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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 2, 531–539

Persistent URL: `http://dml.cz/dmlcz/141551`

Czechoslovak Mathematical Journal, 61 (136) (2011), 531–539

THE STRUCTURE OF THE UNIT GROUP OF THE GROUP ALGEBRA $\mathbb{F}_{2^k} A_4$

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(Received March 23, 2010)

Abstract. The structure of the unit group of the group algebra of the group A_4 over any finite field of characteristic 2 is established in terms of split extensions of cyclic groups.

Keywords: group ring, group algebra, dihedral group, cyclic group

MSC 2010: 16U60, 16S34, 20C05, 15A33

1. Introduction

Let $\mathscr{U}(KG)$ be the unit group of the group algebra KG of the field K over the group G. The homomorphism $\varepsilon\colon KG \longrightarrow K$ given by $\varepsilon\Big(\sum\limits_{g\in G} a_g g\Big) = \sum\limits_{g\in G} a_g$ is called the augmentation mapping of KG. The normalized unit group of KG denoted by $V(KG)$ consists of all the invertible elements of RG of augmentation 1. For further details on group algebras see [9].

It is well known that if G is a finite p-group and K is a field of characteristic p, then $V(KG)$ is a finite p-group of order $|K|^{|G|-1}$. Sandling in [10] provides a basis for $V(\mathbb{F}_p G)$ where G is an abelian p-group and $\mathbb{F}_p$ is the Galois field of p-elements. Let D_8 be the dihedral group of order 8. The structures of $\mathscr{U}(\mathbb{F}_2 D_8)$ and $\mathscr{U}(\mathbb{F}_{2^k} D_8)$ are established in [11] and [5], respectively.

The map $*\colon KG \longrightarrow KG$ defined by $\Big(\sum\limits_{g\in G} a_g g\Big)^* = \sum\limits_{g\in G} a_g g^{-1}$ is an antiautomorphism of KG of order 2. An element v of $V(KG)$ satisfying $v^{-1} = v^*$ is called unitary. We denote by $V_*(KG)$ the subgroup of $V(KG)$ formed by the unitary elements of KG. In [2] a basis for $V_*(KG)$ is constructed for any field of characteristic $p > 2$ and any finite abelian p-group.

The structure of $V_*(\mathbb{F}_2G)$ is established in [1] for all groups of order 8 and 16 and the structure of $V_*(\mathbb{F}_2Q_8)$ is established in [6] where Q_8 is the quaternion group of order 8. Additionally, the order of $V_*(\mathbb{F}_{2^k}G)$ is determined for special cases of G in [4]. In [3], Bovdi and Kovács give conditions for $V_*(KG)$ to be normal in $V(KG)$.

Let $M_n(R)$ be the ring of $n \times n$ matrices over a ring R. Using an isomorphism between RG and a subring of $M_n(R)$ and other techniques, we establish the structure of $\mathscr{U}(\mathbb{F}_{2^k}A_4)$ where A_4 is the group of even permutations on 4 elements. Our main result is

$$\mathscr{U}(\mathbb{F}_{2^k}A_4) \cong \begin{cases} [((C_2 \times {C_4}^2) \rtimes C_4) \rtimes C_4] \rtimes C_3 & \text{when } k = 1, \\ [[((C_2{}^k \times {C_4}^{2k}) \rtimes {C_4}^k) \rtimes {C_4}^k] \rtimes {C_{2^k-1}}^2] \times C_{2^k-1} & \text{when } 3 \mid (2^k - 1), \\ [[((C_2{}^k \times {C_4}^{2k}) \rtimes {C_4}^k) \rtimes {C_4}^k] \rtimes C_{2^{2k}-1}] \times C_{2^k-1} & \text{otherwise.} \end{cases}$$

In [12] it is shown that $V_1 = 1 + J(FA_4)$ is a nilpotent group of class 2 where J is the Jacobson Radical of FA_4 and F is any field of characteristic 2.

2. Background

Definition. A circulant matrix over a ring R is a square $n \times n$ matrix of the form

$$\operatorname{circ}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}$$

where $a_i \in R$.

Definition. Define the 2×2 circulant block matrix over a ring R to be

$$\mathrm{CB}_{2,2}(a, b, c, d) = \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{pmatrix}$$

where $a, b, c, d \in R$.

For further details on circulant matrices see Davis [7].

If $G = \{g_1, \ldots, g_n\}$, then denote the matrix $M(G) = ({g_i}^{-1}g_j)$ where $i, j = 1, \ldots, n$. Similarly, if $w = \sum_{i=1}^{n} \alpha_{g_i} g_i \in RG$, then denote the matrix $M(RG, w) = (\alpha_{{g_i}^{-1}g_j})$, which is called the RG-matrix of w.

Lemma 2.1 (see [8])**.** *Let G be a finite group of order n. There is a ring isomorphism between RG and the $n \times n$ G-matrices over R, which is given by $\sigma\colon\ w \mapsto M(RG, w)$.*

Definition. Define the alternating group A_4 to be the group of even permutations on 4 elements.

Example. Let $a = (12)(34)$, $b = (13)(24)$, $c = (123)$ and let

$$\kappa = \sum_{i=1}^{3} (\alpha_{4i-3} + \alpha_{4i-2}a + \alpha_{4i-1}b + \alpha_{4i}ab)c^{i-1} \in \mathbb{F}_{2^k}A_4,$$

where $\alpha_i \in \mathbb{F}_{2^k}$. Then

$$\sigma(\kappa) = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

where

$$\begin{aligned} A &= \mathrm{CB}_{2,2}(\alpha_1, \alpha_2, \alpha_3, \alpha_4), & B &= \mathrm{CB}_{2,2}(\alpha_5, \alpha_6, \alpha_7, \alpha_8), \\ C &= \mathrm{CB}_{2,2}(\alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}), & D &= \mathrm{CB}_{2,2}(\alpha_9, \alpha_{12}, \alpha_{10}, \alpha_{11}), \\ E &= \mathrm{CB}_{2,2}(\alpha_1, \alpha_4, \alpha_2, \alpha_3), & F &= \mathrm{CB}_{2,2}(\alpha_5, \alpha_8, \alpha_6, \alpha_7), \\ G &= \mathrm{CB}_{2,2}(\alpha_5, \alpha_7, \alpha_8, \alpha_6), & H &= \mathrm{CB}_{2,2}(\alpha_9, \alpha_{11}, \alpha_{12}, \alpha_{10}), \\ I &= \mathrm{CB}_{2,2}(\alpha_1, \alpha_3, \alpha_4, \alpha_2) \end{aligned}$$

where $\alpha_i \in \mathbb{F}_{2^k}$.

Let R_1 and R_2 be rings. Then $R_1 \oplus R_2$ is the direct sum of R_1 and R_2. It is well known that $\mathbb{F}_{p^k}C_3 \cong \mathbb{F}_{p^k} \oplus \mathbb{F}_{p^k} \oplus \mathbb{F}_{p^k}$ if $3 \mid (p^k - 1)$ and $\mathbb{F}_{p^k}C_3 \cong \mathbb{F}_{p^k} \oplus \mathbb{F}_{p^{2k}}$ if $3 \nmid (p^k - 1)$.

3. The Structure of $\mathscr{U}(\mathbb{F}_{2^k}A_4)$

Define the group epimorphism $\theta\colon\ \mathscr{U}(\mathbb{F}_{2^k}A_4) \longrightarrow \mathscr{U}(\mathbb{F}_{2^k}C_3)$ by

$$\sum_{i=1}^{3} (\alpha_{4i-3} + \alpha_{4i-2}a + \alpha_{4i-1}b + \alpha_{4i}ab)c^{i-1} \mapsto \sum_{i=1}^{4} \alpha_i + \sum_{j=1}^{4} \alpha_{j+4}x + \sum_{k=1}^{4} \alpha_{k+8}x^2$$

where $C_3 = \langle x \mid x^3 = 1\rangle$ and $\alpha_i \in \mathbb{F}_{2^k}$.

Define the group homomorphism $\psi\colon \mathscr{U}(\mathbb{F}_{2^k}C_3) \longrightarrow \mathscr{U}(\mathbb{F}_{2^k}A_4)$ by $\gamma+\beta x+\delta x^2 \mapsto \gamma+\beta c+\delta c^2$ where $\gamma,\beta,\delta\in\mathbb{F}_{2^k}$. Then

$$\theta\circ\psi(\gamma+\beta x+\delta x^2)=\theta(\gamma+\beta c+\delta c^2)=\gamma+\beta x+\delta x^2$$

where $\gamma,\beta,\delta\in\mathbb{F}_{2^k}$. Therefore, $\mathscr{U}(\mathbb{F}_{2^k}A_4)$ is a split extension of $\mathscr{U}(\mathbb{F}_{2^k}C_3)$ by $\ker(\theta)$.

Therefore, $\mathscr{U}(\mathbb{F}_{2^k}A_4)\cong H\rtimes\mathscr{U}(\mathbb{F}_{2^k}C_3)$ where $H\cong\ker(\theta)$. Let

$$\kappa=\sum_{i=1}^{3}(\alpha_{4i-3}+\alpha_{4i-2}a+\alpha_{4i-1}b+\alpha_{4i}ab)c^{i-1}\in\mathscr{U}(\mathbb{F}_{2^k}A_4),$$

then $\kappa\in H$ if and only if $\sum\limits_{i=1}^{4}\alpha_i=1$, $\sum\limits_{j=1}^{4}\alpha_{j+4}=0$, $\sum\limits_{l=1}^{4}\alpha_{l+8}=0$ where $\alpha_i\in\mathbb{F}_{2^k}$. Therefore, $|H|=(2^{3k})^3=2^{9k}$.

Lemma 3.1. *H has exponent 4.*

Proof. Let

$$h=1+\sum_{i=1}^{3}[\alpha_i+\alpha_{i+3}c+\alpha_{i+6}c^2+(\alpha_{3i-2}a+\alpha_{3i-1}b+\alpha_{3i}ab)c^{i-1}]\in H,$$

where $\alpha_i\in\mathbb{F}_{2^k}$. Then

$$\sigma(h^4)=\begin{pmatrix}A^4&0&0\\0&E^4&0\\0&0&I^4\end{pmatrix}$$

where $A=\mathrm{CB}_{2,2}((1+\alpha_1+\alpha_2+\alpha_3),\alpha_1,\alpha_2,\alpha_3)$, $E=\mathrm{CB}_{2,2}((1+\alpha_1+\alpha_2+\alpha_3),\alpha_3,\alpha_1,\alpha_2)$, $I=\mathrm{CB}_{2,2}((1+\alpha_1+\alpha_2+\alpha_3),\alpha_2,\alpha_3,\alpha_1)$ where $\alpha_i\in\mathbb{F}_{2^k}$.

It can be shown easily that if $M=\mathrm{CB}_{2,2}(\tau_1,\tau_2,\tau_3,\tau_4)$, then $M^4=\Big(\sum\limits_{i=1}^{4}{\tau_i}^4\Big)I_4$ where $\tau_i\in\mathbb{F}_{2^k}$. Therefore

$$A^4=(1+\alpha_1^3+\alpha_2^3+\alpha_3^3+\alpha_1^3+\alpha_2^3+\alpha_2^3)I_4=I_4=E^4=I^4.$$

Additionally, it can be shown easily that $h^2\neq1$. Therefore H has exponent 4. □

Lemma 3.2. *Let R be the subset of H consisting of elements of the form*

$$1+(1+a)(\alpha_1(1+b)+\alpha_2c+\alpha_3bc)+[\alpha_4(1+ab)+\alpha_5(a+b)]c^2$$

where $\alpha_i\in\mathbb{F}_{2^k}$. Then R is a group and $R\cong{C_2}^k\times{C_4}^{2k}$.

P r o o f. Let

$$r_1 = 1 + (1+a)(\alpha_1(1+b) + \alpha_2 c + \alpha_3 bc) + [\alpha_4(1+ab) + \alpha_5(a+b)]c^2 \in R$$

and

$$r_2 = 1 + (1+a)(\beta_1(1+b) + \beta_2 c + \beta_3 bc) + [\beta_4(1+ab) + \beta_5(a+b)]c^2 \in R$$

where $\alpha_i, \beta_j \in \mathbb{F}_{2^k}$. Then

$$\begin{aligned} r_1 r_2 = 1 &+ (1+a)((\alpha_1+\beta_1)(1+b) + (\alpha_2+\beta_2+\delta_1)c + (\alpha_3+\beta_3+\delta_1)bc) \\ &+ [(\alpha_4+\beta_4+\delta_2)(1+ab) + (\alpha_5+\beta_5+\delta_2)(a+b)]c^2 \end{aligned}$$

where $\delta_1 = (\alpha_4+\alpha_5)(\beta_4+\beta_5)$ and $\delta_2 = (\alpha_2+\alpha_3)(\beta_2+\beta_3)$. Therefore, R is closed under multiplication. Clearly $R \cong {C_2}^l \times {C_4}^m$ for some $l, m \in \mathbb{N}$.

Consider ${C_2}^l \times {C_4}^m$. The number of elements of order 2 or 1 is 2^{l+m} and the number of elements of order 4 is $2^{l+2m} - 2^{l+m} = 2^{l+m}(2^m - 1)$. Let

$$r = 1 + (1+a)(\alpha_1(1+b) + \alpha_2 c + \alpha_3 bc) + [\alpha_4(1+ab) + \alpha_5(a+b)]c^2 \in R,$$

where $\alpha_i \in \mathbb{F}_{2^k}$. Then $r^2 = 1$ if and only if $\alpha_2 = \alpha_3$ and $\alpha_4 = \alpha_5$. Therefore the number of elements of order 4 in R is $2^{5k} - 2^{3k} = 2^{3k}(2^{2k} - 1)$. Thus, $R \cong {C_2}^k \times {C_4}^{2k}$. □

Lemma 3.3. *Let S be the subset of H consisting of elements of the form*

$$1 + \alpha_1(1+b) + \alpha_2(1+a)(1+b)c + (\alpha_3 + \alpha_4 a)(1+b)c^2$$

where $\alpha_i \in \mathbb{F}_{2^k}$. Then S is a group and $S \cong {C_2}^{2k} \times {C_4}^k$.

P r o o f. Let

$$s_1 = 1 + \alpha_1(1+b) + \alpha_2(1+a)(1+b)c + (\alpha_3 + \alpha_4 a)(1+b)c^2 \in S$$

and

$$s_2 = 1 + \beta_1(1+b) + \beta_2(1+a)(1+b)c + (\beta_3 + \beta_4 a)(1+b)c^2 \in S$$

where $\alpha_i, \beta_j \in \mathbb{F}_{2^k}$. Then

$$\begin{aligned} s_1 s_2 = 1 &+ (\alpha_1+\beta_1)(1+b) + (\alpha_2+\beta_2+\delta_1)(1+a)(1+b)c \\ &+ ((\alpha_3+\beta_3+\delta_2) + (\alpha_4+\beta_4+\delta_2)a)(1+b)c^2 \end{aligned}$$

where $\delta_1 = (\alpha_3+\alpha_4)(\beta_3+\beta_4)$ and $\delta_2 = (\alpha_3+\alpha_4)\beta_1$. Therefore, S is closed under multiplication. Let

$$s = 1 + \alpha_1(1+b) + \alpha_2(1+a)(1+b)c + (\alpha_3 + \alpha_4 a)(1+b)c^2 \in S$$

where $\alpha_i \in \mathbb{F}_{2^k}$. Then $s^2 = 1$ if and only if $\alpha_3 = \alpha_4$. Thus the number of elements of order 4 in S is $2^{4k} - 2^{3k} = 2^{3k}(2^k - 1)$. Therefore $S \cong {C_2}^{2k} \times {C_4}^k$. □

Lemma 3.4. *Let T be the subset of H consisting of elements of the form*

$$1+(\alpha_1+\alpha_2 a)(1+b)+(1+a)(\alpha_3+\alpha_4 b)c+\Big(\sum_{i=1}^{3}\alpha_{i+4}+\alpha_5 a+\alpha_6 b+\alpha_7 ab\Big)c^2$$

where $\alpha_i \in \mathbb{F}_{2^k}$. Then $T \cong ({C_2}^k \times {C_4}^{2k}) \rtimes {C_4}^k$.

P r o o f. It can be shown easily that T is closed under multiplication. Clearly $R < T$ and $S < T$. Let

$$r = 1+(1+a)(\alpha_1(1+b)+\alpha_2 c+\alpha_3 bc)+[\alpha_4(1+ab)+\alpha_5(a+b)]c^2 \in R$$

and

$$s = 1+\beta_1(1+b)+\beta_2(1+a)(1+b)c+(\beta_3+\beta_4 a)(1+b)c^2 \in S$$

where $\alpha_i, \beta_j \in \mathbb{F}_{2^k}$. Then

$$\sigma(r^s) = \begin{pmatrix} A & B & C \\ D & A & E \\ F & G & A \end{pmatrix}$$

where $A = \mathrm{CB}_{2,2}(1+\alpha_1, \alpha_1, \alpha_1, \alpha_1)$, $B = \mathrm{CB}_{2,2}(\alpha_2+\delta_1, \alpha_2+\delta_1, \alpha_3+\delta_1, \alpha_3+\delta_1)$, $C = \mathrm{CB}_{2,2}(\alpha_4, \alpha_5, \alpha_5, \alpha_4)$, $D = \mathrm{CB}_{2,2}(\alpha_4, \alpha_4, \alpha_5, \alpha_5)$, $E = \mathrm{CB}_{2,2}(\alpha_2+\delta_1, \alpha_3+\delta_1, \alpha_2+\delta_1, \alpha_3+\delta_1)$, $F = \mathrm{CB}_{2,2}(\alpha_2+\delta_1, \alpha_3+\delta_1, \alpha_3+\delta_1, \alpha_2+\delta_1)$, $G = \mathrm{CB}_{2,2}(\alpha_4, \alpha_5, \alpha_4, \alpha_5)$ and $\delta_1 = (\alpha_4+\alpha_5)(\beta_3+\beta_4)$.

Clearly $r^s \in R$ and S normalizes R. Let

$$M = R \cap S = \{1+(1+a)(1+b)(uc+vc^2)\}$$

where $u, v \in \mathbb{F}_{2^k}$. By the second Isomorphism Theorem, $RS/R \cong S/R \cap S$. Now $|R \cap S| = 2^{2k}$. Therefore $|RS| = 2^{7k} = T$. Clearly S is an elementary abelian 2-group and therefore S completely reduces. Let $S \cong M \times W \cong {C_2}^{2k} \times {C_4}^k$. Clearly $W \cap R = \{1\}$ and W normalizes R. Thus, $T \cong R \rtimes W \cong ({C_2}^{2k} \times {C_4}^{2k}) \rtimes {C_4}^k$. $\square$

Lemma 3.5. *Let L be the subset of H consisting of elements of the form*

$$1+\alpha_1(1+ab)+(\alpha_2+\alpha_3 a)(1+b)c+\alpha_4(1+a)(1+b)c^2$$

where $\alpha_i \in \mathbb{F}_{2^k}$. Then L is a group and $L \cong {C_2}^{2k} \times {C_4}^k$.

P r o o f. Let

$$l_1 = 1+\alpha_1(1+ab)+(\alpha_2+\alpha_3 a)(1+b)c+\alpha_4(1+a)(1+b)c^2 \in L$$

and

$$l_2 = 1 + \beta_1(1+ab) + (\beta_2 + \beta_3 a)(1+b)c + \beta_4(1+a)(1+b)c^2 \in L$$

where $\alpha_i, \beta_j \in \mathbb{F}_{2^k}$. Then

$$\begin{aligned} l_1 l_2 = 1 &+ (\alpha_1+\beta_1)(1+ab) + ((\alpha_2+\beta_2+\delta_1) + (\alpha_3+\beta_3+\delta_1)a)(1+b)c \\ &+ (\alpha_4+\delta_4+\delta_2)(1+a)(1+b)c^2 \end{aligned}$$

where $\delta_1 = \alpha_1(\beta_2+\beta_3) + (\alpha_2+\alpha_3)\beta_1$ and $\delta_2 = (\alpha_2+\alpha_3)(\beta_2+\beta_3)$. Therefore L is closed under multiplication. It can be shown easily that the number of elements of order 4 in L is $2^{4k} - 2^{3k} = 2^{3k}(2^k-1)$. Therefore $L \cong {C_2}^{2k} \times {C_4}^{k}$. □

Lemma 3.6. *$H \cong (({C_2}^k \times {C_4}^{2k}) \rtimes {C_4}^k) \rtimes {C_4}^k$.*

P r o o f. Let

$$\begin{aligned} t = 1 &+ (\alpha_1+\alpha_2 a)(1+b) + (1+a)(\alpha_3+\alpha_4 b)c \\ &+ \left(\sum_{i=1}^{3} \alpha_{i+4} + \alpha_5 a + \alpha_6 b + \alpha_7 ab\right)c^2 \in T \end{aligned}$$

and

$$l = 1 + \beta_1(1+ab) + (\beta_2+\beta_3 a)(1+b)c + \beta_4(1+a)(1+b)c^2 \in L$$

where $\alpha_i, \beta_j \in \mathbb{F}_{2^k}$. Then

$$\sigma(t^l) = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

where

$$\begin{aligned} A &= \mathrm{CB}_{2,2}(1+\alpha_1, \alpha_2, \alpha_1, \alpha_2), \\ B &= \mathrm{CB}_{2,2}(\alpha_3+\delta_1, \alpha_3+\delta_1, \alpha_4+\delta_1, \alpha_4+\delta_1), \\ C &= \mathrm{CB}_{2,2}(\alpha_5+\alpha_6+\alpha_7+\delta_2, \alpha_5+\delta_2, \alpha_6+\delta_2, \alpha_7+\delta_2), \\ D &= \mathrm{CB}_{2,2}(\alpha_5+\alpha_6+\alpha_7+\delta_2, \alpha_7+\delta_2, \alpha_5+\delta_2, \alpha_6+\delta_2), \\ E &= \mathrm{CB}_{2,2}(1+\alpha_1, \alpha_2, \alpha_2, \alpha_1), \\ F &= \mathrm{CB}_{2,2}(\alpha_3+\delta_1, \alpha_4+\delta_1, \alpha_4+\delta_1, \alpha_3+\delta_1), \\ G &= \mathrm{CB}_{2,2}(\alpha_3+\delta_1, \alpha_4+\delta_1, \alpha_4+\delta_1, \alpha_3+\delta_1), \\ H &= \mathrm{CB}_{2,2}(\alpha_5+\alpha_6+\alpha_7+\delta_2, \alpha_6+\delta_2, \alpha_7+\delta_2, \alpha_5+\delta_2), \\ I &= \mathrm{CB}_{2,2}(1+\alpha_1, \alpha_1, \alpha_2, \alpha_2), \end{aligned}$$

$\delta_1 = (\alpha_3+\alpha_4)\beta_1 + (\alpha_1+\alpha_2)(\beta_2+\beta_3)$ and $\delta_2 = (\alpha_6+\alpha_7)\beta_1 + (\alpha_3+\alpha_4)(\beta_2+\beta_3)$.

Clearly $t^l \in T$ and L normalizes T. By the second Isomorphism Theorem, $TL = H$ and $L \cong M \times Q \cong {C_2}^{2k} \times {C_4}^k$. Clearly $T \cap Q = \{1\}$ and Q normalizes T. Therefore $H \cong T \rtimes Q \cong (({C_2}^k \times {C_4}^{2k}) \rtimes {C_4}^k) \rtimes {C_4}^k$. □

Theorem 3.1.

$$\mathscr{U}(\mathbb{F}_{2^k}A_4) \cong \begin{cases} [((C_2 \times {C_4}^2) \rtimes C_4) \rtimes C_4] \rtimes C_3 & \textit{when } k = 1, \\ [[(({C_2}^k \times {C_4}^{2k}) \rtimes {C_4}^k) \rtimes {C_4}^k] \rtimes {C_{2^k-1}}^2] \times C_{2^k-1} & \textit{when } 3 \mid (2^k - 1), \\ [[(({C_2}^k \times {C_4}^{2k}) \rtimes {C_4}^k) \rtimes {C_4}^k] \rtimes C_{2^{2k}-1}] \times C_{2^k-1} & \textit{otherwise.} \end{cases}$$

Proof. Recall that $\mathscr{U}(\mathbb{F}_{2^k}A_4) \cong H \rtimes \mathscr{U}(\mathbb{F}_{2^k}C_3)$. Now consider $\mathbb{F}_{2^k}C_3$.

1. Let $k = 1$. Using The LAGUNA package (V. Bovdi, A. Konovalov, C. Schneider: LAGUNA, Lie AlGebras and UNits of group Algebras (2003), http://www.gap-system.org/Packages/laguna.html) for the GAP system (GAP Groups, Algorithms, and Programming, Version 4.4.10. (2003), http://www.gap-system.org), it can be shown easily that $\mathscr{U}(\mathbb{F}_2C_3) \cong C_3$. Therefore

$$\mathscr{U}(\mathbb{F}_2A_4) \cong [((C_2 \times {C_4}^2) \rtimes C_4) \rtimes C_4] \rtimes C_3.$$

2. $\mathbb{F}_{2^k}C_3 \cong \mathbb{F}_{2^k} \oplus \mathbb{F}_{2^k} \oplus \mathbb{F}_{2^k}$ when $3 \mid (2^k - 1)$. Therefore

$$\begin{aligned} \mathscr{U}(\mathbb{F}_{2^k}A_4) &\cong [(({C_2}^k \times {C_4}^{2k}) \rtimes {C_4}^k) \rtimes {C_4}^k] \rtimes {C_{2^k-1}}^3 \\ &\cong [[(({C_2}^k \times {C_4}^{2k}) \rtimes {C_4}^k) \rtimes {C_4}^k] \rtimes {C_{2^k-1}}^2] \rtimes C_{2^k-1} \end{aligned}$$

since C_{2^k-1} corresponds to $\mathscr{U}(\mathbb{F}_{2^k})$.

3. $\mathbb{F}_{2^k}C_3 \cong \mathbb{F}_{2^k} \oplus \mathbb{F}_{2^{2k}}$ when $3 \nmid (2^k - 1)$. Therefore

$$\begin{aligned} \mathscr{U}(\mathbb{F}_{2^k}A_4) &\cong [(({C_2}^k \times {C_4}^{2k}) \rtimes {C_4}^k) \rtimes {C_4}^k] \rtimes (C_{2^k-1} \times C_{2^{2k}-1}) \\ &\cong [[(({C_2}^k \times {C_4}^{2k}) \rtimes {C_4}^k) \rtimes {C_4}^k] \rtimes C_{2^{2k}-1}] \rtimes C_{2^k-1} \end{aligned}$$

since C_{2^k-1} corresponds to $\mathscr{U}(\mathbb{F}_{2^k})$. □

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