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## STANLEY DECOMPOSITIONS AND POLARIZATION

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*Abstract.* We define nice partitions of the multicomplex associated with a Stanley ideal. As the main result we show that if the monomial ideal  $I$  is a CM Stanley ideal, then  $I^P$  is a Stanley ideal as well, where  $I^P$  is the polarization of  $I$ .

*Keywords:* monomial ideals, partitionable simplicial complexes, multicomplexes, Stanley ideals, polarization

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## 1. INTRODUCTION

Let  $K$  be a field and  $S = K[x_1, \dots, x_n]$  a polynomial ring in  $n$  variables. Let  $I \subset S$  be a monomial ideal,  $u \in S/I$  a monomial and  $Z \subseteq \{x_1, \dots, x_n\}$ . We denote by  $uK[Z]$  the  $K$ -subspace of  $S/I$  generated by all elements  $uv$  where  $v$  is a monomial in  $K[Z]$ . The  $K$ -subspace  $uK[Z] \subset S/I$  is called a *Stanley space* of dimension  $|Z|$ , if  $uK[Z]$  is a free  $K[Z]$ -module. A decomposition of  $S/I$  as a finite direct sum of Stanley spaces  $\mathcal{P}: S/I = \bigoplus_{i=1}^r u_i K[Z_i]$  is called a *Stanley decomposition*. Stanley [15] conjectured that there always exists such a decomposition with  $|Z_i| \geq \text{depth}(S/I)$ . If Stanley conjecture holds for  $S/I$  then  $I$  is called a *Stanley ideal*. The conjecture is still open but true in some special cases [1], [2], [4], [5], [6], [7], [9], [11], [12], [13], [14].

Let  $\Gamma$  be a subset of  $\mathbb{N}_{\infty}^n$ . An element  $m \in \Gamma$  is called maximal if there is no  $a \in \Gamma$  with  $a > m$ . We denote by  $\mathcal{M}(\Gamma)$  the set of maximal elements of  $\Gamma$ . If  $a \in \Gamma$ , we write  $\text{infpt}(a) = \{i: a(i) = \infty\}$ . An element  $a \in \Gamma$  is called a facet of  $\Gamma$  if for all  $m \in \mathcal{M}(\Gamma)$  with  $a \leq m$  one has  $|\text{infpt}(a)| = |\text{infpt}(m)|$ . Herzog and Popescu [8]

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modify Stanley's definition of multicomplexes [15].  $\Gamma$  is called a multicomplex if for all  $a \in \Gamma$  and for all  $b \in \mathbb{N}_\infty^n$  with  $b \leq a$  it follows that  $b \in \Gamma$  and for all  $a \in \Gamma$  there is a maximal element  $m$  in  $\Gamma$  such that  $a \leq m$ . We define an interval  $\mathcal{I}$  of  $\Gamma$  as a subset of  $\Gamma$  for which there exists  $a \leq b$  in  $\Gamma$  such that  $\mathcal{I} = [a, b] = \{c \in \Gamma : a \leq c \leq b\}$ . A partition  $\mathcal{P} : \Gamma = \bigcup_{i=1}^t [a_i, b_i]$  of  $\Gamma$  is a presentation of  $\Gamma$  as a finite disjoint union of intervals  $[a_i, b_i]$ .

Monomial ideals  $I$  in the polynomial ring  $S = K[x_1, \dots, x_n]$  and multicomplexes in  $\mathbb{N}_\infty^n$  correspond to each other bijectively. The multicomplex associated with a monomial ideal  $I$  is denoted by  $\Gamma(I)$  and similarly,  $I(\Gamma)$  denotes the monomial ideal associated with the multicomplex  $\Gamma$ . We show that Stanley's conjecture holds for  $S/I$  if and only if there exists a partition of the multicomplex  $\Gamma(I)$  such that  $|\text{infpt}(b_i)| \geq \text{depth}(S/I)$  for all  $i$ . Any partition of a multicomplex satisfying this condition will be called nice.

Let  $I \subset S = K[x_1, \dots, x_n]$  be a monomial ideal and  $\Gamma(I)$  the multicomplex associated with  $I$ . In Proposition 1.3 we show that a partition  $\mathcal{P} : \Gamma(I) = \bigcup_{i=1}^t [a_i, b_i]$  of  $\Gamma(I)$  is nice if all  $b_i$ 's are facets of  $\Gamma(I)$ . Also, when  $S/I$  is Cohen-Macaulay, we have this result in both directions (see Corollary 1.4).

Let  $I^p$  be the polarization of the monomial ideal  $I$  and let  $\Gamma^p$  be the multicomplex associated with  $I^p$ . In Theorem 2.5 we prove that in the case of Cohen-Macaulay monomial ideals, if  $\Gamma$  has a nice partition then  $\Gamma^p$  has a nice partition. The converse of this theorem is still open. In [7] it is shown that Stanley's conjecture on Stanley decompositions of  $S/I$  holds provided it holds whenever  $S/I$  is Cohen-Macaulay. As a consequence, Theorem 2.5 is true even for all monomial ideals  $I$  (see Remark 2.6).

## 2. PARTITIONS OF MULTICOMPLEXES

Let  $\Gamma$  be a subset of  $\mathbb{N}^n$ . We define on  $\mathbb{N}^n$  a partial order given by

$$(a(1), \dots, a(n)) \leq (b(1), \dots, b(n))$$

if  $a(i) \leq b(i)$  for all  $i$ . According to Stanley [15]  $\Gamma$  is a multicomplex if for all  $a \in \Gamma$  and all  $b \in \mathbb{N}^n$  with  $b \leq a$ , it follows that  $b \in \Gamma$ . The elements of  $\Gamma$  are called faces.

Herzog and Popescu [8] modify Stanley's definition of multicomplexes. Before giving this definition we introduce some notation. We set  $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ . As usual we set  $a \leq \infty$  for all  $a \in \mathbb{N}$ , and extend the partial order on  $\mathbb{N}^n$  naturally to  $\mathbb{N}_\infty^n$ . Thus now we take  $\Gamma$  as a subset of  $\mathbb{N}_\infty^n$ .

An element  $m \in \Gamma$  is called *maximal* if there is no  $a \in \Gamma$  with  $a > m$ . We denote by  $\mathcal{M}(\Gamma)$  the set of maximal elements of  $\Gamma$ . If  $a \in \Gamma$ , we call

$$\text{infpt}(a) = \{i: a(i) = \infty\}$$

the *infinite part* of  $a$ .

**Definition 2.1.** A subset  $\Gamma \subset \mathbb{N}_\infty^n$  is called a multicomplex if

- (1) for all  $a \in \Gamma$  and for all  $b \in \mathbb{N}_\infty^n$  with  $b \leq a$  it follows that  $b \in \Gamma$ ,
- (2) for all  $a \in \Gamma$  there exists an element  $m \in \mathcal{M}(\Gamma)$  such that  $a \leq m$ .

An element  $a \in \Gamma$  is called a *facet* of  $\Gamma$  if for all  $m \in \mathcal{M}(\Gamma)$  with  $a \leq m$  one has  $\text{infpt}(a) = \text{infpt}(m)$ . The set of all facets of  $\Gamma$  will be denoted by  $\mathcal{F}(\Gamma)$ . In [8] it is shown that each multicomplex has only a finite number of facets.

Monomial ideals  $I$  in the polynomial ring  $S = K[x_1, \dots, x_n]$  and multicomplexes in  $\mathbb{N}_\infty^n$  correspond to each other bijectively. The bijection is defined as follows: Let  $\Gamma$  be a multicomplex, and let  $I(\Gamma)$  be the  $K$ -subspace in  $S = K[x_1, \dots, x_n]$  spanned by all monomials  $x^a$  such that  $a \notin \Gamma$ . Note that if  $a \in \mathbb{N}_\infty^n$  and  $b \in \mathbb{N}_\infty^n \setminus \Gamma$ , then  $a + b \in \mathbb{N}_\infty^n \setminus \Gamma$ , that is, if  $x^b \in I(\Gamma)$  then  $x^a x^b \in I(\Gamma)$  for all  $x^a \in S$ . In other words,  $I(\Gamma)$  is a monomial ideal. In particular, the monomials  $x^a$  with  $a \in \Gamma$  form a  $K$ -basis of  $S/I(\Gamma)$ .

Conversely, given an arbitrary monomial ideal  $I \subset S$ , there is a unique multicomplex  $\Gamma$  with  $I = I(\Gamma)$ , namely the smallest multicomplex (with respect to inclusion) which contains  $A = \{a \in \mathbb{N}_\infty^n: x^a \notin I\}$ . Such a multicomplex exists and is uniquely determined since an arbitrary intersection of multicomplexes is again a multicomplex.

One has the following obvious rules: let  $\{\Gamma_j, j \in J\}$  be a family of multicomplexes. Then

- (a)  $I\left(\bigcap_{j \in J} \Gamma_j\right) = \sum_{j \in J} I(\Gamma_j)$ ,
- (b) if  $J$  is finite, then  $I\left(\bigcup_{j \in J} \Gamma_j\right) = \bigcap_{j \in J} I(\Gamma_j)$ .

Let  $\Gamma \subset \mathbb{N}_\infty^n$  be a multicomplex. We define an *interval*  $\mathcal{I}$  of  $\Gamma$  as a subset of  $\Gamma$  for which there exists  $a \leq b$  in  $\Gamma$  such that  $\mathcal{I} = \{c \in \Gamma: a \leq c \leq b\}$ . We denote an interval given by faces  $a$  and  $b$  by  $[a, b]$ . A *partition*  $\mathcal{P}$  of  $\Gamma$  is a presentation of  $\Gamma$  as a finite disjoint union of intervals.

**Lemma 2.2.** Let  $\mathcal{P}: \Gamma = \bigcup_{i=1}^t [a_i, b_i]$  be a partition of  $\Gamma$ . Then  $\text{infpt}(a_i) = \emptyset$  for all  $i$ .

*Proof.* Assume that for some  $i$ , say for  $i = 1$ , we have  $\text{infpt}(a_1) \neq \emptyset$ . We may assume that  $a_1(1) = \infty$ . Set  $a = a_1$  and let  $c$  be any integer. None of the faces  $(c, a(2), \dots, a(n))$  belong to  $[a_1, b_1]$ . Thus for each  $c$  there exists an  $i \in \{2, \dots, t\}$  such that  $(c, a(2), \dots, a(n)) \in [a_i, b_i]$ . Hence for some  $j > 1$ , infinitely many of the vectors  $(c, a(2), \dots, a(n))$  belong to  $[a_j, b_j]$ . This is only possible if  $(\infty, a(2), \dots, a(n))$  belongs to  $[a_j, b_j]$ . This is a contradiction, since  $a_1 = (\infty, a(2), \dots, a(n)) \in [a_1, b_1]$ .  $\square$

Next we describe how Stanley decompositions and partitions are related to each other. Let  $\Gamma \subset \mathbb{N}_\infty^n$  be a multicomplex,  $[a, b] \subset \Gamma$  an interval and  $U_{[a,b]}$  the  $K$ -subspace of  $S$  generated by all monomials  $u = x_1^{c(1)} \dots x_n^{c(n)}$  such that  $c = (c(1), \dots, c(n)) \in [a, b]$ . Then obviously  $U_{[a,b]}$  is a Stanley space if and only if

- (i)  $\text{infpt}(a) = \emptyset$ ,
- (ii)  $i \notin \text{infpt}(b) \Rightarrow a(i) = b(i)$ .

Indeed, in this case  $U_{[a,b]} = x^a K[Z_b]$ , where  $Z_b = \{x_i : b(i) = \infty\}$ .

Let  $I \subset S$  be a monomial ideal and  $\Gamma(I)$  the multicomplex associated with  $I$ . Also let  $S/I = \bigoplus_{i=1}^r x^{a_i} K[Z_i]$  be a Stanley decomposition of  $S/I$ . Set  $b_i(j) = \infty$  if  $x_j \in Z_i$  and  $b_i(j) = a_i(j)$  if  $x_j \notin Z_i$ . Then  $\bigcup_{i=1}^r [a_i, b_i]$  is a partition of  $\Gamma(I)$ . For instance, if  $a \in [a_i, b_i] \cap [a_j, b_j] \cap \mathbb{N}^n$  for  $i, j \in \{1, \dots, r\}$  and  $i \neq j$ , then  $x^a \in a_i K[Z_i] \cup a_j K[Z_j]$ , a contradiction. Thus  $\bigcup_{i=1}^r [a_i, b_i]$  is disjoint.

Conversely, we observe that each interval  $[a, b]$  with  $\text{infpt}(a) = \emptyset$  can be written as a disjoint union of intervals

$$(2.1) \quad [a, b] = \bigcup [c_i, b_i]$$

such that each  $[c_i, b_i]$  corresponds to a Stanley space. Indeed, if, as we may assume, for some integer  $r$  we have that  $b(k) < \infty$  for  $k \leq r$  and  $b(k) = \infty$  for  $k > r$ , then  $[a, b]$  is the disjoint union of the intervals

$$[(c(1), \dots, c(r), a(r+1), \dots, a(n)), (c(1), \dots, c(r), \infty, \dots, \infty)]$$

with  $a(k) \leq c(k) \leq b(k)$  for  $k = 1, \dots, r$ , and each of these intervals satisfies (i) and (ii). Therefore, due to (2.1) and Lemma 2.2, Stanley's conjecture holds for  $S/I$  if and only if there exists a partition  $\mathcal{P}: \Gamma = \bigcup_{i=1}^t [a_i, b_i]$  of the multicomplex  $\Gamma = \Gamma(I)$  such that

$$(2.2) \quad |\text{infpt}(b_i)| \geq \text{depth}(S/I(\Gamma)) \quad \text{for all } i.$$

Any partition of a multicomplex satisfying condition (2.2) will be called *nice*.

**Proposition 2.3.** *A partition  $\mathcal{P}: \Gamma = \bigcup_{i=1}^t [a_i, b_i]$  of the multicomplex  $\Gamma$  is a nice partition if  $b_i \in \mathcal{F}(\Gamma)$  for all  $i$ .*

**Proof.** Let  $I(\Gamma) = \bigcap_{i=1}^m Q_i$  be the unique irredundant presentation of  $I$  as an intersection of irreducible monomial ideals, and let  $P_i = \sqrt{Q_i}$  for  $i = 1, \dots, m$ . Then  $\text{Ass}(S/I) = \{P_1, \dots, P_m\}$ .

By [8, Proposition 9.12] there is a bijection between  $Q_i$  and the set of  $\mathcal{M}(\Gamma)$  of maximal faces of  $\Gamma$ . In fact, for each  $i$  there is a unique  $m_i \in \mathcal{M}(\Gamma)$  such that  $Q_i = I(\Gamma(m_i))$  where  $\Gamma(m_i)$  denotes the smallest multicomplex containing  $m_i$ . The assignment  $Q_i \mapsto m_i$  establishes this bijection. Moreover,  $\dim S/P_i = \text{infpt}(m_i)$  for all  $i$ . Therefore,

$$\begin{aligned} \min\{|\text{infpt}(b_i)|: b_i \in \mathcal{F}(\Gamma)\} &= \min\{|\text{infpt}(m_j)|: m_j \in \mathcal{M}(\Gamma)\} \\ &= \min\{\dim(S/P_j): P_j \in \text{Ass}(S/I(\Gamma))\} \\ &\geq \text{depth}(S/I(\Gamma)). \end{aligned}$$

The first equation follows from the definition of the facets, while the last inequality is a basic fact of commutative algebra, see [3, Proposition 1.2.13]. These considerations show that the given partition is nice.  $\square$

**Corollary 2.4.** *Let  $I \subset S$  be a monomial ideal such that  $S/I$  is Cohen-Macaulay. Let  $\Gamma$  be the multicomplex associated with  $I$  and let  $\mathcal{P}: \Gamma = \bigcup_{i=1}^t [a_i, b_i]$  be a partition of  $\Gamma$ . Then the following conditions are equivalent.*

- (a)  $\mathcal{P}$  is nice.
- (b)  $\{b_1, \dots, b_t\} \subseteq \mathcal{F}(\Gamma)$ .
- (c)  $\mathcal{M}(\Gamma) \subseteq \{b_1, \dots, b_t\} \subseteq \mathcal{F}(\Gamma)$ .

**Proof.** (a)  $\Rightarrow$  (b): In case  $S/I$  is Cohen-Macaulay we have  $|\text{infpt}(b)| \leq \text{depth}(S/I)$  for all faces of  $\Gamma$ , and equality holds for  $b$  if and only if  $b$  is a facet. Thus  $\mathcal{P}$  can be nice only if  $\{b_1, \dots, b_t\} \subseteq \mathcal{F}(\Gamma)$ .

(b)  $\Rightarrow$  (c): Let  $m \in \mathcal{M}(\Gamma)$ ; then  $m \in [a_i, b_i]$  for some  $i$ . Since  $m \leq b_i$  and since  $m$  is maximal it follows that  $m = b_i$ . Thus  $\mathcal{M}(\Gamma) \subseteq \{b_1, \dots, b_t\}$ .

(c)  $\Rightarrow$  (a) follows from Proposition 2.3.  $\square$

**Remark 2.5.** In the above corollary if  $\mathcal{P}$  is nice then we can refine it in such a way that for the refinement

$$\mathcal{P}': \Gamma = \bigcup_{i=1}^{t'} [a'_i, b'_i]$$

we have  $\{b'_1, \dots, b'_{t'}\} = \mathcal{F}(\Gamma)$ . To prove this fact we first observe that  $|\text{infpt}(a_i)| = 0$  for all  $i$ , see Lemma 1.2. Since  $\mathcal{F}(\Gamma) = \bigcup_{i=1}^t (\mathcal{F}(\Gamma) \cap [a_i, b_i])$ , it is enough to write each interval  $[a_i, b_i]$  as a disjoint union of intervals  $\bigcup_{j=1}^l [c_j, e_j]$  where  $\{e_1, e_2, \dots, e_l\} = \mathcal{F}(\Gamma) \cap [a_i, b_i]$ .

For simplicity, we may assume that  $b_i(k) < \infty$  for  $k \leq r$  and  $b_i(k) = \infty$  for  $k > r$ . Then  $e \in [a_i, b_i]$  is a facet of  $\Gamma$  if and only if  $a_i(k) \leq e(k) \leq b_i(k)$  for  $k \leq r$  and  $e(k) = \infty$  for  $k > r$ . Thus if we set  $c_j(k) = e_j(k)$  for  $k \leq r$  and  $c_j(k) = a_i(k)$  for  $k > r$ , then  $[a_i, b_i] = \bigcup_{j=1}^l [c_j, e_j]$  is the desired refinement of  $[a_i, b_i]$ .

### 3. PARTITIONS AND POLARIZATION

Let  $S = K[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over the field  $K$ , and let  $u = \prod_{i=1}^n x_i^{a_i}$  be a monomial in  $S$ . Then

$$u^p = \prod_{i=1}^n \prod_{j=1}^{a_i} x_{ij} \in K[x_{11}, \dots, x_{1a_1}, \dots, x_{n1}, \dots, x_{na_n}]$$

is called the *polarization* of  $u$ .

Let  $I$  be a monomial ideal in  $S$  with monomial generators  $u_1, \dots, u_r$ . Then  $(u_1^p, \dots, u_r^p)$  is called a *polarization* of  $I$  and is denoted by  $I^p$ . It is known that  $I$  is Cohen-Macaulay if and only if  $I^p$  is Cohen-Macaulay. Indeed, the elements  $x_{ij} - x_{i1}$ ,  $i = 1, \dots, n$  and  $j = 1, 2, \dots$  form a regular sequence on  $T/I^p$ , and  $T/I^p$  modulo this regular sequence is isomorphic to  $S/I$ .

Let  $I = (u_1, \dots, u_s) \subset S$  be a monomial ideal. We may assume that for each  $i \in [n]$  there exists  $j$  such that  $x_i$  divides  $u_j$ . Let  $u_j = x_1^{a_j^1} \dots x_n^{a_j^n}$  for  $j = 1, \dots, s$  and set  $r_i = \max a_{ji}$ :  $j = 1, \dots, s$  for  $i = 1, \dots, n$ . Moreover, set  $r = \sum_{i=1}^n r_i$ .

Let  $I = \bigcap_{i=1}^t Q_i$  be the unique irredundant presentation of  $I$  as an intersection of irreducible monomial ideals. In particular, each  $Q_i$  is generated by pure powers of some of the variables. Then  $I^p = \bigcap_{i=1}^{t_1} Q_i^p$  is an ideal in the polynomial ring

$$T = K[x_{11}, \dots, x_{1r_1}, x_{21}, \dots, x_{n1}, \dots, x_{nr_n}]$$

in  $r$  variables.

We denote by  $\Gamma$ ,  $\Gamma^p$ ,  $\Gamma_i$  and  $\Gamma_i^p$  the multicomplexes associated with  $I$ ,  $I^p$ ,  $Q_i$  and  $Q_i^p$ , respectively, and by  $\mathcal{F}$ ,  $\mathcal{F}^p$ ,  $\mathcal{F}_i$  and  $\mathcal{F}_i^p$  the set of facets of  $\Gamma$ ,  $\Gamma^p$ ,  $\Gamma_i$  and  $\Gamma_i^p$ , respectively.

Each  $\Gamma_i$  has only one maximal facet, say  $m_i$ , and  $m_i(k) \leq r_k - 1$  for all  $k$  with  $m_i(k) \neq \infty$ . Moreover,  $\mathcal{M}(\Gamma) = \{m_1, \dots, m_t\}$ . It follows that the set of facets of  $\Gamma$  is a subset of the set

$$\mathcal{B} = \{b \in \mathbb{N}_\infty^n : b(i) < r_i \text{ if } b(i) \neq \infty\}.$$

We define the map

$$\beta: \mathcal{B} \rightarrow \{0, \infty\}^r, \quad b \mapsto b',$$

where the components of the vectors  $b'$  are indexed by pairs of numbers  $ij$ , where for each  $i = 1, \dots, n$  the second index  $j$  runs in the range  $j = 1, \dots, r_i$ . The map  $\beta$  is defined as follows:

$$b'(ij) = \begin{cases} 0, & \text{if } b(i) < \infty \text{ and } j = b(i) + 1, \\ \infty, & \text{otherwise.} \end{cases}$$

We quote the following result by Soleyman Jahan [10, Proposition 3.8].

**Proposition 3.1.** *With the above assumptions and notation the restriction of the map  $\beta$  to  $\mathcal{F}$  induces a bijection  $\mathcal{F} \rightarrow \mathcal{F}^p$ .*

The following example demonstrates this bijection: let  $I = (x_1^2, x_1x_2, x_3^2) = (x_1, x_3^2) \cap (x_1^2, x_2, x_3^2) \subset K[x_1, x_2, x_3]$ . Then the multicomplex  $\Gamma$  associated with  $I$  has the facets

$$(0, \infty, 0), (0, \infty, 1), (1, 0, 0), (1, 0, 1),$$

while the multicomplex of the polarized ideal

$$I^p = (x_{11}x_{12}, x_{11}x_{21}, x_{31}x_{32}) \subset K[x_{11}, x_{12}, x_{21}, x_{31}, x_{32}]$$

has the facets

$$(0, \infty, \infty, 0, \infty), (0, \infty, \infty, \infty, 0), (\infty, 0, 0, 0, \infty), (\infty, 0, 0, \infty, 0).$$

Let  $\Gamma = \bigcup_{i=1}^t [a_i, b_i]$  be a nice partition of  $\Gamma$  with  $\mathcal{F}(\Gamma) = \{b_1, \dots, b_t\}$ . With the notation introduced above we have

**Lemma 3.2.**  $a_i(j) \leq r_j$  for all  $i$  and  $j$ .



Proof. Suppose without loss of generality that  $a_1(1) > r_1$ . Then  $b_1(1) = \infty$ , because if  $b_1(1) < \infty$  then it follows that  $a_1(1) \leq b_1(1) < r_1$ , a contradiction. Now since  $\Gamma = \bigcup_{i=1}^t [a_i, b_i]$  and since  $a = (r_1, a_1(2), \dots, a_1(n)) \in \Gamma \setminus [a_1, b_1]$ , there exists  $i > 1$  such that  $a \in [a_i, b_i]$ . As above,  $b_i(1) = \infty$  because if  $b_i(1) < \infty$  then  $r_1 \leq b_i(1) < r_1$ , which is not possible. Hence we conclude that  $a_i \leq a < a_1 < b_i \Rightarrow a_1 \in [a_i, b_i]$ , a contradiction.  $\square$

We want to “polarize” the nice partition  $\Gamma = \bigcup_{i=1}^t [a_i, b_i]$ . For this purpose we consider the set  $\mathcal{A} = \{a \in \mathbb{N} : a(i) \leq r_i\}$  and the map  $\gamma : \mathcal{A} \rightarrow \{0, 1\}^r$  with

$$\gamma(a)(ij) = \begin{cases} 0, & \text{if } j > a(i), \\ 1, & \text{otherwise.} \end{cases}$$

We observe that  $\gamma$  is injective. Indeed, for  $a \neq a'$  there exists  $i$  such that  $a(i) \neq a'(i)$ , say,  $a(i) < a'(i)$ . Then  $a(ij) = 0$  for  $j = a(i) + 1$ , while  $a'(ij) = 1$  for  $j = a(i) + 1$ .

Let  $\mathcal{I} = [a, b] \subset \Gamma \subset \mathbb{N}_\infty^n$  be an interval such that  $a = (a(1), a(2), \dots, a(n))$  and  $b = (b(1), b(2), \dots, b(n))$ . We define an  $i$ -subinterval as

$$\{c \in \mathbb{N}_\infty : a(i) \leq c \leq b(i)\}$$

and denote it by  $\mathcal{I}(i) = [a(i), b(i)]$ .

**Example 3.3.** Let  $a, b \in \Gamma \subset \mathbb{N}_\infty^2$ ,  $a = (2, 5)$ ,  $b = (4, \infty)$ . Then

$$\begin{aligned} \mathcal{I}(1) &= [a(1), b(1)] = [2, 4] \quad \text{i.e. } \mathcal{I}(1) = \{2, 3, 4\}, \\ \mathcal{I}(2) &= [a(2), b(2)] = [5, \infty] \quad \text{i.e. } \mathcal{I}(2) = \{5, 6, \dots\}. \end{aligned}$$

Now we need the following elementary lemma.

**Lemma 3.4.** Let  $\mathcal{I}_1, \mathcal{I}_2$  be two intervals of a multicomplex  $\Gamma \subset \mathbb{N}_\infty^n$  such that  $\mathcal{I}_1 = [a, b]$  and  $\mathcal{I}_2 = [c, d]$ . Suppose  $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$ . Then there exists  $i$  such that  $\mathcal{I}_1(i) \cap \mathcal{I}_2(i) = \emptyset$ .

Let  $I \subset S$  be a monomial ideal and let  $S/I = \bigoplus_{i=1}^r u_i K[Z_i]$  be its Stanley decomposition, where  $u_i = x^{a_i}$  for  $i = 1, \dots, r$ . Then the Hilbert series is given by  $H(S/I) = \sum_{i=1}^r t^{|a_i|} / (1-t)^{|Z_i|}$ , where  $|a_i|$  denotes the sum of the components of  $a_i$  and  $|Z_i|$  the cardinality of  $Z_i$ . Thus if  $\Gamma$  is the multicomplex associated with  $I$  and

$\Gamma = \bigcup_{i=1}^t [a_i, b_i]$  is the corresponding partition (with  $b_i(j) = a_i(j)$  for  $x_j \notin Z_i$  and  $b_i(j) = \infty$  for  $x_j \in Z_i$ ), then  $H(S/I) = \sum_{i=1}^r t^{|a_i|} / (1-t)^{|b_i|_\infty}$ , where  $|b_i|_\infty = |\text{infpt } b_i|$ .

**Theorem 3.5.** *Let  $I \subset S$  be a monomial ideal such that  $S/I$  is Cohen-Macaulay, and let  $I^p$  be the polarization of  $I$ . Suppose  $I$  satisfies the Stanley Conjecture. Then  $I^p$  satisfies it too.*

*Proof.* Let  $\Gamma$  be the multicomplex associated with  $I$ . Since  $I$  satisfies the Stanley Conjecture,  $\Gamma$  has a nice partition. Let  $\Gamma^p$  be the multicomplex associated with  $I^p$ . Then we show that  $\Gamma^p$  has a nice partition.

Let  $\Gamma = \bigcup_{i=1}^{\hat{t}} [\hat{a}_i, \hat{b}_i]$  be a nice partition of  $\Gamma$ . Then by Corollary 1.4,  $\hat{b}_i \in \mathcal{F}(\Gamma)$  for all  $i$ . Again by Remark 1.5, we can refine this partition to another nice partition, say  $\mathcal{P}: \Gamma = \bigcup_{i=1}^t [a_i, b_i]$ , such that  $\{b_1, \dots, b_t\} = \mathcal{F}(\Gamma)$ .

Let  $\beta$  and  $\gamma$  be the functions defined above and set  $\beta(b_i) = \bar{b}_i$  and  $\gamma(a_i) = \bar{a}_i$  for all  $i = 1, \dots, t'$ . We will show that  $\mathcal{P}^p: \Gamma^p = \bigcup_{i=1}^t [\bar{a}_i, \bar{b}_i]$  is a nice partition of  $\Gamma^p$ .

$\mathcal{P}^p$  is a partition if the intervals  $[\bar{a}_i, \bar{b}_i]$  are disjoint for all  $i = 1, \dots, t$  and  $\mathcal{P}^p$  covers all faces of  $\Gamma^p$ .

Suppose that the intervals are not disjoint and, say, there exists a face  $a \in [\bar{a}_i, \bar{b}_i] \cap [\bar{a}_j, \bar{b}_j]$  for some  $i \neq j, i, j \in \{1, \dots, t\}$ . Since  $a_i \neq a_j$  we get  $\bar{a}_i \neq \bar{a}_j$ ,  $\gamma$  being injective.

The intervals  $[a_i, b_i]$  and  $[a_j, b_j]$  are disjoint and so by Lemma 2.4 there exists at least one pair of  $i_1$ -subintervals, say  $[a_i(i_1), b_i(i_1)]$  and  $[a_j(i_1), b_j(i_1)]$  for  $i_1 \in \{1, \dots, n\}$ , such that  $[a_i(i_1), b_i(i_1)] \cap [a_j(i_1), b_j(i_1)] = \emptyset$ .

So at least one of  $b_i(i_1), b_j(i_1)$  is finite, say  $b_i(i_1) \neq \infty$ , thus  $i_1 \notin \text{infpt}(b_i)$ , so by condition (ii) of being Stanley space,  $b_i(i_1) = a_i(i_1)$ . Also we can assume that  $a_i(i_1) < a_j(i_1)$ . If not and  $b_j(i_1) = \infty$  then  $[a_i(i_1), b_i(i_1)] \subset [a_j(i_1), b_j(i_1)]$ , which is not possible; if  $b_j(i_1) < \infty$  then change  $i$  by  $j$ .

Let  $a_i(i_1) = b_i(i_1) = k$  and  $a_j(i_1) = m > k$ . Then by definition of  $\gamma$  and  $\beta$  we have  $\bar{a}_i(\overline{i_1 k + 1}) = 0 = \bar{b}_i(\overline{i_1 k + 1})$  and  $\bar{a}_j(i_1 l) = 1$  for  $l \leq m$ , thus  $\bar{a}_j(\overline{i_1 k + 1}) = 1$ .

It follows that  $a(\overline{i_1 k + 1}) = 0$ . On the other hand, since  $a \geq \bar{a}_j$  we get  $a(\overline{i_1 k + 1}) \geq \bar{a}_j(\overline{i_1 k + 1}) = 1$ , a contradiction.

Now for the second part of the proof, we will use the Hilbert series. We have  $H(S/I) = \sum_{i=1}^t s^{|a_i|} / (1-s)^{|b_i|_\infty}$ . The definition of the function  $\gamma$  implies that  $|a_i| = |\bar{a}_i|$  for all  $i \in \{1, \dots, t\}$ . Now for each polarization step, the depth of  $S/I$  increases by 1. Also by the definition of  $\beta$  for each polarization step the number of infinite points increases by 1. Thus after  $n_1$  polarization steps we have  $|\text{infpt}(\bar{b}_i)| = |\text{infpt}(b_i)| + n_1$ .

So

$$H\left(\bigcup_{i=1}^t [\bar{a}_i, \bar{b}_i]\right) = \sum_{i=1}^t \frac{s^{|a_i|}}{(1-s)^{|b_i|_\infty + n_1}} = \frac{1}{(1-s)^{n_1}} H(S/I)$$

is in fact the Hilbert series of  $H(S^p/I^p)$ . Hence  $S^p/I^p = \bigcup_{i=1}^t [\bar{a}_i, \bar{b}_i]$ .

Note that  $\mathcal{P}^p$  is a nice partition because  $|\bar{b}_i|_\infty = |b_i|_\infty + n_1 \geq \text{depth}_S(S/I) + n_1 = \text{depth}_{S^p}(S^p/I^p)$  for all  $i$ .  $\square$

**Remark 3.6.** In the above theorem, the condition for  $S/I$  to be Cohen-Macaulay is even not necessary. As in [7], in Corollary 2.2 it is shown that for each monomial ideal  $I$ , Stanley's conjecture holds for  $S/I$  provided it holds whenever  $S/I$  is Cohen-Macaulay.

The converse of Theorem 2.5 is still open. If one can prove the converse, then Stanley's Conjecture will reduce to the case of squarefree monomial ideals  $I$ .

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