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MOMENTS OF VECTOR MEASURES AND PETTIS INTEGRABLE FUNCTIONS

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Abstract. Conditions, under which the elements of a locally convex vector space are the moments of a regular vector-valued measure and of a Pettis integrable function, both with values in a locally convex vector space, are investigated.

Keywords: locally convex vector space, vector valued measure, Pettis integrable function, moments of such measures and functions

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1. Introduction

The Hausdorff moment problem [1], [6], [7], [9], [11] reads as follows: given a prescribed set of real numbers $\{a_n\}_0^\infty$, find a bounded non-decreasing function $u(t)$ on the closed interval $[0,1]$ such that its moments are equal to the prescribed values; that is,

$$\int_0^1 t^n \, \mathrm{d}u(t) = a_n, \qquad n = 0, 1, 2, \ldots.$$

The integral is a Riemann-Stieltjes integral. Equivalently, find a nonnegative measure μ on borelian subsets in $[0,1]$ with

$$\int_{[0,1]} t^n \, \mathrm{d}\mu(t) = a_n, \qquad n = 0, 1, 2, \ldots.$$

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We shall need the operator ∇^k $(k = 0, 1, 2, \ldots)$ defined by

$$\nabla^0 a_n = a_n, \quad \nabla^1 a_n = a_n - a_{n+1},$$
$$\nabla^k a_n = a_n - \binom{k}{1} a_{n+1} + \binom{k}{2} a_{n+2} - \ldots + (-1)^k a_{n+k}, \quad n = 1, 2, \ldots.$$

for any sequence of real or complex numbers $\{a_n\}_0^\infty$.

Note that we confine ourselves to the interval $[0, 1]$ for simplicity, this is, however, no limitation of generality, the results hold for any bounded interval $[a, b]$.

Definition 1.1. For each $f \in L_1([0, 1])$, the elements

$$a_n = \int_0^1 t^n f(t)\,\mathrm{d}t, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

are called the moments of f. For a measure μ on $[0, 1]$, the elements

$$a_n = \int_0^1 t^n \mu(\mathrm{d}t), \quad n = 0, 1, 2, \ldots \tag{1.2}$$

are called the moments of μ.

Put

$$l_{k,m} = \binom{k}{m} \nabla^{k-m} a_m \quad (k, m = 0, 1, 2, \ldots).$$

See [11] where $\Delta^k a_n = (-1)^k \nabla^k a_m$ is used.

For a sequence of (scalar, possibly vector) elements $\{a_n\}$, define an operator $L_N(t)[a]$ by

$$L_N(t)[a] = (1 + N) l_{N([Nt])}(a) \quad (N = 1, 2, \ldots).$$

$[Nt]$ means the largest integer contained in Nt. We have

$$\int_0^1 |L_N(t)[a]|\,\mathrm{d}t = \frac{N+1}{N} \sum_{m=0}^{N-1} |l_{N,m}(a)|, \quad N = 1, 2, \ldots.$$

If f is in $L_1([0, 1])$ (similarly a measure μ) and a_n are its moments, then

$$\lim_{N\to\infty} \int_0^1 t^n L_N(t)[a]\,\mathrm{d}t = a_n, \; n = 0, 1, \ldots. \tag{1.3}$$

Hence for every continuous function $g(t)$ there exists

$$\lim_{N\to\infty} \int_0^1 g(t) L_N(t)[a]\,\mathrm{d}t = \int_0^1 g(t)\,\mathrm{d}\mu(t).$$

If a_n are moments of f (similarly of μ), then

$$(1.4)\qquad \int_0^1 L_N(t)[a]\,\mathrm{d}t = (N+1)\int_0^1\int_0^1 \binom{N}{[Nt]} s^{[Nt]}(1-s)^{N-[Nt]} f(s)\,\mathrm{d}s\,\mathrm{d}t$$
$$= \int_0^1\int_0^1 K_N(t,s)\,\mathrm{d}t f(s)\,\mathrm{d}s \quad (N=1,2,\dots,\ 0\leqslant s,t\leqslant 1)$$

where

$$K_N(t,s) = (1+N)\binom{N}{[Nt]} s^{[Nt]}(1-s)^{N-[Nt]}\ .$$

It is easy to show that, for every t,

$$\int_0^1 K_N(t,s)\,\mathrm{d}s = \frac{N+1}{N}\sum_{m=0}^{N-1}\int_0^1 \binom{N}{m} s^m(1-s)^{N-m}\,\mathrm{d}s \leqslant M < \infty,$$

and, for every s,

$$(1.5)\qquad \int_0^1 K_N(t,s)\,\mathrm{d}t = \frac{N+1}{N}\sum_{m=0}^{N-1}\int_0^1 \binom{N}{m} s^m(1-s)^{N-m}\,\mathrm{d}t \leqslant M < \infty,$$

for $N=1,2,\dots$

We shall use a part of the following theorem (see [11], pp. 100–114, [4], Theorem 1) giving four equivalence statements.

Theorem 1.2. *Given a sequence a_n, $n=0,1,2,\dots$, of complex numbers, there exists*

(1a) *a function $f\in L_1$ such that a_n are the moments of f;*

(2a) *a function $f\in L_p$, $1<p\leqslant\infty$, such that a_n are the moments of f;*

(3a) *a complex, regular Borel measure μ such that a_n are the moments of μ;*

(4a) *a nonnegative regular Borel measure μ such that a_n are the moments of μ;*
if and only if the functions $L_k(t)\{a_n\}$

(1b) *converge in the L_1-norm;*

(2b) *are bounded in the L_p-norm;*

(3b) *are bounded in L_1-norm;*

(4b) *are nonnegative.*

2. Vector-valued measures

Let X be a quasi-complete, locally convex topological vector space. For each N, let $\Phi_N\colon C([0,1]) \to X$ be a linear mapping. The set of maps Φ_N is said to be weakly equi-compact if there is a weakly compact subset W of X such that

$$\{\Phi_N(\psi);\ \psi \in C([0,1]), \|\psi\| \leqslant 1,\ N = 1, 2, \ldots\} \subset W.$$

Let $\mathscr{B}([0,1])$ stand for the σ-algebra of all Borel sets in $[0,1]$.

Definition 2.1. Given a sequence a_n, $n = 0, 1, 2, \ldots$ of elements of X, we say that a_n, $n = 0, 1, 2, \ldots$ are the moments of a regular measure $\mu\colon \mathscr{B}([0,1]) \to X$ if a_n are of the form

$$a_n = \int_0^1 t^n \mu(\mathrm{d}t).$$

First we derive a necessary and sufficient condition for a sequence a_n, $n = 0, 1, 2, \ldots$ to be the moments of a regular measure μ.

Theorem 2.2. *Given a sequence a_n, $n = 0, 1, 2, \ldots$ of elements of X, there exists a regular measure $\mu\colon \mathscr{B}([0,1]) \to X$ such that a_n are the moments of μ if and only if the set of maps $\Phi_N\colon C([0;1]) \to X$, $N = 1, 2, \ldots$ defined by*

$$\Phi_N(\psi) = \int_0^1 \psi(t) L_N(t)\,\mathrm{d}t, \quad \psi \in C([0,1])$$

is weakly equi-compact.

P r o o f. Necessity. Suppose that such a measure exists. Then, for each $\psi \in C([0,1])$, see (1.4),

$$\Phi_N(\psi) = \int_0^1 \psi(t) \int_0^1 K_N(t,s)\mu(\mathrm{d}s)\,\mathrm{d}t = \int_0^1 \left(\int_0^1 K_N(t,s)\psi(t)\,\mathrm{d}t\right)\mu(\mathrm{d}s).$$

Let now $R(\mu) = \{\mu(A)\colon\ A \in \mathscr{B}([0,1])\}$, the range of μ, and let Q be the closed, absolutely convex hull of $R(\mu)$. Then $R(\mu)$ is relatively weakly compact in X (see [10]) and so by the Krein theorem (e.g., [10]), Q is weakly compact. Now, for all $\psi \in C([0,1]))$ with $\|\psi\| \leqslant 1$, we have, for $s \in [0,1]$, by (1.5),

$$\left|\int_0^1 K_N(t,s)\psi(t)\,\mathrm{d}t\right| \leqslant \|\psi\| \int_0^1 |K_N(t,s)|(t)\,\mathrm{d}t \leqslant M.$$

But for every measurable φ with $|\varphi(t)| \leqslant 1$ for all $t \in [0,1]$, we have

$$\int_0^1 \varphi(t)\mu(\mathrm{d}t) \in Q.$$

Therefore, $\Phi_N(\psi)$ is in MQ for all N and all $\psi \in C([0,1])$ with $\|\psi\| \leqslant 1$. That is, the set of Φ_N is weakly equi-compact.

Sufficiency. Suppose now that the set of Φ_N is weakly equi-compact. Then there exists a weakly compact subset W of X such that $\{\Phi_N(\psi)\colon\ \psi \in C([0,1]),\ \|\psi\| \leqslant 1,\ N = 1, 2, \ldots\} \subset W$. Take $x' \in X'$, the dual of X. Then there exists a constant $\alpha_{x'}$ such that

$$|\left\langle \Phi_N(\psi), x' \right\rangle| \leqslant \alpha_{x'}$$

for all N and all ψ with $\|\psi\| \leqslant 1$. Therefore, for each N,

$$\sup_{\|\psi\| \leqslant 1} \left| \int_0^1 \psi(t) \left\langle L_N(t), x' \right\rangle \, \mathrm{d}t \right| \leqslant \alpha_{x'};$$

that is

$$\int_0^1 |\left\langle L_N(t), x' \right\rangle| \, \mathrm{d}t \leqslant \alpha_{x'}.$$

Therefore, part (3a)–(3b) of Theorem 1.2 implies that there exists a scalar-valued measure $\mu_{x'}$ such that

$$\left\langle a_n, x' \right\rangle = \int_0^1 t^n \mu_{x'}(\mathrm{d}t) \tag{2.1}$$

and by (1.3),

$$\lim_N \left\langle \Phi_N(\psi), x' \right\rangle = \int_0^1 \psi(t) \mu_{x'}(\mathrm{d}t) \tag{2.2}$$

for all $\psi \in C([0,1])$. That is, for each fixed ψ, $\{\left\langle \Phi_N(\psi), x' \right\rangle\}$ is convergent for all $x' \in X'$. Thus $\{\Phi_N(\psi)\}$ is weakly Cauchy and therefore weakly convergent since $\{\Phi_N(\psi)\colon\ N = 1, 2, \ldots\}$ is contained in the weakly compact set $\|\psi\| W$. Denote its weak limit by $\Phi(\psi)$. Then, for all ψ with $\|\psi\| \leqslant 1$, $\Phi(\psi) \in W$. Since W is weakly compact, Φ is weakly compact (i.e., it takes the unit ball of $C([0,1])$ to a relatively weakly compact set). So, by extensions [3], [8] of a theorem of Bartle, Dunford and Schwartz, [2], Proposition l, to (quasi-complete) locally convex spaces, there exists a regular measure $\mu\colon\ \mathscr{B}([0,1]) \to X$ such that

$$\Phi(\psi) = \int_0^1 \psi(t) \mu(\mathrm{d}t)$$

for all $\psi \in C([0,1])$. Taking $\psi = t^n$ gives, for all $x' \in X'$,

$$\left\langle \Phi(t^n), x' \right\rangle = \int_0^1 t^n \left\langle \mu(\mathrm{d}t), x' \right\rangle.$$

But, by (2.1) and (2.2)

$$\langle \Phi(t^n), x' \rangle = \int_0^1 t^n \mu_{x'}(\mathrm{d}t) = \langle a_n, x' \rangle .$$

Hence

$$a_n = \int_0^1 t^n \mu(\mathrm{d}t).$$

□

Now we shall prove a theorem concerning moments of Pettis integrable functions. Recall that a function $f(t)\colon [a,b] \to X$ is called Pettis integrable if to every Borel set E in $[a,b]$ there corresponds an element $x_E \in X$ such that for all $x' \in X'$

$$x'(x_E) = \int_E x'[f(t)]\,\mathrm{d}t.$$

We then write

$$\int_E f(t)\,\mathrm{d}t = x_E.$$

In particular, if the space X is reflexive, then $f(t)$ is Pettis integrable if and only if $x'[f(t)] \in L_1$ for all $x' \in X'$.

If $f\colon [0.1] \to X$ is a Pettis (or Bochner) integrable function, then the members of the sequence a_n of elements in X are called the moments of f if they are of the form

$$a_n = \int_0^1 t^n f(t)\,\mathrm{d}t, \quad n = 1, 2, \ldots$$

Theorem 2.3. *Given a sequence a_n, $n = 0, 1, 2, \ldots$, of elements of X and a Pettis integrable function f, then a_n are the moments of f if and only if*

$$\lim_N \int_0^1 \psi(t)(L_N(t) - f(t))\,\mathrm{d}t = 0 \tag{2.3}$$

for all $\psi \in C([0,1])$ with $\|\psi\| \leqslant 1$.

Proof. Suppose that the a_n are the moments of f. Let V be an absorbent neighborhood of 0 in X. For all $\psi \in C([0,1])$,

$$\begin{aligned}\int_0^1 \psi(t)(L_N(t) - f(t))\,\mathrm{d}t &= \int_0^1 \psi(t)\Big(\int_0^1 K_N(t,s)f(s)\,\mathrm{d}s - f(t)\Big)\,\mathrm{d}t \\ &= \int_0^1 f(s)\Big(\int_0^1 K_N(t,s)\psi(t)\,\mathrm{d}t - \psi(s)\Big)\,\mathrm{d}s.\end{aligned}$$

Also, there exists a constant $\delta > 0$ such that for all γ with $|\gamma| < \delta$,

$$\int_0^1 \gamma f(s)\,\mathrm{d}s \in V.$$

But for each $\psi \in C([0,1])$ with $\|\psi\| \leqslant 1$, there exists an integer $N_0(\psi)$ such that for all $N > N_0(\psi)$,

$$\left| \int_0^1 K_N(t,s)\psi(t)\,\mathrm{d}t - \psi(s) \right| < \delta.$$

Hence, if ψ is in $C([0,1])$ with $\|\psi\| \leqslant 1$, $N > N_0(\psi)$ implies

$$\int_0^1 \psi(t)(L_n(t) - f(t))\,\mathrm{d}t \in V.$$

Conversely, define

$$\Phi_N(\psi) = \int_0^1 \psi(t)L_n(t)\,\mathrm{d}t, \quad N = 1, 2, \ldots,$$
$$\Phi(\psi) = \int_0^1 \psi(t)f(t)\,\mathrm{d}t, \quad \psi \in C([0,1]),$$

and suppose that $\lim_N \Phi_N(\psi) = \Phi(\psi)$ for all $\psi \in C([0,1])$ with $\|\psi\| \leqslant 1$. Then, for all $x' \in X'$ and all such ψ,

$$\lim_N \langle \Phi_N(\psi), x' \rangle = \langle \Phi(\psi), x' \rangle\,.$$

So, for every $x' \in X'$,

$$\langle a_n, x' \rangle = \lim_N l_{N[Nt]} \langle a_n, x' \rangle = \lim_N \int_0^1 t^n \langle L_N(t), x' \rangle \,\mathrm{d}t = \left\langle \int_0^1 t^n f(t)\,\mathrm{d}t, x' \right\rangle$$

and hence

$$a_n = \int_0^1 t^n f(t)\,\mathrm{d}t.$$

□

For the sake of completeness we shall introduce here the theorems concerning the problems of moments for vector measures with finite variation and Bochner integrable functions with values in a Banach space X ([4]).

Theorem 2.4. *Given a sequence a_n, $n = 0, 1, 2, \dots$ of elements of X, there exists a regular measure $\mu\colon \mathscr{B}([0,1]) \to X$ of finite total variation such that a_n are the moments of μ if and only if there exists a finite constant H such that*

$$\int_0^1 \|L_N(t)\| \, \mathrm{d}t \leqslant H, \quad N = 1, 2, \dots$$

Recall the definition of a Bochner integrable function. If $f\colon [a,b] \to X$ is simple, i.e., $f(s) = \sum_{i=1}^{n} \chi_{E_i}(s) x_i$, where χ_E denotes the indicator function of the set $E \subset [a,b]$, $x_i \in X$, then for any $E \in \mathscr{B}([a,b])$

$$\int_E f \, \mathrm{d}\lambda = \sum_{i=1}^{n} \lambda(E \cap E_i) x_i,$$

where λ is a probability measure on $[a,b]$. Such functions are λ-measurable. Any function $f\colon [a,b] \to X$ which is the λ-almost everywhere limit of a sequence of simple functions is (called) λ-measurable.

A λ-measurable function $f\colon [a,b] \to X$ is called Bochner integrable if there exists a sequence of simple functions (f_n) such that

$$\lim \int_{[a,b]} \|f_n(s) - f(s)\| \, \mathrm{d}\lambda(s) = 0.$$

In this case $\int_E f \, \mathrm{d}\lambda$ is defined for each measurable set E in $[a,b]$ by

$$\int_E f \, \mathrm{d}\lambda = \lim_n \int_E f_n \, \mathrm{d}\lambda.$$

A λ-measurable function $f\colon [a,b] \to X$ is Bochner integrable if and only if $\int_{[a,b]} \|f\| \, \mathrm{d}\lambda < \infty$ ([5]).

Theorem 2.5. *Given a sequence a_n, $n = 0, 1, 2, \dots$, of elements of X, there exists an X-valued Bochner integrable function f on $[0,1]$ such that a_n are the moments of f if and only if*

$$\lim_{N,J \to \infty} \int_0^1 \|L_N(t)(a) - L_J(t)(a)\| \, \mathrm{d}t = 0.$$

References

[1] *Ch. Berg, J. P. P. Christensen and P. Ressel*: Harmonic Analysis on Semigroups. Springer-Verlag, Berlin, Germany, 1954.
[2] *R. G. Bartle, N. Dunford and J. C. Schwartz*: Weak compactness and vector measures. Canad. J. Math. *7* (1955), 289–305.
[3] *C. Debieve*: Integration par rapport a une mesure vectorielle. Ann. de la Societé Scientif. de Bruxelles. *11* (1973), 165–185.
[4] *M. Duchoň, C. Debieve*: Moments of vector-valued functions and measures. Tatra Mt. Math. Publ. *42* (2009), 199–210.
[5] *N. Dunford, J. T. Schwartz*: Linear Operators. Part I, Interscience, New York, USA, 1966.
[6] *F. Hausdorff*: Summationsmethoden und Momentenfolgen II. Math. Z. *9* (1921), 280–299.
[7] *F. Hausdorff*: Momentprobleme fuer ein endliches Interval. Math. Z. *16* (1923), 220–248.
[8] *D. R. Lewis*: Integration with respect to vector measures. Pac. J. Math. *33* (1970), 157–165.
[9] *G. G. Lorentz*: Bernstein Polynomials. Toronto University Press, Toronto, Canada, 1953.
[10] *I. Tweddle*: Weak compactness in locally convex spaces. Glasgow Math. J. *9* (1968), 123–127.
[11] *D. V. Widder*: The Laplace Transform. Princeton University Press, Princeton, N.J., 1941.

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