Tejinder S. Neelon
On Boman's theorem on partial regularity of mappings

Commentationes Mathematicae Universitatis Carolinae, Vol. 52 (2011), No. 3, 349--357

Persistent URL: http://dml.cz/dmlcz/141607

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 2011

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library
http://project.dml.cz
On Boman’s theorem on partial regularity of mappings

Tejinder S. Neelon

Abstract. Let $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$ and $k$ be a positive integer. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally bounded map such that for each $(\xi, \eta) \in \Lambda$, the derivatives $D^j_{\xi} f(x) := \frac{d}{dt} f(x + t\xi) \bigg|_{t=0}, j = 1, 2, \ldots k$, exist and are continuous. In order to conclude that any such map $f$ is necessarily of class $C^k$ it is necessary and sufficient that $\Lambda$ be not contained in the zero-set of a nonzero homogenous polynomial $\Phi(\xi, \eta)$ which is linear in $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ and homogeneous of degree $k$ in $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.

This generalizes a result of J. Boman for the case $k = 1$. The statement and the proof of a theorem of Boman for the case $k = \infty$ is also extended to include the Carleman classes $C\{M_k\}$ and the Beurling classes $C(M_k)$ (Boman J., Partial regularity of mappings between Euclidean spaces, Acta Math. 119 (1967), 1-25).

Keywords: $C^k$ maps, partial regularity, Carleman classes, Beurling classes

Classification: 26B12, 26B35

A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ that is differentiable when restricted to arbitrary differentiable curves is not necessarily differentiable as a function of several variables (see [12]). Indeed, there are discontinuous functions $f : \mathbb{R}^n \to \mathbb{R}$ whose restrictions to arbitrary analytic arcs are analytic [2]. But a $C^\infty$ function $f : \mathbb{R}^n \to \mathbb{R}$ whose restriction to every line segment is real analytic is necessarily real analytic ([13]). In [8], [9], [10] and [11] this result was extended by considering restrictions to algebraic curves and surfaces of functions belonging to more general classes of infinitely differentiable functions. It is also well known that a function $f : \mathbb{R}^n \to \mathbb{R}$ that is infinitely differentiable in each variable separately may be no better than measurable ([7]). In [4], the obverse problem is considered; for vector valued functions hypothesis is made on the source as well as the target space. In this note, Theorem 4 of [4] is generalized to $C^k, k \geq 1$, the class of functions that have continuous derivatives up to order $k$.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally bounded map. For $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$, set

$$D^j_{\xi} \langle f, \eta \rangle (x) := \frac{d}{dt} \langle f(x + t\xi), \eta \rangle \bigg|_{t=0} \quad \text{in the sense of distributions},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^m$. By the Leibniz Integral rule, we have

$$\frac{d}{dt} \int f(x + t\xi), \eta \rangle dx = \int \frac{d}{dt} \langle f(x + t\xi), \eta \rangle dx.$$
Let \( k, 1 \leq k < \infty \), be fixed. For \( \xi \in \mathbb{R}^n \), denote by \( C^k_\xi(\mathbb{R}^n) \) the space of all continuous functions \( f : \mathbb{R}^n \to \mathbb{R} \) such that the derivatives \( D^j_\xi f(x) := \frac{d^j}{dt^j} f(x + t\xi) \big|_{t=0}, \ j = 1, 2, \ldots, k \), exist and are continuous. Similarly, \( C^\infty(\mathbb{R}^n) := \bigcap_{k=0}^\infty C^k(\mathbb{R}^n) \).

We are interested in finding the necessary and sufficient conditions on a subset \( \Lambda \subset \mathbb{R}^n \times \mathbb{R}^m \) to have the following property:

\[
\text{if } f : \mathbb{R}^n \to \mathbb{R}^m \text{ is locally bounded such that } (f, \eta) \in C^k_\xi(\mathbb{R}^n), \forall (\xi, \eta) \in \Lambda, \text{ then } f \in C^k(\mathbb{R}^n).
\]

The case \( k = 1 \) and \( k = \infty \) was dealt in [4].

Let \( Z^n_+ \) denote all \( n \)-tuples of nonnegative integers. For \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in Z^n_+ \), set \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \). The set \( Z^n_+ \) of multi-indices is assumed to be ordered lexicographically i.e. for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in Z^n_+ \), define \( \alpha < \beta \) if there is \( i, 1 \leq i \leq n \), such that \( \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i \).

Let \( K_n = \binom{k+n-1}{k} \) denote the number of monomials of degree \( k \) in \( n \) variables.

Then for any \( \varphi \in C^\infty(\mathbb{R}^n) \), we have

\[
\int D_\xi \langle f, \eta \rangle (x) \varphi(x) \, dx = \frac{d}{dt} \int \langle f(x + t\xi), \eta \rangle \varphi(x) \, dx \bigg|_{t=0} = \frac{d}{dt} \langle f(x + t\xi) \varphi(x) \rangle \bigg|_{t=0} = -\sum_i \xi_i \langle f(x) \partial_i \varphi(x - t\xi) \rangle \bigg|_{t=0} = \sum_{i,j} \xi_i \eta_j \int \partial_j f_j(x) \varphi(x) \, dx.
\]

By iteration, we obtain the formula for higher-order distributional derivatives:

\[
D^p_\xi \langle f, \eta \rangle (x) = \sum_{|\alpha| = p} \sum_{j=1}^m \xi^\alpha \eta_j \partial^\alpha f_j(x).
\]

Let

\[
\mathcal{B}_k := \left\{ \Phi(\xi, \eta) = \sum_{j=1}^m \sum_{|\alpha| = k} \varphi_{\alpha j} \xi^\alpha \eta_j : \varphi_{\alpha j} \in \mathbb{R}, \alpha \in Z^n_+, j \in Z_+ \right\}.
\]

For any function \( \Phi(\xi, \eta) \), set \( \|\Phi\| := \max_{|\xi| \leq 1, |\eta| \leq 1} |\Phi(\xi, \eta)| \). For a subset \( K \subset \subset \Lambda, \subset \subset \text{ denotes the compact inclusion} \) put \( \|\Phi\|_K := \max_{(\xi, \eta) \in K} |\Phi(\xi, \eta)| \).

**Theorem 1.** Let \( \Lambda \subset \mathbb{R}^n \times \mathbb{R}^m \) be a subset and \( k \) be a positive integer. The following conditions are equivalent:

(i) \( \Lambda \) is not contained in an algebraic hypersurface defined by an element of \( \mathcal{B}_k \) i.e.

\[
\Phi \in \mathcal{B}_k, \Phi|_{\Lambda} \equiv 0 \Rightarrow \Phi \equiv 0;
\]
(ii) there exists a set consisting of \( m \cdot k_n \) points

\[
(\xi^*, \eta^*) = \left\{ (\xi^{(p)}, \eta^{(p)}) \in \Lambda, \ p = 1, 2, \ldots, mk_n \right\}
\]
such that \( \det \Delta(\xi^*, \eta^*) \neq 0 \), where

\[
\Delta(\xi^*, \eta^*) := \left[ \left( \xi^{(p)} \right)^{\alpha_j} \eta^{(p)} \right]_{|\alpha|=k, 1 \leq j \leq m, 1 \leq p \leq mk_n}
\]

(iii) if \( f : \mathbb{R}^n \to \mathbb{R}^m \) is locally bounded and \( \langle f, \eta \rangle \in C^k(\mathbb{R}^n), \ \forall (\xi, \eta) \in \Lambda \), then \( f \in C^k(\mathbb{R}^n, \mathbb{R}^m) \).

If any one of the above equivalent conditions is satisfied, then there exists a constant \( B \) depending only on \( \Lambda \) such that the following inequality holds for all locally bounded maps \( f : \mathbb{R}^n \to \mathbb{R}^m \):

\[
(2) \quad \max_{1 \leq j \leq m} \max_{|\alpha|=k} |\partial^\alpha f_j(x)| \leq B \cdot \sup_{(\xi, \eta) \in \Lambda} |D^k \langle f, \eta \rangle(x)|, \forall x \in \mathbb{R}^n.
\]

**Proof:** We will prove (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (i).

(i) \( \Rightarrow \) (ii). Suppose \( \det \Delta(\xi^*, \eta^*) = 0 \) for every set of \( mk_n \) elements \( (\xi^*, \eta^*) = \{ (\xi^{(p)}, \eta^{(p)}) \}_{1 \leq p \leq mk_n} \) in \( \Lambda \). Fix one such set \( (\xi^*, \eta^*) \) so that the rank \( l := \text{rank} \Delta(\xi^*, \eta^*) \) is positive. Let \( \Delta^{(l)} \) denote some \( l \times l \) submatrix of \( \Delta(\xi^*, \eta^*) \) such that the minor \( \det \Delta^{(l)} \) is nonzero. Let \( \Delta^{(l+1)} \) be a \((l+1) \times (l+1)\) submatrix of \( \Delta(\xi^*, \eta^*) \) that contains \( \Delta^{(l)} \) as a submatrix. Replace the point \( (\xi^{(p)}, \eta^{(p)}) \) in \( \Delta^{(l+1)} \) which does not appear in \( \Delta^{(l)} \) by variables \( (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m \). By expanding \( \Delta^{(l+1)} \) along the row where the replacement took place we obtain an element

\[
\Phi(\xi, \eta) = \sum_{\alpha, j} \varphi_{\alpha j} \xi^\alpha \eta_j,
\]

of \( B_k \) which is nonzero since one of its coefficients coincides with \( \det \Delta^{(l)} \) up to a sign.

Since \( \det \Delta(\xi^*, \eta^*) \) has rank \( l \), we find that \( \Phi(\xi, \eta) = 0 \) for all \( (\xi, \eta) \in (\xi^*, \eta^*) \). If \( \Phi(\xi, \eta) = 0 \) for all \( (\xi, \eta) \in \Lambda \), we are done. Otherwise, choose a point \( (\xi, \eta) \in \Lambda \setminus (\xi^*, \eta^*) \) with \( \Phi(\xi, \eta) \neq 0 \).

Let \( (\xi^*, \eta^*) \) be the set which is obtained from \( (\xi^*, \eta^*) \) by replacing the point \( (\xi^{(p)}, \eta^{(p)}) \) by \( (\xi, \eta) \). Then, the rank \( \Delta(\xi^*, \eta^*) \geq l + 1 \). By repeating above procedure, we find a sequence of subsets \( (\xi^*, \eta^*)^{(i)} \subset \Lambda, i = 1, 2, 3, \ldots, \) each with \( mk_n \) elements such that the rank \( \Delta(\xi^*, \eta^*)^{(j)} \) is a strictly increasing sequence of nonnegative integers. After finitely many steps we obtain a nonzero element of \( B_k \) which vanishes on the entire \( \Lambda \).


(ii)⇒(iii). Let \((\xi^*,\eta^*) = \{(\xi^{(p)},\eta^{(p)}) \in \Lambda\}_{1 \leq p \leq mk}^\Lambda\) be a set of points such that 
\[
\det \Delta(\xi^*,\eta^*) \neq 0.
\]
By applying Cramer’s rule to (1), we get
\[
\partial^\alpha f_j(x) = \sum_{p=1}^{mk} \frac{\det \Delta_{\alpha j}^{(p)}}{\det \Delta} D_{\xi^{(p)}}^j \langle f,\eta^{(p)} \rangle(x) \quad \text{in the distributional sense},
\]
where \(\Delta_{\alpha j}^{(p)}\) denotes the cofactor obtained by deleting the \((\alpha,j)\)-th row and the \(p\)-th column. Since \(D_{\xi}^j(f,\eta) \in C^0\) for all \((\xi,\eta) \in \Lambda\), we have
\[
\partial^\alpha f_j(x) = \sum_{p=1}^{mk} \frac{\det \Delta_{\alpha j}^{(p)}}{\det \Delta} D_{\xi^{(p)}}^j \langle f,\eta^{(p)} \rangle(x) \in C^0.
\]
Furthermore, there exists a constant \(B = B(k,f,\Lambda)\) such that
\[
|\partial^\alpha f_j(x)| \leq \sum_{p=1}^{mk} \frac{|\det \Delta_{\alpha j}^{(p)}}{\det \Delta} \left| D_{\xi^{(p)}}^j \langle f,\eta^{(p)} \rangle(x) \right| \leq B \sup_{(\xi,\eta) \in \Lambda} \left| D_{\xi}^j \langle f,\eta \rangle(x) \right|,
\]
for all \(\alpha\) with \(|\alpha| = k\), and all \(j = 1,2,\ldots,m\).

(iii)⇒(i). Suppose (i) does not hold. Let \(\Phi \in B_k\) be such that \(\Phi|\Lambda = 0\.

We can write \(\Phi(\xi,\eta) = \langle \varphi(\xi),\eta \rangle\), where \(\varphi(\xi) := (\varphi_1(\xi),\varphi_2(\xi),\ldots,\varphi_m(\xi))\) and \(\varphi_j(\xi) = \sum_{|\alpha|=k} \varphi_{\alpha_j} \xi^\alpha, j = 1,2,\ldots,m\), homogeneous polynomials of degree \(k\).

Define the map
\[
f(x) := \begin{cases} 
(\ln|\ln|x||)\varphi,(x) & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]
Clearly \(f \notin C^k\) and \(f\) is \(C^\infty\) in \(\{x \in \mathbb{R}^n : 0 < |x| < 1\}\). We will prove that \(D_{\xi}^j \langle f(x),\eta \rangle\) exists at \(x = 0\), for all \((\xi,\eta) \in \Lambda\). It is easy to see that here are constants \(C_{\alpha}\) such that
\[
|\partial^\alpha \ln|\ln|x||| \leq \frac{C_{\alpha}}{|x|^{|\alpha|}|\ln|x||}, \forall \alpha,|\alpha| \geq 1.
\]

Since the \(\varphi_j(x)\)'s are homogeneous polynomials of degree \(k\), when the Leibniz’s formula is applied to the products \(\langle \ln|\ln|x||\rangle \varphi_j(x)\), it is clear that all terms in \(D_{\xi}^j \langle f(x),\eta \rangle\), \(1 \leq p \leq k\), except possibly
\[
(\ln|\ln|x||) \langle D_{\xi}^k \varphi(x),\eta \rangle
\]
tend to 0 as \(x \to 0\). We only need to prove that the function in (3) also tends to 0 as \(x \to 0\). By expanding \((x_1 + t\xi_1)^{n_1}(x_2 + t\xi_2)^{n_2} \cdots (x_n + t\xi_n)^{n_n}\) binomially, we can write
\[
\varphi(x + t\xi) := \varphi(x) + P(x,\xi,t) + \varphi(\xi)t^k.
\]
But since \((\xi, \eta) \in \Lambda\),
\[
\langle D_\xi^k \varphi(x), \eta \rangle = k! \langle \varphi(\xi), \eta \rangle = 0.
\]

It follows that \(|D_\xi^p (f(0), \eta)| = 0\) for \(p \leq k\). Thus, \(f \in C_\xi^k\) for all \((\xi, \eta) \in \Lambda\), but \(f \notin C^k\).

\[\square\]

**Remark 1** (cf. [6]). Suppose (i) is satisfied for all \(k \geq 0\). It would be of interest to know whether there exists a constant \(\rho = \rho(\Lambda)\), depending only on some appropriate notion of capacity of \(\Lambda\), so that (2) is satisfied with \(B = (\rho(\Lambda))^{-k}\) for all \(f\) and all \(k\).

**Remark 2.** Suppose \(\Lambda\) satisfies (i) or (ii). The proof of Theorem 1 shows that if \(f\) is continuous and \(D_\xi^k (f, \eta) = 0\), \(\forall (\xi, \eta) \in \Lambda\), then \(f\) is a polynomial. The assumption of continuity of \(f\) is not necessary but our proof is valid only if \(f\) is continuous (see [4]).

**Remark 3.** If \(\Lambda\) satisfies (i), then \(\Lambda\) contains at least \(mk_n\) elements. Furthermore, if (i) holds for \(k\) then (i) also holds for all \(j \leq k\). Suppose there exists \(\Phi \in B_j\), \(j < k\) such that \(\Phi \equiv 0\) but \(\Phi \not\equiv 0\). Then, \(\xi_1^{k-j} \Phi \in B_k\), \(\xi_1^{k-j} \Phi \equiv 0\) but this is a contradiction.

Let \(\{M_k\}_{k=0}^\infty\) be a sequence of nonnegative numbers. For \(h > 0\) and \(K \subset \subset \mathbb{R}^n\) define the seminorm on \(C^\infty(\mathbb{R}^n)\),
\[
|D^{\alpha} f(x)| \cdot M_{|\alpha|}.
\]

The spaces
\[
C \{M_k\} = \{f \in C^\infty(\mathbb{R}^n) : \forall K \subset \subset \mathbb{R}^n, \exists h > 0, \text{ s.t. } p_{h,K}(f) < \infty\}
\]
and
\[
C (M_k) = \{f \in C^\infty(\mathbb{R}^n) : p_{h,K}(f) < \infty, \forall K \subset \subset \mathbb{R}^n, \forall h > 0\}
\]
are called the Carleman and Beurling classes, respectively. The classes \(C \{(k!)^\nu\}, \nu > 1\), known as Gevrey classes, are especially important in partial differential equations and harmonic analysis. The class \(C \{k!\}\) is precisely the class of real analytic functions.

We assume that
\[
M_0 = 1 \quad \text{and} \quad M_k \geq k!, \forall k; \quad (4)
\]
and
\[
M_k^{1/k} \quad \text{is strictly increasing; (5)}
\]
$\exists C > 0$ such that $M_{k+1} \leq C^k M_k$, $\forall k.$

These conditions insure that the classes $C\{M_k\}$ and $C(M_k)$ are nontrivial and are closed under product and differentiation of functions. For more properties of these spaces, see [5], [11] and references therein.

It is well known that $f \in C^\infty(\mathbb{R}^n)$ if and only if $\sup_{\xi \in \mathbb{R}^n} |\xi| |\hat{f}(\xi)| < \infty$, $\forall \chi \in C_c^\infty(\mathbb{R}^n)$, $j \geq 1$. A similar characterization is also available for $C\{M_k\}$ (see [5]) a routine modification of which yields an analogous characterization of $C(M_k)$.

Let $r > 0$. Choose a sequence of cut-off functions $\chi_j(x) \in C_c^\infty$, $j = 1, 2, \ldots$, such that $\chi_j(x) = 1$ if $|x - x_0| < r$, $\chi_j(x) = 0$ if $|x - x_0| > 3r$ and

$$|\partial^\alpha\chi_j(x)| \leq (C_1j)^{|\alpha|}, \forall j, \forall |\alpha| \leq j, \forall x,$$

where the constant $C_1$ is independent of $j$. Then $f \in C\{M_k\}$ (resp. $C(M_k)$) in a neighborhood of $x_0 \in \mathbb{R}^n$ if and only if there exists a constant $h > 0$ (resp. for every $h > 0$) such that

$$\sup_{\xi \in \mathbb{R}^n} \sup_{j \geq 1} h^{-j} M_j^{-1} |\xi| |\hat{f}(\chi_j)(\xi)| < \infty.$$

Call a subset $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$ a determining set for bilinear forms of rank 1 if there is no nonzero bilinear form $\varphi(\xi, \eta), \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^m$ of rank 1 such that $\varphi(\xi, \eta) = 0$ for all $(\xi, \eta) \in \Lambda$.

Clearly $\Lambda$ is a determining set for bilinear forms of rank 1 if and only if

$$\langle u, \xi \rangle \langle v, \eta \rangle = 0, \forall (\xi, \eta) \in \Lambda \Rightarrow |u||v| = 0$$

(here $\langle u, \xi \rangle$ and $\langle v, \eta \rangle$ are dot products on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively), or equivalently,

$$\bigcap_{(\xi, \eta) \in \Lambda} \{ (u, v) \in \mathbb{R}^n \times \mathbb{R}^m : \langle u, \xi \rangle \langle v, \eta \rangle = 0 \} = (\mathbb{R}^n \times 0) \cup (0 \times \mathbb{R}^m).$$

Since $\mathbb{R}[u, v]$ is a Noetherian ring, $\Lambda$ contains a finite subset $\Lambda'$ such that the sets $\{ \langle u, \xi \rangle \langle v, \eta \rangle : (\xi, \eta) \in \Lambda \}$ and $\{ \langle u, \xi \rangle \langle v, \eta \rangle : (\xi, \eta) \in \Lambda' \}$ generate the same ideal in $\mathbb{R}[u, v]$ and thus define the same varieties:

$$\bigcap_{(\xi, \eta) \in \Lambda} \{ (u, v) \in \mathbb{R}^n \times \mathbb{R}^m : \langle u, \xi \rangle \langle v, \eta \rangle = 0 \} = \bigcap_{(\xi, \eta) \in \Lambda'} \{ (u, v) \in \mathbb{R}^n \times \mathbb{R}^m : \langle u, \xi \rangle \langle v, \eta \rangle = 0 \}.$$

Thus, any determining set for bilinear forms of rank 1 contains a finite determining set for bilinear forms of rank 1.
Let \( C\{M_k\}\{\xi\} \) (resp. \( C(M_k)\{\xi\} \)) denote the set of all \( f \in C^\infty(\mathbb{R}^n) \) such that for every subset \( K \subset \mathbb{R}^n \), \( \sup_{j,x \in K} |D_j^l f(x)| h^{-j} M_j^{-1} < \infty, \forall j, \) for some \( h > 0 \) (resp. for every \( h > 0 \)).

**Theorem 2.** Let \( \{M_k\}_{k=0}^\infty \) be a sequence of nonnegative numbers satisfying the conditions (4), (5) and (6). The following statements are equivalent:

(i) \( \Lambda \) is a determining set for bilinear forms of rank 1;

(ii) for any locally bounded map \( f : \mathbb{R}^n \to \mathbb{R}^m \),

\[ \langle \eta, f \rangle \in C \{M_k\}\{\xi\}, \forall (\eta, \xi) \in \Lambda \Rightarrow f \in C \{M_k\} ; \]

(iii) for any locally bounded map \( f : \mathbb{R}^n \to \mathbb{R}^m \),

\[ \langle \eta, f \rangle \in C (M_k)\{\xi\}, \forall (\eta, \xi) \in \Lambda \Rightarrow f \in C (M_k) ; \]

(iv) for any locally bounded map \( f : \mathbb{R}^n \to \mathbb{R}^m \),

\[ \langle \eta, f \rangle \in C^\infty (\xi), \forall (\eta, \xi) \in \Lambda \Rightarrow f \in C^\infty . \]

**Proof:** (cf. Theorem 4 in [4]) Assume (i) holds. By the remark above, by replacing \( \Lambda \) by a subset, if necessary, we may assume \( \Lambda \) is finite. Suppose for every \( \eta, \xi \in \Lambda \), \( \langle \eta, f \rangle \in C(M_k)(\xi) \) (resp. \( \langle \eta, f \rangle \in C(M_k)(\xi) \)). Now for a suitable function \( f \),

\[ \langle \xi, z \rangle \langle \eta, f(z) \rangle = \langle \xi, z \rangle \langle \eta, f(z) \rangle = \left\langle \eta, i \int [\langle \xi, \partial_x \rangle e^{-i(x,z)} f(x) \, dx \right\rangle \]

\[ = \left\langle \eta, -i \int e^{-i(x,z)} \langle \xi, \partial_x f \rangle (x) \, dx \right\rangle = \left\langle \eta, -i \int e^{-i(x,z)} D_\xi f(x) \, dx \right\rangle . \]

Let \( g_{(j)} := f(x) \in C\{M_k\} \) near a fixed point \( x_0 \). Assume, without loss of generality, \( x_0 = 0 \). By assumption, for all \( (\xi, \eta) \in \Lambda \) there exist constants \( C = C_{\xi,\eta} \) and \( h = h_{\xi,\eta} > 0 \) (resp. for all \( (\xi, \eta) \in \Lambda \) and for all \( h > 0 \) there exists a constant \( C = C_{\xi,\eta,h} \) such that

\[ \left| \left\langle \eta, g_{(j)} \right\rangle (\xi) \right| \leq \left| \left\langle \eta, g_{(j)} \right\rangle (\xi) \right| \leq C h^j M_j, \]

\[ \forall (\xi, \eta) \in \Lambda, \xi \in \mathbb{R}^n, j \in \mathbb{Z}_+. \]

The function

\[ R^n \times R^m \ni (u, v) \to \sum_{(\xi, \eta) \in \Lambda} |\langle \eta, v \rangle| ||\xi, u||^l , \]

is homogeneous of degree 1 in \( v \), of homogeneous degree \( l \) in \( u \). Since none of the terms \( ||\eta, v|| ||\xi, u|| \) can vanish on all of \( \Lambda \), the function in (7) has a positive
minimum on the compact set \( \{(u, v) : |u| = 1, |v| = 1\} \). Thus, there is an \( \varepsilon > 0 \) such that
\[
\sum_{(\xi, \eta) \in \Lambda} |\langle \xi, u \rangle| |\langle \eta, v \rangle| \geq \varepsilon |v||u|,
\]
(see [Lemma 1][4]). Applying this to \( u = \zeta, v = \hat{g}(j)(\zeta) \), we get
\[
|\hat{g}(j)(\zeta)| |\zeta| \leq \varepsilon - 1 \sum_{(\xi, \eta) \in \Lambda} |\langle \eta, \hat{g}(j)(\zeta) \rangle| |\langle \xi, \zeta \rangle| \leq C \hbar M_j,
\]
where \( h = \max_{(\xi, \eta) \in \Lambda} h_{\xi\eta} \) (resp. for all \( h > 0 \)) and \( C = \varepsilon - 1 \sum_{(\xi, \eta) \in \Lambda} C_{\xi\eta} \). Thus (ii) and (iii) hold. By setting \( h = 1 \) and \( M_j = 1, \forall j \), in the above argument, it is clear that (iii) holds as well.

Conversely if \( \Lambda \) is not a determinant set for bilinear forms of rank 1, there exist \( u \neq 0 \) and \( v \neq 0 \) such that
\[
\langle u, \xi \rangle \langle v, \eta \rangle = 0, \ \forall (\xi, \eta) \in \Lambda.
\]
Let \( h : \mathbb{R} \to \mathbb{R} \) be an arbitrary continuous function. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be defined as \( f(z) = h(\langle u, z \rangle) \cdot v \). Then
\[
\left( \frac{d}{dt} \langle \eta, f(z + t\xi) \rangle \right)_{t=0}^{\langle \eta, v \rangle \langle u, \xi \rangle} h' \langle (u, z + t\xi) \rangle_{t=0} \equiv 0.
\]
Thus \( \langle \eta, f \rangle \in C(M_k)(\xi) \subset C(M_k)(\xi) \subset C^\infty(\xi), \forall (\xi, \eta) \in \Lambda \) but \( f \) need not be even differentiable. \( \square \)

References


DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY SAN MARCOS, SAN MARCOS, CA 92096-0001, USA

E-mail: neelon@csusm.edu

URL: http://www.csusm.edu/neelon/neelon.html

(Received February 14, 2011, revised July 14, 2011)