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CONTINUOUS DEPENDENCE ON PARAMETERS OF CERTAIN SELF-AFFINE MEASURES, AND THEIR SINGULARITY

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Abstract. In this paper, we first prove that the self-affine sets depend continuously on the expanding matrix and the digit set, and the corresponding self-affine measures with respect to the probability weight behave in much the same way. Moreover, we obtain some sufficient conditions for certain self-affine measures to be singular.

Keywords: iterated function system, self-affine set, self-affine measure, singularity

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1. Introduction

Let $A \in M_d(\mathbb{R})$ be an expanding real matrix. Here a $d \times d$ real matrix A (i.e., $A \in M_d(\mathbb{R})$) is expanding if all its eigenvalues have absolute values strictly bigger than one. For a finite subset $D = \{d_1 = 0, d_2, \dots, d_N\} \subset \mathbb{R}^d$ of cardinality N, we will consider the iterated function system (IFS) $\{\varphi_j\}_{j=1}^N$ defined by

$$\varphi_j(x) = A^{-1}(x + d_j), \quad 1 \leqslant j \leqslant N \ \ (x \in \mathbb{R}^d). \tag{1.1}$$

We first know from [8] that there exists a unique compact set $T := T(A, D)$, called the attractor (or self-affine set) of the IFS, with the property that $T = \bigcup_{j=1}^N \varphi_j(T)$. D is called the digit set of the IFS. Then, for a probability weight $P = (p_1, p_2, \dots, p_N)$, i.e., $0 < p_j < 1$ $(j = 1, 2, \dots, N)$, $\sum_{j=1}^N p_j = 1$, there exists a unique probability

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measure $\mu := \mu_{A,D,P}$ satisfying the self-affine identity

$$\mu = \sum_{j=1}^{N} p_j \mu \circ \varphi_j^{-1}. \tag{1.2}$$

Such a measure $\mu_{A,D,P}$ is supported on $T(A,D)$, and is called a self-affine measure. For more details on IFSs, we refer to [2], [3], [4], [8].

The self-affine measures $\mu_{A,D,P}$, including self-similar measures as a special case, have received much attention in recent years. The previous research on such a measure and its Fourier transform revealed some surprising connections with a number of areas in mathematics, such as harmonic analysis, number theory, dynamical systems, and others (see, e.g. [5], [9], [13], [15]). The previous studies have also left some well-known open problems, such as the nature of the Bernoulli convolutions (cf. [1], [6], [13]), and how to determine the singularity or absolute continuity of $\mu_{A,D,P}$, which have motivated the present research.

In this note, we will consider the following two questions:

(1) When some parameters of IFS change continuously, what happens to the corresponding attractors and self-affine measures?
(2) On what conditions with respect to the parameters of IFS, the corresponding self-affine measures are singular with respect to the Lebesgue measure?

We organize the paper as follows. In Section 2 we prove that the self-affine sets depend continuously on the expanding matrix and the digit set in the sense of the Hausdorff metric, and the self-affine measures also depend continuously on the expanding matrix, the digit set and the probability weight in the sense of the Hutchinson metric. In Section 3 we give some properties of singularity of the self-affine measures, and prove that the class of self-affine measures is singular.

2. Continuous dependence on parameters of self-affine sets and self-affine measures

2.1. Continuous dependence on parameters of self-affine sets

Let (X, ϱ) be a complete metric space, and let $H(X)$ denote the collection of all non-empty compact subsets of X. We first introduce some notation:

$$\varrho(x,B) := \min\{\varrho(x,y)\colon\ y \in B\},\quad x \in X,\quad B \in H(X);$$
$$\varrho(A,B) := \max\{\varrho(x,B)\colon\ x \in A\},\quad A,B \in H(X);$$
$$a \vee b := \max\{a,b\},\quad a,b \in \mathbb{R}.$$

Definition 2.1. Let $A, B \in H(X)$. The Hausdorff metric is defined by

$$h(A,B) := \varrho(A,B) \vee \varrho(B,A).$$

An alternative definition is given by

$$h(A,B) = \inf\{\delta\colon\ A \subset B_\delta \quad \text{and} \quad B \subset A_\delta\}$$

where A_δ is the δ-neighborhood of A given by $A_\delta = \{y\colon \inf\limits_{x\in A} \varrho(x,y) \leqslant \delta\}$. It is easy to show that h is a complete metric on $H(X)$ and $(H(X), h)$ is a complete metric space which is often called a fractal space (see [4]).

We first introduce two lemmas on the Hausdorff metric (see [4]).

Lemma 2.1. *Let $A_i, B_i \in H(X)$, $i = 1, 2, \ldots, N$. Then*

$$h\Big(\bigcup_{i=1}^{N} A_i, \bigcup_{i=1}^{N} B_i\Big) \leqslant \sup_{1\leqslant i\leqslant N} h(A_i, B_i).$$

Lemma 2.2. *Let f be a contractive mapping on (X, ϱ) with ratio s. Define $f(B) := \{f(x)\colon\ x \in B\}$, $B \in H(X)$. Then f is a contractive mapping of $H(X) \to H(X)$ with the same ratio s, i.e., $h(f(A), f(B)) \leqslant s \cdot h(A,B)$ for all $A, B \in H(X)$.*

For $x \in \mathbb{R}^d$, $|x|$ denotes the Euclidean norm of x. Now we substitute $(\mathbb{R}^d, \varrho)$ for (X, ϱ) where $\varrho(x,y) := |x-y|$ for all $x, y \in \mathbb{R}^d$. For $A \in M_d(\mathbb{R})$, the norm of A is denoted by $\|A\| := \sup\limits_{|x|=1} |Ax|$.

Theorem 2.3. *Given any $n \in \mathbb{N}$ suppose that $T_n := T_n(A_n, D_n)$ is the self-affine set of the IFS: $\{\varphi_{jn}\}_{j=1}^N$ with the expanding matrix $A_n \in M_d(\mathbb{R})$ and the digit set $D_n = \{d_{1n}, d_{2n}, \ldots, d_{Nn}\} \subset \mathbb{R}^d$. Let $T := T(A, D)$ be the self-affine set of the IFS: $\{\varphi_j\}_{j=1}^N$ with the expanding matrix $A \in M_d(\mathbb{R})$ and the digit set $D = \{d_1, d_2, \ldots, d_N\} \subset \mathbb{R}^d$. If $\|A_n - A\| \to 0$, $|d_{jn} - d_j| \to 0$ $(j = 1, 2, \ldots, N)$ as $n \to \infty$, then T_n converges to T in the Hausdorff metric.*

P r o o f. For each $y \in T$ we have

$$\begin{aligned} \varrho(\varphi_{jn}(y), \varphi_j(y)) &= |A_n^{-1}(y + d_{jn}) - A^{-1}(y + d_j)| \\ &\leqslant \|A_n^{-1} - A^{-1}\| \cdot |y| + \|A_n^{-1} - A^{-1}\| \cdot |d_{jn}| + \|A^{-1}\| \cdot |d_{jn} - d_j|. \end{aligned}$$

Since T is a compact set, there exists a positive constant C_1 such that $|y| \leqslant C_1$ for all $y \in T$. By the convergence of $\{d_{jn}\}$, there exists a positive constant C_2 such that $|d_{jn}| \leqslant C_2$ for all $n \in \mathbb{N}$ and $1 \leqslant j \leqslant N$. Thus we have

$$\begin{aligned}
\varrho(\varphi_{jn}(T), \varphi_j(T)) &= \max_{x\in T}\min_{y\in T} \varrho(\varphi_{jn}(x), \varphi_j(y)) \\
&\leqslant \max_{x\in T}\min_{y\in T}(\varrho(\varphi_{jn}(x), \varphi_{jn}(y)) + \varrho(\varphi_{jn}(y), \varphi_j(y))) \\
&\leqslant \max_{x\in T}\min_{y\in T}(\varrho(\varphi_{jn}(x), \varphi_{jn}(y)) + \|A_n^{-1} - A^{-1}\| \cdot |y| \\
&\quad + \|A_n^{-1} - A^{-1}\| \cdot |d_{jn}| + \|A^{-1}\| \cdot |d_{jn} - d_j|) \\
&\leqslant \max_{x\in T}\min_{y\in T}(\varrho(\varphi_{jn}(x), \varphi_{jn}(y))) + \|A_n^{-1} - A^{-1}\| \cdot (C_1 + C_2) \\
&\quad + \|A^{-1}\| \cdot |d_{jn} - d_j| \\
&= \varrho(\varphi_{jn}(T), \varphi_{jn}(T)) + \|A_n^{-1} - A^{-1}\| \cdot (C_1 + C_2) \\
&\quad + \|A^{-1}\| \cdot |d_{jn} - d_j| \\
&= \|A_n^{-1} - A^{-1}\| \cdot (C_1 + C_2) + \|A^{-1}\| \cdot |d_{jn} - d_j|.
\end{aligned}$$

Similarly, we get

$$\varrho(\varphi_j(T), \varphi_{jn}(T)) \leqslant \|A_n^{-1} - A^{-1}\| \cdot (C_1 + C_2) + \|A^{-1}\| \cdot |d_{jn} - d_j|.$$

Therefore

$$\begin{aligned}
h(\varphi_{jn}(T), \varphi_j(T)) &= \varrho(\varphi_{jn}(T), \varphi_j(T)) \vee \varrho(\varphi_j(T), \varphi_{jn}(T)) \\
&\leqslant \|A_n^{-1} - A^{-1}\| \cdot (C_1 + C_2) + \|A^{-1}\| \cdot |d_{jn} - d_j|.
\end{aligned} \tag{2.1}$$

By Lemma 2.1, Lemma 2.2 and (2.1) we have

$$\begin{aligned}
h(T_n, T) &= h\Big(\bigcup_{j=1}^N \varphi_{jn}(T_n), \bigcup_{j=1}^N \varphi_j(T)\Big) \\
&\leqslant \sup_{1\leqslant j\leqslant N} h(\varphi_{jn}(T_n), \varphi_j(T)) \\
&\leqslant \sup_{1\leqslant j\leqslant N} (h(\varphi_{jn}(T_n), \varphi_{jn}(T)) + h(\varphi_{jn}(T), \varphi_j(T))) \\
&\leqslant \sup_{1\leqslant j\leqslant N} (\|A_n^{-1}\| \cdot h(T_n, T) + h(\varphi_{jn}(T), \varphi_j(T))) \\
&\leqslant \|A_n^{-1}\| \cdot h(T_n, T) + \|A_n^{-1} - A^{-1}\| \cdot (C_1 + C_2) \\
&\quad + \|A^{-1}\| \cdot \sup_{1\leqslant j\leqslant N} |d_{jn} - d_j|.
\end{aligned}$$

Thus it follows that

$$h(T_n, T) \leqslant \frac{\|A_n^{-1} - A^{-1}\| \cdot (C_1 + C_2) + \|A^{-1}\| \cdot \sup_{1\leqslant j\leqslant N} |d_{jn} - d_j|}{1 - \|A_n^{-1}\|}. \tag{2.2}$$

If

$$\|A_n - A\| \to 0 \quad \text{and} \quad |d_{jn} - d_j| \to 0 \quad (j = 1, 2, \ldots, N)$$

as $n \to \infty$, then we have

$$1 - \|A_n^{-1}\| \to 1 - \|A^{-1}\| > 0, \ \|A_n^{-1} - A^{-1}\| \to 0 \ \text{and} \sup_{1 \leqslant j \leqslant N} |d_{jn} - d_j| \to 0.$$

Hence, when $n \to \infty$, it follows from (2.2) that

$$h(T_n, T) \to 0.$$

We have completed the proof. □

Remark 2.4. Let $\{K_n\}_{n\in\mathbb{N}} \subset H(X)$ and $K \in H(X)$. If K_n is convergent to K in the Hausdorff metric, we know from [2] that

$$K = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} K_i}.$$

Hence, by Theorem 2.3, the self-affine set T can be constructed by a sequence of self-affine sets $\{T_n\}_{n\in\mathbb{N}}$, that is,

$$T = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} T_i}.$$

Thus we obtain an approach to constructing a self-affine set by choosing the expanding matrix and the digit set.

2.2. Continuous dependence on parameters of self-affine measures

In order to investigate the continuous dependence of self-affine measures on parameters of IFS, we now introduce the Hutchinson metric. Let (X, ϱ) be a compact metric space. We denote by $\mathfrak{M}$ the collection of all probability measures on X, and by $C(X)$ the collection of all continuous functions mapping X to $\mathbb{R}$. $f \in C(X)$ is called a Lipschitz function if there exists a constant M_f such that

$$|f(x) - f(y)| \leqslant M_f \cdot \varrho(x, y) \quad \text{for all } x, y \in X,$$

where M_f is called the Lipschitz constant of f. In particular, if $M_f = 1$, we write $f \in \mathrm{Lip}1$.

Definition 2.2. The Hutchinson metric d_H on $\mathfrak{M}$ is defined by

$$d_H(\mu,\nu) := \sup\biggl\{\int_X f\,\mathrm{d}\mu - \int_X f\,\mathrm{d}\nu\colon\ f \in \mathrm{Lip1}\biggr\} \quad \text{for all} \quad \mu,\nu \in \mathfrak{M}.$$

It may be shown that d_H is a metric on $\mathfrak{M}$ and $(\mathfrak{M}, d_H)$ is a complete metric space (see [4]). Now we recall the result on self-affine sets. Under the assumption of Theorem 2.3, we know that T_n is convergent to T in the Hausdorff metric. Hence, there exists a compact subset E of $\mathbb{R}^d$ such that $T \subset E$ and $T_n \subset E$ for all $n \in \mathbb{N}$. Taking $X = E$, we have the following theorem.

Theorem 2.5. *Keep the assumption of Theorem 2.3. Given any $n \in \mathbb{N}$ suppose that $\mu_n := \mu_{A_n,D_n,P_n}$ is the self-affine measure of the IFS: $\{\varphi_{jn}\}_{j=1}^N$ with the probability weight $P_n = (p_{1n}, p_{2n}, \ldots, p_{Nn})$. Let $\mu := \mu_{A,D,P}$ be the self-affine measure of the IFS: $\{\varphi_j\}_{j=1}^N$ with the probability weight $P = (p_1, p_2, \ldots, p_N)$. If $\|A_n - A\| \to 0$, $|p_{jn} - p_j| \to 0$ and $|d_{jn} - d_j| \to 0$ ($j = 1, 2, \ldots, N$) as $n \to \infty$, then μ_n converges to μ in the Hutchinson metric.*

Proof. We first claim that

$$\sup_{g\in\mathrm{Lip1}}\biggl(\sum_{j=1}^N p_{jn}\int g\circ\varphi_j\,\mathrm{d}\mu - \sum_{j=1}^N p_j\int g\circ\varphi_j\,\mathrm{d}\mu\biggr) \leqslant |T|\cdot\sum_{j=1}^N |p_{jn} - p_j| \tag{2.3}$$

where $|T|$ denotes the diameter of T. In fact, taking $x_0 \in T$, we write $\tilde g(x) = g(x) - g(x_0)$ for all $g \in \mathrm{Lip1}$. Then $\tilde g \in \mathrm{Lip1}$ and $\tilde g(x_0) = 0$. Therefore we have

$$\begin{aligned}
&\sum_{j=1}^N p_{jn}\int g\circ\varphi_j\,\mathrm{d}\mu - \sum_{j=1}^N p_j\int g\circ\varphi_j\,\mathrm{d}\mu\\
&\quad= \sum_{j=1}^N p_{jn}\int \tilde g\circ\varphi_j\,\mathrm{d}\mu - \sum_{j=1}^N p_j\int \tilde g\circ\varphi_j\,\mathrm{d}\mu\\
&\quad= \sum_{j=1}^N (p_{jn} - p_j)\int \tilde g(\varphi_j(x)) - \tilde g(x_0)\,\mathrm{d}\mu(x)\\
&\quad\leqslant \sum_{j=1}^N |p_{jn} - p_i|\int |\varphi_j(x) - x_0|\,\mathrm{d}\mu(x)\\
&\quad\leqslant |T|\cdot\sum_{j=1}^N |p_{jn} - p_j|,
\end{aligned}$$

which yields (2.3) since g is arbitrary.

For each $g \in \mathrm{Lip}1$, by (1.2) we get

$$\begin{aligned}
(2.4)\qquad \int g\,\mathrm{d}\mu_n - \int g\,\mathrm{d}\mu &= \sum_{j=1}^{N} p_{jn} \int g\circ\varphi_{jn}\,\mathrm{d}\mu_n - \sum_{j=1}^{N} p_j \int g\circ\varphi_j\,\mathrm{d}\mu \\
&= \sum_{j=1}^{N} p_{jn}\Big(\int g\circ\varphi_{jn}\,\mathrm{d}\mu_n - \int g\circ\varphi_{jn}\,\mathrm{d}\mu\Big) \\
&\quad + \sum_{j=1}^{N} p_{jn}\Big(\int g\circ\varphi_{jn}\,\mathrm{d}\mu - \int g\circ\varphi_j\,\mathrm{d}\mu\Big) \\
&\quad + \sum_{j=1}^{N} p_{jn} \int g\circ\varphi_j\,\mathrm{d}\mu - \sum_{j=1}^{N} p_j \int g\circ\varphi_j\,\mathrm{d}\mu \\
&= \|A_n^{-1}\| \cdot \sum_{j=1}^{N} p_{jn} \\
&\quad \times \Big(\int \|A_n^{-1}\|^{-1} g\circ\varphi_{jn}\,\mathrm{d}\mu_n - \int \|A_n^{-1}\|^{-1} g\circ\varphi_{jn}\,\mathrm{d}\mu\Big) \\
&\quad + \sum_{j=1}^{N} p_{jn}\Big(\int g\circ\varphi_{jn}\,\mathrm{d}\mu - \int g\circ\varphi_j\,\mathrm{d}\mu\Big) \\
&\quad + \sum_{j=1}^{N} p_{jn} \int g\circ\varphi_j\,\mathrm{d}\mu - \sum_{j=1}^{N} p_j \int g\circ\varphi_j\,\mathrm{d}\mu.
\end{aligned}$$

Since $\|A_n^{-1}\|^{-1} g\circ\varphi_{jn} \in \mathrm{Lip}1$, it follows from (2.3) and (2.4) that

$$\begin{aligned}
(2.5)\qquad d_H(\mu_n,\mu) &= \sup_{g\in\mathrm{Lip}1}\Big\{\int g\,\mathrm{d}\mu_n - \int g\,\mathrm{d}\mu\Big\} \\
&\leqslant \|A_n^{-1}\| \cdot d_H(\mu_n,\mu) \\
&\quad + \sup_{g\in\mathrm{Lip}1} \sum_{j=1}^{N} p_{jn}\Big(\int g\circ\varphi_{jn}\,\mathrm{d}\mu - \int g\circ\varphi_j\,\mathrm{d}\mu\Big) \\
&\quad + |T| \sum_{j=1}^{N} |p_{jn} - p_j|.
\end{aligned}$$

Since T is a compact set, there exists a positive constant C_1 such that

$$|y| \leqslant C_1 \quad \text{for all } y \in T.$$

By the convergence of $\{d_{jn}\}$, there exists a positive constant C_2 such that

$$|d_{jn}| \leqslant C_2 \quad \text{for all } n \in \mathbb{N} \text{ and } 1 \leqslant j \leqslant N.$$

It follows from (2.5) that

$$
\begin{aligned}
&d_H(\mu_n, \mu) \\
&\quad \leqslant \frac{1}{1-\|A_n^{-1}\|}\Big(\sup_{g\in \mathrm{Lip}1}\sum_{j=1}^{N} p_{jn}\int (g\circ\varphi_{jn} - g\circ\varphi_j)\,\mathrm{d}\mu + |T|\sum_{j=1}^{N}|p_{jn}-p_j|\Big) \\
&\quad \leqslant \frac{1}{1-\|A_n^{-1}\|}\Big(\sum_{j=1}^{N} p_{jn}\int |(A_n^{-1}-A^{-1})(x+d_{jn}) + A^{-1}(d_{jn}-d_j)|\,\mathrm{d}\mu(x)\Big) \\
&\qquad + \frac{1}{1-\|A_n^{-1}\|}\Big(|T|\sum_{j=1}^{N}|p_{jn}-p_j|\Big) \\
&\quad \leqslant \frac{1}{1-\|A_n^{-1}\|}\Big(\|A_n^{-1}-A^{-1}\|\cdot(C_1+C_2) + \|A^{-1}\|\cdot\sum_{j=1}^{N} p_{jn}|(d_{jn}-d_j)|\Big) \\
&\qquad + \frac{1}{1-\|A_n^{-1}\|}\Big(|T|\sum_{j=1}^{N}|p_{jn}-p_j|\Big).
\end{aligned}
$$

If

$$\|A_n - A\| \to 0,\ \ |p_{jn}-p_j| \to 0 \quad\text{and}\quad |d_{jn}-d_j| \to 0\ \ (j=1,2,\ldots,N)$$

as $n\to\infty$, then

$$1-\|A_n^{-1}\| \to 1-\|A^{-1}\| > 0, \quad \|A_n^{-1}-A^{-1}\| \to 0, \quad |p_{jn}-p_j| \to 0$$

and

$$|d_{jn}-d_j| \to 0 \quad \text{for} \quad 1\leqslant j\leqslant N,$$

which yields

$$d_H(\mu_n,\mu)\to 0 \quad \text{as} \quad n\to\infty.$$

The proof is completed. □

In Section 3, we will give an example illustrating that there exists a Borel set B such that $\mu_n(B)$ is not convergent to $\mu(B)$, even though μ_n is convergent to μ in the Hutchinson metric. Actually, μ_n converges to μ in Hutchinson metric if and only if μ_n converges weakly to μ (see [4]).

3. Singularity of self-affine measures

Let M be the expanding real matrix of the IFS, $D = \{d_1 = 0, d_2, \ldots, d_N\} \subset \mathbb{R}^d$ the digit set, and $P = (p_1, p_2, \ldots, p_N)$ the probability weight. We define the function $m_{D,P}(x)$ by putting

$$m_{D,P}(x) = \sum_{j=1}^{N} p_j \mathrm{e}^{2\pi \mathrm{i} \langle d_j, x\rangle}, \quad x \in \mathbb{R}^d.$$

Let M^* denote the conjugate transpose of M, in fact $M^* = M^T$. We first introduce the following lemma established by Li [11].

Lemma 3.1. *With the same notation as above, if there exists a nonzero point $\xi_0 \in \mathbb{R}^d$ such that*

$$m_{D,P}(M^{*k}\xi_0) \neq 0 \quad \text{for all } k \in \mathbb{Z} \tag{3.1}$$

and

$$\sum_{k=0}^{\infty} |1 - |m_{D,P}(M^{*k}\xi_0)|| < +\infty, \tag{3.2}$$

then the self-affine measure $\mu_{M,D,P}$ is singular.

Corollary 3.2. *Suppose that $M \in M_d(\mathbb{Z})$ is an integer-valued expanding matrix and $D = \{d_1 = 0, d_2, \ldots, d_N\} \subset \mathbb{Z}^d$. For any probability weight $P = (p_1, \ldots, p_N)$, if there exists a non-zero integer point $\xi_0 \in \mathbb{Z}^d$ such that for any positive integer k, $m_{D,P}(M^{*-k}\xi_0) \neq 0$, then the corresponding self-affine measure $\mu_{M,D,P}$ is singular.*

Proof. Since $M \in M_d(\mathbb{Z})$ and $D = \{d_1 = 0, d_2, \ldots, d_N\} \subset \mathbb{Z}^d$, for the non-zero integer point $\xi_0 \in \mathbb{Z}^d$ we have

$$m_{D,P}(M^{*k}\xi_0) = 1 \quad \text{and} \quad |1 - |m_{D,P}(M^{*k}\xi_0)|| = 0 \quad (k = 0, 1, 2, \ldots).$$

Therefore Corollary 3.2 follows from Lemma 3.1 directly. □

The above result in the case of the dimension $d = 1$ was also obtained by Hu [7] and Niu [12] by using different techniques. From this corollary, we get the following proposition.

Proposition 3.3. *Suppose that $M \in M_d(\mathbb{Z})$ is an integer-valued expanding matrix and $D = \{d_1 = 0, d_2, \ldots, d_N\} \subset \mathbb{Z}^d$. Let $\mu_{M,D,P}$ be the self-affine measure with respect to the probability weight $P = \{p_1, p_2, \ldots, p_N\}$. If there exists $j \in \{1, 2, \ldots, N\}$ such that $p_j > 1/2$, then $\mu_{M,D,P}$ is singular.*

Proof. If there exists j, say $j = 1$, such that $p_1 > 1/2$, we claim that for any positive integer k and $\xi \in \mathbb{R}^d$, $m_{D,P}(M^{*-k}\xi) \neq 0$. We argue by contradiction to verify the claim. Assume that there exist k and ξ such that $m_{D,P}(M^{*-k}\xi) = 0$, then

$$p_1 + \sum_{j=2}^{N} p_j \mathrm{e}^{2\pi \mathrm{i}\langle d_j, M^{*-k}\xi\rangle} = 0.$$

Therefore

$$p_1 = \left|\sum_{j=2}^{N} p_j \mathrm{e}^{2\pi \mathrm{i}\langle d_j, M^{*-k}\xi\rangle}\right| \leqslant p_2 + \ldots + p_N = 1 - p_1,$$

which yields $p_1 \leqslant 1/2$, a contradiction. Thus we have, for any positive integer k,

$$m_{D,P}(M^{*-k}\xi) \neq 0, \quad \text{for all } \xi \in \mathbb{R}^d,$$

which implies that $\mu_{M,D,P}$ is singular by Corollary 3.2. The proof is completed. □

Now we give an example illustrating that the fact that a self-affine measure sequence $\{\mu_n\}_{n=1}^{\infty}$ converges to μ in the Hutchinson metric does not imply that $\{\mu_n(A)\}_{n=1}^{\infty}$ converges to $\mu(A)$ for every Borel set A.

Example 3.1. Taking $M = 2$, $D = \{0, 1\}$, $P_n = (\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n})$, and $P = (\frac{1}{2}, \frac{1}{2})$, we write $\mu_n = \mu_{M,D,P_n}$ and $\mu = \mu_{M,D,P}$, then $\operatorname{supp}(\mu_n) = \operatorname{supp}(\mu) = [0, 1]$. By Theorem 2.5, μ_n is convergent to μ in the Hutchinson metric. However, we see that there exists a Borel set $B \subset \mathbb{R}$ such that the sequence $\{\mu_n(B)\}_{n=3}^{\infty}$ does not converge to $\mu(B)$. In fact, by Proposition 3.3, μ_n is singular with respect to the Lebesgue measure restricted to $[0, 1]$ denoted by $\mathcal{L}$. Then there exist Borel sets B_n such that

$$\mu_n(B_n) = 0 \quad \text{and} \quad \mathcal{L}([0, 1] \setminus B_n) = 0.$$

Writing $B = \bigcap_{n=3}^{\infty} B_n$, we have

$$\mu_n(B) = 0 \quad \text{and} \quad \mathcal{L}([0, 1] \setminus B) = 0.$$

Evidently, μ is equal to $\mathcal{L}$. Since $\mathcal{L}(B) = 1 = \mu(B)$, we have

$$\lim_{n\to\infty} \mu_n(B) \neq \mu(B).$$

Let $M = pI_d$ where I_d is the $d \times d$ identity matrix on $\mathbb{R}^d$ and $p \geqslant 2$ is a natural number, and let $D = \{d_1 = 0, d_2, \ldots, d_N\} \subset \mathbb{Z}^d$ with $d_j = (a_{j1}, a_{j2}, \ldots, a_{jd})^T \subset \mathbb{R}^d$ $(2 \leqslant j \leqslant N)$. Then the following result is obtained.

Proposition 3.4. *Let $\mu_{M,D,P}$ be the self-affine measure with respect to the probability weight $P = \{p_1, p_2, \ldots, p_N\}$. If there exists $l \in \{1, 2, \ldots, d\}$ such that g.c.d.$(a_{2l}, a_{3l}, \ldots, a_{Nl}) = 1$ where g.c.d. is the abbreviation of greatest common divisor, then $\mu_{M,D,P}$ is singular for almost all probability weights.*

Proof. Since g.c.d.$(a_{2l}, a_{3l}, \ldots, a_{Nl}) = 1$, there exists j such that $p \nmid a_{jl}$. Without loss of generality, we may assume $j = 2$. By Corollary 3.2, if μ is not singular, then for a given integer point $e_l = (0, \ldots, 0, 1, 0, \ldots, 0) \subset \mathbb{R}^d$ where the lth coordinate is 1, there exists a positive integer k such that $m_{D,P}(M^{*-k} e_l) = 0$, i.e., $P = \{p_1, p_2, \ldots, p_N\}$ satisfies the equations

$$\begin{cases} p_1 + p_2 \mathrm{e}^{2\pi \mathrm{i} a_{2l}/p^{-k}} + \sum\limits_{j=3}^{N} p_j \mathrm{e}^{2\pi \mathrm{i} a_{jl}/p^{-k}} = 0, \\ p_1 + p_2 + \sum\limits_{j=3}^{N} p_j = 1. \end{cases}$$

Note that $\mathrm{e}^{2\pi \mathrm{i} a_{2l}/p^{-k}} \neq 1$ as $p \nmid a_{2l}$. When $p_3, \ldots, p_N$ are fixed, the above set of linear equations has a unique solution (p_1, p_2). By Fubini's theorem, the set of all weights whose corresponding measures are not singular has $(N-1)$-dimensional Lebesgue measure 0. In other words, for almost all weights, the self-affine measure $\mu_{M,D,P}$ is singular. $\square$

Now we wish to investigate the singularity of the self-affine measures concerned with Pisot numbers. An algebraic integer is a root of a polynomial whose leading coefficient is 1 and the rest of the coefficients are all integers. The algebraic integer $\beta > 1$ is a Pisot number if all its algebraic conjugates have modulus less than 1 (cf. [14]), e.g. the golden ratio $(\sqrt{5}+1)/2$ is a Pisot number, being a root of $x^2 - x - 1 = 0$. We first state two lemmas on the Pisot number β (see [10], [14]).

Lemma 3.5. *Let $\beta > 1$ be a Pisot number. Then there exists $0 < \theta < 1$ such that $\|\beta^k\| < \theta^k$ for large k, where $\|x\|$ denotes the distance from x to the nearest integer.*

Lemma 3.6. *Let $\beta > 1$ be a Pisot number. Consider the trigonometric polynomial $Q(x) = \sum\limits_{j=1}^{N} c_j \mathrm{e}^{2\pi \mathrm{i} b_j x}$, where $c_j \in \mathbb{R}$, $b_j \in \mathbb{Q}$ and $\sum\limits_{j=1}^{N} c_j \neq 0$. Let $B \in \mathbb{Z} \setminus \{0\}$ be such that $B_j = Bb_j$, $1 \leqslant j \leqslant N$, are integers. Then there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $Q(mB\beta^k) \neq 0$ for all $k \in \mathbb{Z}$.*

Using the above properties of the Pisot number, we prove that a class of self-affine measures are singular.

Theorem 3.7. *Let $M = (c_{ij}) \in M_d(\mathbb{R})$ and $D = \{d_1 = 0, d_2, \ldots, d_N\}$, where M is a triangular matrix with $c_{ii} = \beta > 1$ for $1 \leqslant i \leqslant d$, β is a Pisot number and $d_j = (a_{j1}, a_{j2}, \ldots, a_{jd})^T \subset \mathbb{R}^d$ $(2 \leqslant j \leqslant N)$. If one of the following two conditions holds,*

(1) *M is a lower triangular matrix and $a_{j1} \in \mathbb{Q}$ for $2 \leqslant j \leqslant N$;*
(2) *M is an upper triangular matrix and $a_{jd} \in \mathbb{Q}$ for $2 \leqslant j \leqslant N$,*

then for any weight $P = (p_1, p_2, \ldots, p_N)$, $\mu_{M,D,P}$ is singular.

Proof. (i) If the condition (1) holds, we consider the trigonometric polynomial

$$Q(x) = p_1 + \sum_{j=2}^{N} p_j \mathrm{e}^{2\pi \mathrm{i} a_{j1} x}.$$

Let $B \in \mathbb{Z} \setminus \{0\}$ be such that $B_j = B a_{j1}$, $2 \leqslant j \leqslant N$, are integers. By Lemma 3.6, there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $Q(mB\beta^k) \neq 0$ for all $k \in \mathbb{Z}$. Take

$$\xi_0 = (mB, 0, \ldots, 0)^T \in \mathbb{R}^d \setminus \{0\} \quad \text{and} \quad A_k := M^{*k} - \beta^k I_d \text{ for all } k \in \mathbb{Z}.$$

Then $A_k \xi_0 = 0$ for all $k \in \mathbb{Z}$. Thus we conclude that there exists $\xi_0 = (mB, 0, \ldots, 0)^T \in \mathbb{R}^d \setminus \{0\}$ such that

$$\begin{aligned} m_{D,P}(M^{*k}\xi_0) &= p_1 + \sum_{j=2}^{N} p_j \mathrm{e}^{2\pi \mathrm{i} \langle d_j, M^{*k}\xi_0 \rangle} \\ &= p_1 + \sum_{j=2}^{N} p_j \mathrm{e}^{2\pi \mathrm{i} \langle d_j, \beta^k I_d \xi_0 \rangle} \mathrm{e}^{2\pi \mathrm{i} \langle d_j, A_k \xi_0 \rangle} \\ &= p_1 + \sum_{j=2}^{N} p_j \mathrm{e}^{2\pi \mathrm{i} m B a_{j1} \beta^k} \\ &= Q(mB\beta^k) \neq 0 \end{aligned}$$

for all $k \in \mathbb{Z}$. Hence the condition (3.1) is satisfied. Next we employ Lemma 3.5 to verify the condition (3.2). If l_k is the integer nearest to β^k, we can write $\beta^k = l_k + \{\beta^k\}$ so that $\|\beta^k\| = |\{\beta^k\}|$. Furthermore, it follows from the above equality and Lemma 3.5 that for large k,

$$\begin{aligned}\big|1-|m_{D,P}(M^{*k}\xi_0)|\big| &= \left|1-\left|p_1+\sum_{j=2}^{N}p_j\mathrm{e}^{2\pi\mathrm{i}mBa_{j1}\beta^k}\right|\right|\\ &\leqslant \sum_{j=2}^{N}|p_j|\cdot|1-\mathrm{e}^{2\pi\mathrm{i}mBa_{j1}\beta^k}|\\ &\leqslant \theta^k\sum_{j=2}^{N}|p_j|\cdot|2\pi mBa_{j1}|,\end{aligned}$$

which yields

$$\sum_{k=0}^{\infty}\big|1-|m_{D,P}(M^{*k}\xi_0)|\big|<+\infty.$$

That is, the condition (3.2) is satisfied, thus $\mu_{M,D,P}$ is singular by Lemma 3.1.

(ii) If the condition (2) holds, we consider the trigonometric polynomial

$$Q(x)=p_1+\sum_{j=2}^{N}p_j\mathrm{e}^{2\pi\mathrm{i}a_{jd}x}.$$

Let $B\in\mathbb{Z}\setminus\{0\}$ be such that $B_j=Ba_{jd}$, $2\leqslant j\leqslant N$, are integers. By Lemma 3.6, there exists $m\in\mathbb{Z}\setminus\{0\}$ such that $Q(mB\beta^k)\neq 0$ for all $k\in\mathbb{Z}$. Take

$$\xi_0=(0,0,\ldots,0,mB)^T\in\mathbb{R}^d\setminus\{0\}\quad\text{and}\quad A_k:=M^{*k}-\beta^kI_d\ \text{ for all } k\in\mathbb{Z}.$$

Then $A_k\xi_0=0$ for all $k\in\mathbb{Z}$. The remainder of the proof is the same as in (i). We have completed the proof. □

Remark 3.8. In the case of the dimension $d=1$, $M=\beta$ and $D=\{b_1=0,b_2,\ldots,b_N\}\subset\mathbb{Q}$, Lau, Ngai and Rao [10] proved that for any weight $P=(p_1,p_2,\ldots,p_N)$ with $\sum\limits_{j=1}^{N}p_j=1$, $\mu_{M,D,P}$ is singular. In addition, for $M=\beta I_d$ and $D=\{b_1=0,b_2,\ldots,b_{d+1}\}\subset\mathbb{R}^d$ where $b_2,\ldots,b_{d+1}$ are d linearly independent vectors in $\mathbb{R}^d$, Li [11] proved that for any weight $P=(p_1,p_2,\ldots,p_{d+1})$ with $\sum\limits_{j=1}^{d+1}p_j=1$, $\mu_{M,D,P}$ is singular.

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References

[1] *P. Erdős*: On family of symmetric Bernoulli convolutions. Amer. J. Math. *61* (1939), 974–976.
[2] *K. J. Falconer*: The Geometry of Fractal Sets. Cambridge University Press, Cambridge, 1985.
[3] *K. J. Falconer*: Fractal Geometry: Mathematical Foundations and Applications. John Wiley & Sons, Chichester, 1990.
[4] *K. J. Falconer*: Techniques in Fractal Geometry. John Wiley & Sons, Chichester, 1997.
[5] *D.-J. Feng, Y. Wang*: Bernoulli convolutions associated with certain non-Pisot numbers. Adv. Math. *187* (2004), 173–194.
[6] *A. M. Garsia*: Arithmetic properties of Bernoulli convolutions. Trans. Am. Math. Soc. *102* (1962), 409–432.
[7] *T.-Y. Hu*: Asymptotic behavior of Fourier transforms of self-similar measures. Proc. Am. Math. Soc. *129* (2001), 1713–1720.
[8] *J. E. Hutchinson*: Fractal and self similarity. Indiana Univ. Math. J. *30* (1981), 713–747.
[9] *P. E. T. Jorgensen, K. A. Kornelson, K. L. Shuman*: Affine systems: asymptotics at infinity for fractal measures. Acta Appl. Math. *98* (2007), 181–222.
[10] *K.-S. Lau, S.-M. Ngai, H. Rao*: Iterated function systems with overlaps and self-similar measures. J. Lond. Math. Soc., II. Ser. *63* (2001), 99–116.
[11] *J.-L. Li*: Singularity of certain self-affine measures. J. Math. Anal. Appl. *347* (2008), 375–380.
[12] *M. Niu, L.-F. Xi*: Singularity of a class of self-similar measures. Chaos Solitons Fractals *34* (2007), 376–382.
[13] *Y. Peres, W. Schlag, B. Solomyak*: Sixty years of Bernoulli convolutions. Fractal Geometry and Stochastics, II. Proc. 2nd Conf. (Greifswald/Koserow, Germany, 1998). Birkhäuser, Basel, 2000; Prog. Probab. *46* (2000), 39–65.
[14] *R. Salem*: Algebraic Numbers and Fourier Analysis. D. C. Heath and Company, Boston, 1963.
[15] *R. S. Strichartz*: Self-similarity in harmonic analysis. J. Fourier Anal. Appl. *1* (1994), 1–37.

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