_4^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

and the Hall exponent of the adjacency matrix of its line digraph is 3. $\square$

We now consider the strict Hall exponent of the line digraph of a regular or nearly regular tournament. In [12], it is stated that the primitive exponent of a regular or nearly regular tournament of order n is 3 for $n \geqslant 7$. The proof is based on a lemma which, contrary to what is claimed, does not apply to nearly regular tournaments. We now extend that lemma so that it applies to nearly regular tournaments for all orders $n \geqslant 10$.

Lemma 4.3. *Let T be a tournament of order $n \geqslant 9$ with score sequence $(r_1, r_2, \ldots, r_n)$. If*

$$\frac{n-2}{2} \leqslant r_i \leqslant \frac{n}{2} \quad (i = 1, 2, \ldots, n),$$

then for every ordered pair of vertices i and j there exists a walk from i to j of length 3.

P r o o f. Tournaments whose outdegrees satisfy the inequalities in the lemma are either regular or nearly regular. The tournament T must be strongly connected and hence every vertex is in a cycle of length 3. We now prove that if i and j are distinct vertices, then there is a walk from i to j of length 3. For a vertex p, let

$$N^+(p) = \{k\colon \text{ there is an edge } p \to k \text{ from } p \text{ to } k\},$$

and

$$N^-(p) = \{k\colon \text{ there is an edge } k \to p \text{ from } k \text{ to } p\}.$$

Thus $N^+(i)$ ($N^-(i)$) consists of the vertices *dominated* by the vertex i (respectively, *dominating* the vertex i). Note that since T is a tournament, $N^+(p)$ and $N^-(p)$ are disjoint and their union is $\{1, 2, \ldots, n\} \setminus \{p\}$.

First consider the case when $i \to j$. Since $n \geqslant 9$, $|N^+(i) \setminus \{j\}| \geqslant \frac{1}{2}(n-2) - 1$. If there are distinct vertices $p, q \neq j$ that are dominated by i and dominate j, then since $p \to q$ or $q \to p$, there is a walk of length 3 from i to j. Now assume that there is at most one such vertex. Thus j dominates at least $|N^+(i) \setminus \{j\})| - 1$ vertices in $N^+(i) \setminus \{j\}$. At most $|N^+(j)| - |N^+(i) \setminus \{j\}| - 1$ vertices of $N^-(i)$ are dominated by j, and so j is dominated by at least

$$|N^-(i)| - (|N^+(j)| - (|N^+(i) \setminus \{j\}| - 1)) = n - 1 - |N^+(j)| - 2 \geqslant \frac{n}{2} - 3$$

vertices of $N^-(i)$. Since $n \geqslant 9$, there are at least two such vertices p and q. If $N^+(i) \setminus \{j\} \subseteq N^+(p)$ then, since $i, j \in N^+(u)$, we have $|N^+(p)| \geqslant |N^+(i)| + 1$. Since $|N^+(u)| \leqslant \frac{9}{2}$, we conclude that $N^+(u) = N^+(i) \cup \{i\}$. If we also have $N^+(i) \setminus \{j\} \subseteq N^+(q)$, then using this argument we conclude that $N^+(u) = N^+(v)$, a contradiction since T is a tournament. So e.g. u is dominated by a vertex w in $N^+(i) \setminus \{j\}$. We thus have a walk $i \to w \to u \to j$ of length 3 from i to j.

A very similar argument works if $i \to j$ is not an edge of T. $\square$

Corollary 4.4. *The primitive exponent of a nearly regular tournament of order $n \geqslant 10$ equals* 3.

It is well-known [10] that a strongly connected tournament T of order $n \geqslant 4$ is primitive and that its exponent satisfies

$$d(T) \leqslant e(T) \leqslant d(T) + 3 \leqslant n + 2 \quad (n \geqslant 5),$$

where $d(T)$ is the *diameter* of T. It thus follows that if A is the adjacency matrix of a strongly connected tournament of order $n \geqslant 4$, then A^k is irreducible for every positive integer k. In [12] it is shown that a strongly connected tournament of order $n \geqslant 5$ satisfies $e(T) = n + 2$ if and only if it is isomorphic to T_n^*. As a corollary we get

Corollary 4.5. *Let T be a strongly connected tournament of order $n \geqslant 5$. Then*

$$h(LD(T)) \leqslant h^*(LD(T)) \leqslant e(LD(T)) \leqslant n + 3.$$

Moreover, $e(LD(T)) = n + 3$ if and only if T is isomorphic to T_n^*.

The bound for the Hall exponent of the line digraph of a tournament as given in Corollary 4.5 is rather weak and we now turn to improving it.

Theorem 4.6. *Let T be a strongly connected tournament of order $n \geqslant 4$. Then*

$$h(LD(T)) \leqslant 4 - n + \sqrt{2n^2 - 2n + 1} \approx (\sqrt{2} - 1)n.$$

P r o o f. Let A be the adjacency matrix of T. Since T is strongly connected and $n \geqslant 4$, A^k is irreducible for $k \geqslant 1$. Let $W \subseteq \{1, 2, \ldots, n\}$ be a nonempty subset of vertices of T. We denote the score sequence of T by $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$. The proof begins with four claims.

Claim 1. $|W| = |R_0(W)| \leqslant |R_1(W)| \leqslant |R_2(W)| \leqslant |R_3(W)| \leqslant \ldots.$

Let t be a nonnegative integer. Since T is strongly connected, the claim holds with equality throughout if $W = \{1, 2, \ldots, n\}$. Thus $R_{t+1}(W) = R_t(W)$ if $R_t(W) = \{1, 2, \ldots, n\}$. Suppose that $R_t(W) \subset \{1, 2, \ldots, n\}$. Then T restricted to the vertices in $R_t(W)$ is a tournament. Since $R_{t+1}(W) = R_1(R_t(W))$, we have $|R_{t+1}(W) \cap R_t(W)| \geqslant |R_t(W)| - 1$. Since T is irreducible, $R_{t+1}(W) \cap \overline{R_t(W)} \neq \emptyset$. Hence $|R_{t+1}(W)| \geqslant |R_t(W)|$.

Claim 2. If $|W| + t \leqslant n$, then $|R_{t+4}(W)| \geqslant |W| + t$.

Since T is strongly connected, every vertex belongs to a cycle of each length $3, 4, \ldots, n$, and hence for all $t \geqslant 0$, $R_t(W) \subseteq R_{t+p}(W)$ for $p = 3, 4, \ldots, n$. Thus $\bigcup\limits_{k=1}^{t} R_k(W) \subseteq R_{t+3}(W)$. Strong connectivity also implies that $R_t(W) \setminus \bigcup\limits_{k=1}^{t} R_k(W) \neq$

$\emptyset$ if $\bigcup_{k=0}^{t-1} R_k(W) \neq \{1,2,\dots,n\}$. Thus $\left|\bigcup_{k=0}^{t} R_k(W)\right| > \left|\bigcup_{k=0}^{t-1} R_k(W)\right|$ if $\bigcup_{k=0}^{t-1} R_k(W) \neq \{1,2,\dots,n\}$. By Claim 1, $|R_1(W)| \geqslant |W|$, and we now conclude that $\left|\bigcup_{k=1}^{t} R_k(W)\right| \geqslant |W|+t-1$. Thus $|R_{t+3}(W)| \geqslant |W|+t-1$, and this implies the claim.

Note that Claim 2 implies that $|R_{t+3}(\{i\})| \geqslant t$ for each vertex i.

Claim 3. Let $e = i \to j$ be any edge of T. Then the sum of the outdegrees of the vertices reachable from j by a walk of length $t+3$ is at least $\sum_{i=1}^{t} d_i$. Thus in $LD(T)$ the number of edges reachable from e by a walk of length $t+4$ is at least $\sum_{i=1}^{t} d_i$.

By Claim 2, $|R_{t+3}(j)| \geqslant t$. The smallest sum of t of the outdegrees of T is $\sum_{i=1}^{t} d_i$.

In a similar way one verifies the next claim.

Claim 4. Let $W \subseteq \{1,2\dots,n\}$ and let F be the set of edges whose terminal vertices are in W. Then in $LD(T)$ the number of edges reachable from F by a walk of length $t+5$ is at least $\sum_{i=1}^{\min\{t+|U|,n\}} d_i$.

Continuing with the notation in Claim 4, we have $|F| \leqslant \sum_{i=n-|U|+1|}^{n} d_i$. Since the score sequence $(1,1,2,\dots,n-3,n-2,n-2)$ is the largest in the majorization order, we have

$$|F| \leqslant \sum_{i=n-|U|+1}^{n} (i-1) = \frac{1}{2}(2n|U| - |U|^2 - |U|).$$

By Claim 4, in the line digraph, F can reach by a walk of length $t+5$ at least

$$\sum_{k=1}^{t+|U|-1} k = \frac{1}{2}(t^2 + t(2|U|-1) + |U|^2 - |U|)$$

edges. Thus $h(LD(T)) \leqslant t+5$ provided that

$$\frac{1}{2}(2n|U| - |U|^2 + |U| - 2n) \leqslant \frac{1}{2}(t^2 + t(2|U|-1) + |U|^2 - |U|),$$

and this holds provided that

$$t \geqslant \frac{1}{2}\big(-2|U| + 1 + \sqrt{8n|U| - 4|U|^2 - 4|U| + 1}\big). \tag{4.3}$$

The derivative of the expression on the right hand side of (4.3) is 0 when

$$|U| = \frac{1}{2}\big(2n - 1 - \sqrt{2n^2 - 2n + 1}\big).$$

Using this in (4.3), we get

$$t \geqslant 1 - n + \sqrt{2n^2 - 2n + 1}.$$

Thus

$$h(LD(T)) \leqslant 6 - n + \sqrt{2n^2 - 2n + 1}.$$

$\square$

We now show that the upper bound for the Hall exponent in Theorem 4.6 cannot be improved by much, by obtaining a lower bound for the Hall exponent of the line digraph of the special tournament T_n^*.

Theorem 4.7. *Let $n \geqslant 4$. Then*

$$h(LD(T_n^*)) \geqslant 2 - n + \sqrt{2n^2 - 2n - 7} \approx (\sqrt{2} - 1)n.$$

Proof. The score sequence of T_n^* is $(1, 1, 2, 3, \ldots, n-3, n-2, n-2)$. Let $l \geqslant 3$ and consider the set U of the l vertices with outdegrees $1, 1, 2, 3, \ldots, l-1$ and hence with indegrees $n-2, n-2, n-3, \ldots, n-l$. Let F be the set of edges with terminal vertex in U. Thus

$$|F| = (n-2) + \sum_{i=1}^{l-1}(n-i-1) = -1 + nl - \frac{1}{2}(l^2 + l).$$

Let $1 \leqslant k < n$. It is easy to check that there exist walks of length $k-1$ from each vertex in U to at least one of the vertices with outdegrees $1, 1, 2, 3, \ldots, l+k-2$ and to no other vertices. Thus in $LD(T_n^*)$ the number of edges reachable from F by a walk of length k is at most

$$1 + \sum_{i=1}^{l+k-2} i = 1 + \frac{1}{2}((l+k-2)(l+k-1)).$$

In order that $h(LD(T_n^*)) \leqslant k$ we must have

$$1 + \frac{1}{2}((l+k-2)(l+k-1)) \geqslant -1 + nl - \frac{l^2 + l}{2},$$

equivalently,

$$k^2 + (2l-3)k + (2l^2 - 2l - 2nl + 6) \geqslant 0.$$

This implies that

$$k \geqslant \frac{3}{2} - l + \frac{1}{2}\sqrt{-15 - 4l - 4l^2 + 8nl}. \tag{4.4}$$

The right hand side of (4.4) is maximum when

$$l = -\frac{1}{2} + n - \frac{1}{2}\sqrt{2n^2 - 2n - 7}.$$

Substituting this into (4.4), we get

$$k \geqslant 2 - n + \sqrt{2n^2 - 2n - 7}.$$

It follows that

$$h(LD(T_n^*)) \geqslant 2 - n + \sqrt{2n^2 - 2n - 7}.$$

□

We conclude with some questions for possible further exploration.

1. In view of Theorems 4.6 and 4.7, does $LD(T_n^*)$ have the largest Hall exponent among all strongly connected tournaments of order n?
2. The Hall exponent of every strongly connected tournament of order n is 1. What is the maximum Hall exponent of line digraphs of digraphs of order n with Hall exponent 1?
3. Does $h^*(LD(T)) = h(LD(T))$ for all tournaments T?
4. What is the connection between the degree sequence of a tournament and the Hall exponent of its line digraph?

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