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Applications of Mathematics, Vol. 56 (2011), No. 5, 459--480

Persistent URL: http://dml.cz/dmlcz/141619

56 (2011) APPLICATIONS OF MATHEMATICS No. 5, 459–480

ON THE WORST SCENARIO METHOD: APPLICATION TO A QUASILINEAR ELLIPTIC 2D-PROBLEM WITH UNCERTAIN COEFFICIENTS*

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(Received April 28, 2009)

Abstract. We apply a theoretical framework for solving a class of worst scenario problems to a problem with a nonlinear partial differential equation. In contrast to the one-dimensional problem investigated by P. Harasim in Appl. Math. *53* (2008), No. 6, 583–598, the two-dimensional problem requires stronger assumptions restricting the admissible set to ensure the monotonicity of the nonlinear operator in the examined state problem, and, as a result, to show the existence and uniqueness of the state solution. The existence of the worst scenario is proved through the convergence of a sequence of approximate worst scenarios. Furthermore, it is shown that the Galerkin approximation of the state solution can be calculated by means of the Kachanov method as the limit of a sequence of solutions to linearized problems.

Keywords: worst scenario problem, nonlinear differential equation, uncertain input parameters, Galerkin approximation, Kachanov method

MSC 2010: 35D30, 35G30, 47H05, 47J05, 65J15, 65N30

1. Introduction: Worst scenario problem

In this paper we extend the results obtained in [5] to a problem with an uncertain partial differential equation.

First of all, let us present the worst scenario problem framework that we will use later (see also [5], [8], [9]). Let us consider a real, separable and reflexive Banach space V. Let V^* denote its dual space. We are concerned with state problems that are described by means of the following operator state equation:

$$\mathcal{A}u = b, \quad u \in V, \tag{1.1}$$

* The work was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z 30860518.

where $\mathcal{A}\colon V \to V^*$, $b \in V^*$. The operator $\mathcal{A}$ depends on an input parameter A that belongs to an admissible set $\mathcal{U}_{\rm ad} \subset U$, where U is a Banach space. The set $\mathcal{U}_{\rm ad}$ represents an uncertainty in the input parameter of $\mathcal{A}$. Consequently, the state solution also depends on the parameter A. This A-dependent solution is then evaluated by a criterion functional Φ that can, in general, explicitly depend on input data, so that $\Phi\colon \mathcal{U}_{\rm ad} \times V \to \mathbb{R}$. The goal is to solve the following worst scenario problem: Find $A^0 \in \mathcal{U}_{\rm ad}$ such that

$$A^0 = \arg \max_{A \in \mathcal{U}_{\rm ad}} \Phi(A, u(A)). \tag{1.2}$$

The solution of (1.2) can be obtained as the limit of a sequence of solutions to approximate worst scenario problems [5, Theorem 3.1]. To this end, we replace the admissible set $\mathcal{U}_{\rm ad}$ by its finite-dimensional approximation $\mathcal{U}_{\rm ad}^M \subset \mathcal{U}_{\rm ad} \subset U$, and the space V by its finite-dimenional subspace V_h. Let $u_h(A) \in V_h$ be the Galerkin approximation of the state solution $u(A)$. We define the approximate worst scenario problem in the following way: Find $A_h^{M0} \in \mathcal{U}_{\rm ad}^M$ such that

$$A_h^{M0} = \arg \max_{A^M \in \mathcal{U}_{\rm ad}^M} \Phi(A^M, u_h(A^M)). \tag{1.3}$$

Theorem 3.1 in [5] guarantees the existence of a solution to the problem (1.2) if the following assumptions are fulfilled:

(i) the set $\mathcal{U}_{\rm ad}$ is compact in U;

(ii) a unique state solution $u(A)$ of equation (1.1) exists for any parameter $A \in \mathcal{U}_{\rm ad}$;

(iii) if $A_n \in \mathcal{U}_{\rm ad}$, $A_n \to A$ in U and $v_n \to v$ in V as $n \to \infty$, then

$$\Phi(A_n, v_n) \to \Phi(A, v);$$

(iv) the set $\mathcal{U}_{\rm ad}^M$ is compact in U;

(v) for any $A \in \mathcal{U}_{\rm ad}$, there exists a unique Galerkin approximation $u_h(A)$ of the state solution $u(A)$;

(vi) if $A_n \in \mathcal{U}_{\rm ad}$ and $A_n \to A$ in U as $n \to \infty$, then $u_h(A_n) \to u_h(A)$ in V_h;

(vii) if $A_n \in \mathcal{U}_{\rm ad}$, $A_n \to A$ in U as $n \to \infty$, and if $h_n \to 0$ as $n \to \infty$, then $u_{h_n}(A_n) \to u(A)$ in V, where $\{u_{h_n}(A_n)\}$ is an n-controlled sequence of the Galerkin approximations;

(viii) for any $A \in \mathcal{U}_{\rm ad}$, there exists a sequence $\{A^M\}$, $A^M \in \mathcal{U}_{\rm ad}^M$, $M \to \infty$, such that $A^M \to A$ in U as $M \to \infty$.

The basis assertion concerning the existence of the solution to the problem (1.2) is preserved if we replace the strong convergence $v_n \to v$ in (iii) and $u_{h_n}(A_n) \to u(A)$ in (vii) by the weak convergence.

Quasilinear elliptic boundary value problems with uncertain coefficients were studied in [6], [7], [1], [2], see also [9, Chapter III]. This paper, primarily, generalizes the one-dimensional problem examined in [5] to a two-dimensional uncertain partial differential equation. As well as in the case of the ordinary differential equation, we assume that the equation coefficients depend on the squared gradient of the state solution u. Equations of this kind describe some electromagnetic phenomena, fluid flow phenomena, and the elastoplastic deformation of a body, see [11, p. 212]. Since a common and more straightforward technique fails, we will prove the existence of the worst scenario via the convergence of a sequence of solutions to approximate worst scenario problems.

The crucial problem in this paper is to prove the monotonicity of the nonlinear operator $\mathcal{A}$ in (1.1), which guarantees the existence of a solution to the state problem. In addition, the monotonicity of $\mathcal{A}$ is required for the verification of the assumption (vii) above. Unlike the one-dimensional case, we add an additional requierement on the admissible set $\mathcal{U}_{\rm ad}$. Consequently, the operator $\mathcal{A}$ is even strictly monotone, which guarantees the uniqueness of the state solution.

To solve the approximate nonlinear state problem, the Galerkin approximation $u_h(A)$ of the state solution $u(A)$ can be found by means of the Kachanov Method (or Method of secant modules). We prove, motivated by [10], that a sequence of linearized state problems converges to the Galerkin approximation $u_h(A)$ if an appropriate condition is fulfilled (see (2.26) below).

2. Application to problem with an uncertain partial differential equation

In this section we apply the theoretical framework proposed in the previous section to the following state problem: Find $u \in H_0^1(\Omega)$ such that

$$\iint_\Omega A(|\nabla u|^2)\nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y = \iint_\Omega f v \,\mathrm{d}x\,\mathrm{d}y \quad \forall\, v \in H_0^1(\Omega), \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open domain with a polygonal boundary, $H_0^1(\Omega)$ is the usual Sobolev space on Ω with vanishing traces on $\partial\Omega$, $A = (a_{ij})_{i,j=1}^2$ is a diagonal matrix, a_{ii}, $i \in \{1,2\}$, are Lipschitz continuous fuctions on $\mathbb{R}_0^+$ (nonnegative real numbers), and $f \in L^2(\Omega)$.

The uncertainty in the input parameter A is modeled through the admissible set $\mathcal{U}_{\rm ad}$. This admissible set, whose elements are represented by diagonal matrices, is defined as the Cartesian product $\mathcal{U}_{\rm ad}^1 \times \mathcal{U}_{\rm ad}^2$, where, for $i \in \{1,2\}$, we define

$$\mathcal{U}_{\rm ad}^i := \{a_{ii} \in \mathcal{U}_{\rm ad}^{i0}\colon\ 0 < a_{\min,i} \leqslant a_{ii}(x) \leqslant a_{\max,i}\ \forall\, x \in \mathbb{R}_0^+\}$$

and

$$\mathcal{U}_{\rm ad}^{i0} := \Big\{a_{ii} \in C^{(0),1}(\mathbb{R}_0^+)\colon\ 0 < c_{\min,i} \leqslant \frac{{\rm d}a_{ii}}{{\rm d}x} \leqslant C_{{\rm L},i} \text{ a.e.}, \\ a_{ii}(x) = a_{ii}(x_{\rm C}) \text{ for } x \geqslant x_{\rm C}\Big\},$$

where $C_{{\rm L},i}$, $c_{\min,i}$, $a_{\min,i}$, $a_{\max,i}$, $x_{\rm C}$ are positive constants, and $C^{(0),1}(\mathbb{R}_0^+)$ stands for the Lipschitz continuous functions defined on $\mathbb{R}_0^+$.

We observe that $\mathcal{U}_{\rm ad}$ is a subset of the Cartesian product U^2, where U is the Banach space of functions continuous on $\mathbb{R}_0^+$ and constant for $x \geqslant x_{\rm C}$, with the norm $\|f\|_U := \max\limits_{x\in[0,x_{\rm C}]} |f(x)|$ for $f \in U$. The space U^2 is a Banach space with the norm $\|(f_1,f_2)\|_{U^2} := \max\limits_{1\leqslant i\leqslant 2} \|f_i\|_U$ for $(f_1,f_2) \in U^2$.

R e m a r k 2.1. The state problem (2.1) is the weak formulation of the following boundary value problem: Find a function $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$(2.2)\qquad -\operatorname{div}(A(|\nabla u|^2)\nabla u) = f \quad \text{on } \Omega, \\ u = 0 \quad \text{on } \partial\Omega,$$

where the elements of the matrix A and the right-hand side function f are sufficiently smooth.

The operator equation (1.1) arrises from (2.1) if we set $V := H_0^1(\Omega)$ and define $\mathcal{A}\colon V \to V^*$ and $b \in V^*$ by

$$(2.3)\qquad \langle \mathcal{A}u, v\rangle := \iint_\Omega [a_{11}(|\nabla u|^2)u_x v_x + a_{22}(|\nabla u|^2)u_y v_y]\,{\rm d}x\,{\rm d}y$$

and

$$(2.4)\qquad \langle b, v\rangle := \iint_\Omega f v\,{\rm d}x\,{\rm d}y,$$

where $u, v \in V$, and where u_x, v_x, u_y, v_y denote the partial derivatives of u and v.

It is obvious that the functionals $\mathcal{A}u$ and b are linear. Let us define $a_{\max} := \max\limits_{1\leqslant i\leqslant 2} a_{\max,i}$. Since

$$(2.5)\qquad |\langle \mathcal{A}u, v\rangle| = \left|\iint_\Omega \big[a_{11}(|\nabla u|^2)u_x v_x + a_{22}(|\nabla u|^2)u_y v_y\big]\,{\rm d}x\,{\rm d}y\right| \\ \leqslant a_{\max} \iint_\Omega \big[|u_x||v_x| + |u_y||v_y|\big]\,{\rm d}x\,{\rm d}y \\ \leqslant a_{\max}\big(\|u_x\|_{L^2(\Omega)}\|v_x\|_{L^2(\Omega)} + \|u_y\|_{L^2(\Omega)}\|v_y\|_{L^2(\Omega)}\big) \\ \leqslant C_0\|u\|_V\|v\|_V$$

and

$$|\langle b, v\rangle| = \left|\iint_\Omega f v \,\mathrm{d}x\,\mathrm{d}y\right| \leqslant C_1\|v\|_V, \tag{2.6}$$

where $C_0 := 2a_{\max}$, and $C_1 := \|f\|_{L^2(\Omega)}$, the functionals $\mathcal{A}u$ and b are also bounded.

To be able to apply [5, Theorem 3.1], we have to verify its assumptions, mentioned in Section 1. First we will prove some auxiliary assertions.

Lemma 2.1. *Let us denote* $a_{\min} := \min\limits_{1\leqslant i\leqslant 2} a_{\min,i}$, $C_{\mathrm{L}}^{\max} := \max\limits_{1\leqslant i\leqslant 2} C_{\mathrm{L},i}$. *If we assume that*

$$4x_{\mathrm{C}}C_{\mathrm{L}}^{\max} \leqslant a_{\min}, \tag{2.7}$$

then the operator $\mathcal{A}$ *defined by* (2.3) *is monotone, that is*

$$\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \geqslant 0 \quad \textit{for all } u, v \in V. \tag{2.8}$$

P r o o f. Let us rewrite the left-hand side of (2.8) as follows:

$$\begin{aligned}\iint_\Omega \big[&a_{11}(u_x^2+u_y^2)u_x^2 - a_{11}(u_x^2+u_y^2)u_xv_x - a_{11}(v_x^2+v_y^2)u_xv_x\\ &+ a_{11}(v_x^2+v_y^2)v_x^2 + a_{22}(u_x^2+u_y^2)u_y^2 - a_{22}(u_x^2+u_y^2)u_yv_y\\ &- a_{22}(v_x^2+v_y^2)u_yv_y + a_{22}(v_x^2+v_y^2)v_y^2\big]\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We can write the functions $a_{ii}(x)$, $i \in \{1,2\}$, as

$$a_{ii}(x) = a_i(x) + b_i,$$

where $a_i(x)$ is a Lipschitz continuous function on $\mathbb{R}_0^+$ such that $c_{\min,i} \leqslant \mathrm{d}a_i/\mathrm{d}x \leqslant C_{\mathrm{L},i}$, $a_i(0) = 0$, and $a_i(x) = a_i(x_{\mathrm{C}})$ for $x \geqslant x_{\mathrm{C}}$, and where $b_i \geqslant 4x_{\mathrm{C}}C_{\mathrm{L}}^{\max}$. Now, the left-hand side of (2.8) takes the form

$$\iint_\Omega z(u_x, u_y, v_x, v_y)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.9}$$

where, for $u_1, u_2, v_1, v_2 \in \mathbb{R}$,

$$\begin{aligned} z(u_1,u_2,v_1,v_2) := {} & [a_1(u_1^2+u_2^2)+b_1]u_1^2 - [a_1(u_1^2+u_2^2)+b_1]u_1v_1\\ &- [a_1(v_1^2+v_2^2)+b_1]u_1v_1 + [a_1(v_1^2+v_2^2)+b_1]v_1^2\\ &+ [a_2(u_1^2+u_2^2)+b_2]u_2^2 - [a_2(u_1^2+u_2^2)+b_2]u_2v_2\\ &- [a_2(v_1^2+v_2^2)+b_2]u_2v_2 + [a_2(v_1^2+v_2^2)+b_2]v_2^2.\end{aligned} \tag{2.10}$$

We will show that

$$z(u_1, u_2, v_1, v_2) \geqslant 0 \quad \forall\, u_1, u_2, v_1, v_2 \in \mathbb{R}, \tag{2.11}$$

hence the integral (2.9) will be non-negative and the inequality (2.8) will hold.

1. First we consider the case

$$u_1^2 + u_2^2 \leqslant x_{\mathrm{C}} \quad \text{and} \quad v_1^2 + v_2^2 \leqslant x_{\mathrm{C}}. \tag{2.12}$$

The relation (2.10) can be equivalently written as

$$\begin{aligned} z(u_1, u_2, v_1, v_2) = a_1(u_1^2 + u_2^2)(u_1 - v_1)^2 + a_2(u_1^2 + u_2^2)(u_2 - v_2)^2 \\ + [a_1(v_1^2 + v_2^2) - a_1(u_1^2 + u_2^2)](v_1^2 - u_1 v_1) \\ + [a_2(v_1^2 + v_2^2) - a_2(u_1^2 + u_2^2)](v_2^2 - u_2 v_2) \\ + b_1(u_1 - v_1)^2 + b_2(u_2 - v_2)^2. \end{aligned} \tag{2.13}$$

Let us denote: $\alpha_1 := a_1(v_1^2 + v_2^2) - a_1(u_1^2 + u_2^2)$, $\alpha_2 := a_2(v_1^2 + v_2^2) - a_2(u_1^2 + u_2^2)$, $\beta_1 := v_1^2 - u_1 v_1$, $\beta_2 := v_2^2 - u_2 v_2$. Since the functions a_1 and a_2 are increasing, both α_1 and α_2 are either non-negative or non-positive. Three situations can be distinguished:

(i) Let $\alpha_1, \alpha_2 \geqslant 0$, $\beta_1, \beta_2 \geqslant 0$, or $\alpha_1, \alpha_2 \leqslant 0$, $\beta_1, \beta_2 \leqslant 0$. Then evidently $z(u_1, u_2, v_1, v_2) \geqslant 0$.

(ii) Let $\alpha_1, \alpha_2 \geqslant 0$ and $\beta_1, \beta_2 \leqslant 0$. The case $\alpha_1, \alpha_2 \leqslant 0$ and $\beta_1, \beta_2 \geqslant 0$ can be treated analogously. Since the functions a_i, $i \in \{1, 2\}$, are Lipschitz continuous and increasing, $a_i(v_1^2 + v_2^2) - a_i(u_1^2 + u_2^2)$ and $C_{\mathrm{L},i}(v_1^2 + v_2^2 - u_1^2 - u_2^2)$ have the same sign. Moreover,

$$|a_i(v_1^2 + v_2^2) - a_i(u_1^2 + u_2^2)| \leqslant |C_{\mathrm{L},i}(v_1^2 + v_2^2 - u_1^2 - u_2^2)|.$$

For the function z defined by (2.13) we have

$$\begin{aligned} z(u_1, u_2, v_1, v_2) \geqslant C_{\mathrm{L},1}(v_1^2 + v_2^2 - u_1^2 - u_2^2)(v_1^2 - u_1 v_1) \\ + C_{\mathrm{L},2}(v_1^2 + v_2^2 - u_1^2 - u_2^2)(v_2^2 - u_2 v_2) \\ + b_1(u_1 - v_1)^2 + b_2(u_2 - v_2)^2 \\ =: z_1(u_1, u_2, v_1, v_2). \end{aligned}$$

We will show that z_1 is a non-negative function. We have

$$\begin{aligned}
&z_1(u_1, u_2, v_1, v_2) \\
&\quad = C_{\mathrm{L},1}[(v_1+u_1)(v_1-u_1)+(v_2+u_2)(v_2-u_2)]v_1(v_1-u_1) \\
&\qquad + C_{\mathrm{L},2}[(v_1+u_1)(v_1-u_1)+(v_2+u_2)(v_2-u_2)]v_2(v_2-u_2) \\
&\qquad + b_1(u_1-v_1)^2 + b_2(u_2-v_2)^2 \\
&\quad = C_{\mathrm{L},1}v_1(v_1+u_1)(v_1-u_1)^2 + C_{\mathrm{L},1}v_1(v_2+u_2)(v_1-u_1)(v_2-u_2) \\
&\qquad + C_{\mathrm{L},2}v_2(v_1+u_1)(v_1-u_1)(v_2-u_2) + C_{\mathrm{L},2}v_2(v_2+u_2)(v_2-u_2)^2 \\
&\qquad + b_1(u_1-v_1)^2 + b_2(u_2-v_2)^2.
\end{aligned}$$

We infer from (2.12) that $|u_i| \leqslant \sqrt{x_{\mathrm{C}}}$, $|v_i| \leqslant \sqrt{x_{\mathrm{C}}}$, $i \in \{1,2\}$. Consequently,

$$|C_{\mathrm{L},1}v_1(v_2+u_2) + C_{\mathrm{L},2}v_2(v_1+u_1)| \leqslant 2p,$$

where we have set $p := 2x_{\mathrm{C}}C_{\mathrm{L}}^{\max}$. This implies that

$$\begin{aligned}
&[C_{\mathrm{L},1}v_1(v_2+u_2) + C_{\mathrm{L},2}v_2(v_1+u_1)](v_1-u_1)(v_2-u_2) \\
&\qquad\qquad \geqslant -2p(v_1-u_1)(v_2-u_2)
\end{aligned}$$

for $(v_1-u_1)(v_2-u_2) \geqslant 0$, and

$$\begin{aligned}
&[C_{\mathrm{L},1}v_1(v_2+u_2) + C_{\mathrm{L},2}v_2(v_1+u_1)](v_1-u_1)(v_2-u_2) \\
&\qquad\qquad \geqslant 2p(v_1-u_1)(v_2-u_2)
\end{aligned}$$

for $(v_1-u_1)(v_2-u_2) \leqslant 0$. Moreover, it is obvious that for $i \in \{1,2\}$ we have

$$C_{\mathrm{L},i}v_i(v_i+u_i)(v_i-u_i)^2 \geqslant -p(v_i-u_i)^2,$$

and by virtue of (2.7), $b_i \geqslant 2p$, $i \in \{1,2\}$, and we can write $b_i = 2p + d_i$, where $d_i \geqslant 0$.

Thus, if $(v_1-u_1)(v_2-u_2) \geqslant 0$, then

$$\begin{aligned}
&z_1(u_1, u_2, v_1, v_2) \\
&\quad \geqslant -p(v_1-u_1)^2 - 2p(v_1-u_1)(v_2-u_2) - p(v_2-u_2)^2 \\
&\qquad + 2p(v_1-u_1)^2 + 2p(v_2-u_2)^2 + d_1(v_1-u_1)^2 + d_2(v_2-u_2)^2 \\
&\quad = p[(v_1-u_1)-(v_2-u_2)]^2 + d_1(v_1-u_1)^2 + d_2(v_2-u_2)^2 \geqslant 0.
\end{aligned}$$

If $(v_1 - u_1)(v_2 - u_2) \leqslant 0$, then

$$\begin{aligned}
&z_1(u_1, u_2, v_1, v_2)\\
&\quad\geqslant -p(v_1 - u_1)^2 + 2p(v_1 - u_1)(v_2 - u_2) - p(v_2 - u_2)^2\\
&\qquad + 2p(v_1 - u_1)^2 + 2p(v_2 - u_2)^2 + d_1(v_1 - u_1)^2 + d_2(v_2 - u_2)^2\\
&\quad= p[(v_1 - u_1) + (v_2 - u_2)]^2 + d_1(v_1 - u_1)^2 + d_2(v_2 - u_2)^2 \geqslant 0.
\end{aligned}$$

(iii) We consider the following four groups of assumptions:
(A) $\alpha_1, \alpha_2 \geqslant 0$, $\beta_1 \geqslant 0$, $\beta_2 \leqslant 0$,
(B) $\alpha_1, \alpha_2 \geqslant 0$, $\beta_1 \leqslant 0$, $\beta_2 \geqslant 0$,
(C) $\alpha_1, \alpha_2 \leqslant 0$, $\beta_1 \geqslant 0$, $\beta_2 \leqslant 0$,
(D) $\alpha_1, \alpha_2 \leqslant 0$, $\beta_1 \leqslant 0$, $\beta_2 \geqslant 0$.

They can be analysed in a very similar way. Let us do it for (A) only. We have

$$\begin{aligned}
&z(u_1, u_2, v_1, v_2)\\
&\quad\geqslant [a_1(v_1^2 + v_2^2) - a_1(u_1^2 + u_2^2)](v_1^2 - u_1 v_1)\\
&\qquad + C_{\mathrm{L},2}[(v_1 + u_1)(v_1 - u_1) + (v_2 + u_2)(v_2 - u_2)]v_2(v_2 - u_2)\\
&\qquad + b_1(v_1 - u_1)^2 + b_2(v_2 - u_2)^2\\
&\quad\geqslant C_{\mathrm{L},2}[(v_1 + u_1)(v_1 - u_1) + (v_2 + u_2)(v_2 - u_2)]v_2(v_2 - u_2)\\
&\qquad + b_1(v_1 - u_1)^2 + b_2(v_2 - u_2)^2\\
&\quad= C_{\mathrm{L},2}v_2(v_1 + u_1)(v_1 - u_1)(v_2 - u_2) + C_{\mathrm{L},2}v_2(v_2 + u_2)(v_2 - u_2)^2\\
&\qquad + b_1(v_1 - u_1)^2 + b_2(v_2 - u_2)^2 =: z_2(u_1, u_2, v_1, v_2).
\end{aligned}$$

We can again use the parameters p and d_i, $i \in \{1, 2\}$, defined in (ii), and analogously conclude: If $(v_1 - u_1)(v_2 - u_2) \geqslant 0$, then

$$\begin{aligned}
&z_2(u_1, u_2, v_1, v_2)\\
&\quad\geqslant -2p(v_1 - u_1)(v_2 - u_2) - p(v_2 - u_2)^2\\
&\qquad + p(v_1 - u_1)^2 + 2p(v_2 - u_2)^2 + (p + d_1)(v_1 - u_1)^2 + d_2(v_2 - u_2)^2\\
&\quad= p[(v_1 - u_1) - (v_2 - u_2)]^2 + (p + d_1)(v_1 - u_1)^2\\
&\qquad + d_2(v_2 - u_2)^2 \geqslant 0;
\end{aligned}$$

and if $(v_1 - u_1)(v_2 - u_2) \leqslant 0$, then

$$\begin{aligned}
&z_2(u_1, u_2, v_1, v_2)\\
&\quad\geqslant p[(v_1 - u_1) + (v_2 - u_2)]^2 + (p + d_1)(v_1 - u_1)^2 + d_2(v_2 - u_2)^2 \geqslant 0.
\end{aligned}$$

2. Now, we consider the case

$$u_1^2 + u_2^2 \leqslant x_{\rm C} \quad \text{and} \quad v_1^2 + v_2^2 \geqslant x_{\rm C}. \tag{2.14}$$

The relation (2.10) becomes

$$\begin{aligned} z(u_1, u_2, v_1, v_2) = {} & a_1(u_1^2 + u_2^2)(u_1 - v_1)^2 + a_2(u_1^2 + u_2^2)(u_2 - v_2)^2 \\ & + [a_1(x_{\rm C}) - a_1(u_1^2 + u_2^2)](v_1^2 - u_1 v_1) \\ & + [a_2(x_{\rm C}) - a_2(u_1^2 + u_2^2)](v_2^2 - u_2 v_2) \\ & + b_1(u_1 - v_1)^2 + b_2(u_2 - v_2)^2. \end{aligned}$$

Since the functions a_i, $i \in \{1, 2\}$, are increasing and the condition $u_1^2 + u_2^2 \leqslant x_{\rm C}$ is fulfilled, the expressions $a_i(x_{\rm C}) - a_i(u_1^2 + u_2^2)$, $i \in \{1, 2\}$, are non-negative. As in the previous section, we denote $\beta_1 := v_1^2 - u_1 v_1$, $\beta_2 := v_2^2 - u_2 v_2$. We observe that $\beta_1 < 0$ and $\beta_2 < 0$ is not possible. Indeed, these inequalities would imply $|u_1| > |v_1|$ and $|u_2| > |v_2|$, which contradicts (2.14).

It remains to examine the following situations:

(i) Let $\beta_1 \geqslant 0$, $\beta_2 \geqslant 0$. Then obviously $z(u_1, u_2, v_1, v_2) \geqslant 0$.

(ii) Let $\beta_1 \leqslant 0$, $\beta_2 \geqslant 0$, or $\beta_1 \geqslant 0$, $\beta_2 \leqslant 0$. We examine the first case, the other one is analogical. We have

$$\begin{aligned} z(u_1, u_2, v_1, v_2) & \geqslant [a_1(x_{\rm C}) - a_1(u_1^2 + u_2^2)](v_1^2 - u_1 v_1) \\ & \quad + [a_2(x_{\rm C}) - a_2(u_1^2 + u_2^2)](v_2^2 - u_2 v_2) \\ & \quad + b_1(u_1 - v_1)^2 + b_2(u_2 - v_2)^2 \\ & \geqslant C_{{\rm L},1}(x_{\rm C} - u_1^2 - u_2^2)(v_1^2 - u_1 v_1) \\ & \quad + b_1(u_1 - v_1)^2 + b_2(u_2 - v_2)^2 =: z_3(u_1, u_2, v_1, v_2). \end{aligned}$$

If $u_1 v_1 \leqslant 0$, then $z_3(u_1, u_2, v_1, v_2) \geqslant 0$. Let us concentrate on the case $u_1 v_1 \geqslant 0$. It is sufficient to suppose that $u_1, v_1 \geqslant 0$; the other possibility can be treated analogously. In view of the condition $u_1^2 + u_2^2 \leqslant x_{\rm C}$, the function z_3 is obviously bounded from below. Consequently, there exists a sufficiently large value $v_{2,0} > 0$ such that $z_3(u_1, u_2, v_1, v_2) \geqslant 0$ if $|v_2| \geqslant v_{2,0}$. Now, it is sufficient to show that the minimum of z_3 over the set M, where

$$\begin{aligned} M := \{(u_1, u_2, v_1, v_2) \in \mathbb{R}^4 \colon\ & u_1 \geqslant 0 \wedge v_1 \geqslant 0 \wedge v_1 \leqslant u_1 \\ & \wedge -v_{2,0} \leqslant v_2 \leqslant v_{2,0} \wedge u_1^2 + u_2^2 \leqslant x_{\rm C} \wedge v_1^2 + v_2^2 \geqslant x_{\rm C}\}, \end{aligned}$$

is equal to zero. The minimum of the function z_3 over the compact set M is either a local minimum in the interior of M, or the minimum on the boundary of M. At the point of a local extreme, all partial derivatives are equal to zero. In particular, in our problem, we have

$$\frac{\partial z_3}{\partial v_2} = -2b_2(u_2 - v_2) = 0.$$

Thus, a necessary condition for a local minimum of the function z_3 is $u_2 = v_2$. By the definition of M,

$$u_1^2 + u_2^2 \leqslant v_1^2 + v_2^2 \quad \text{and} \quad u_1, v_1 \geqslant 0,$$

and therefore it has to be $u_1 \leqslant v_1$ at a local minimum. The inequality $v_1 \leqslant u_1$ has to be valid, too (see the definition of M). Consequently, the minimum of z_3 belongs to the boundary of M. The point lies on the boundary of M, if at least one of the inequalities in the definition of M becomes an equality. We will examine the case $v_1^2 + v_2^2 = x_{\mathrm{C}}$ (in the others, it is obvious that $z_3 \geqslant 0$). We obtain

$$\begin{aligned}
&z_3(u_1, u_2, v_1, v_2) \\
&\quad = C_{\mathrm{L},1}(x_{\mathrm{C}} - u_1^2 - u_2^2)(v_1^2 - u_1 v_1) + b_1(u_1 - v_1)^2 + b_2(u_2 - v_2)^2 \\
&\quad = C_{\mathrm{L},1}(v_1^2 + v_2^2 - u_1^2 - u_2^2)(v_1^2 - u_1 v_1) + b_1(u_1 - v_1)^2 + b_2(u_2 - v_2)^2 \\
&\quad = C_{\mathrm{L},1}[(v_1 + u_1)(v_1 - u_1) + (v_2 + u_2)(v_2 - u_2)]v_1(v_1 - u_1) \\
&\qquad + b_1(u_1 - v_1)^2 + b_2(u_2 - v_2)^2 \\
&\quad = C_{\mathrm{L},1}v_1(v_1 + u_1)(v_1 - u_1)^2 + C_{\mathrm{L},1}v_1(v_2 + u_2)(v_1 - u_1)(v_2 - u_2) \\
&\qquad + b_1(u_1 - v_1)^2 + b_2(u_2 - v_2)^2.
\end{aligned}$$

If we use the parameters p and d_i, $i \in \{1, 2\}$, defined in the previous section, we can analogously show: If $(v_1 - u_1)(v_2 - u_2) \leqslant 0$, then

$$\begin{aligned}
&z_3(u_1, u_2, v_1, v_2) \\
&\quad \geqslant C_{\mathrm{L},1}v_1(v_2 + u_2)(v_1 - u_1)(v_2 - u_2) + b_1(u_1 - v_1)^2 + b_2(u_2 - v_2)^2 \\
&\quad \geqslant p[(v_1 - u_1) + (v_2 - u_2)]^2 + d_1(v_1 - u_1)^2 + d_2(v_2 - u_2)^2 \geqslant 0;
\end{aligned}$$

and if $(v_1 - u_1)(v_2 - u_2) \geqslant 0$, then

$$\begin{aligned}
&z_3(u_1, u_2, v_1, v_2) \\
&\quad \geqslant p[(v_1 - u_1) - (v_2 - u_2)]^2 + d_1(v_1 - u_1)^2 + d_2(v_2 - u_2)^2 \geqslant 0.
\end{aligned}$$

3. Finally, by considering

$$u_1^2 + u_2^2 \geqslant x_{\mathrm{C}} \quad \text{and} \quad v_1^2 + v_2^2 \geqslant x_{\mathrm{C}}, \tag{2.15}$$

we arrive at

$$a_{ii}(u_1^2+u_2^2)=a_{ii}(v_1^2+v_2^2)=a_i(x_{\rm C})+b_i=K_i,$$

where K_i, $i\in\{1,2\}$, are positive constants. Now, the left-hand side of the inequality (2.8) becomes

$$\begin{aligned}(2.16)\qquad &\iint_\Omega [K_1{u_x}^2-K_1u_xv_x-K_1u_xv_x+K_1v_x^2\\ &\quad+K_2{u_y}^2-K_2u_yv_y-K_2u_yv_y+K_2v_y^2]\,{\rm d}x\,{\rm d}y\\ &=K_1\iint_\Omega (u_x-v_x)^2\,{\rm d}x\,{\rm d}y+K_2\iint_\Omega (u_y-v_y)^2\,{\rm d}x\,{\rm d}y\geqslant 0.\end{aligned}$$

□

Lemma 2.2. *The operator* $\mathcal{A}$ *defined by* (2.3) *is continuous on* V.

Proof. We can write the operator $\mathcal{A}$ as the sum two operators, namely $\mathcal{A}_1$ and $\mathcal{A}_2$:

$$\begin{aligned}\langle\mathcal{A}u,v\rangle&=\iint_\Omega a_{11}(|\nabla u|^2)u_xv_x\,{\rm d}x\,{\rm d}y+\iint_\Omega a_{22}(|\nabla u|^2)u_yv_y\,{\rm d}x\,{\rm d}y\\ &=\langle\mathcal{A}_1u,v\rangle+\langle\mathcal{A}_2u,v\rangle.\end{aligned}$$

The sum of continuous operators is continuous. That is why it is sufficient to prove the continuity of $\mathcal{A}_1$. The proof of the continuity of $\mathcal{A}_2$ is similar.

The function $q\colon\ \Omega\times\mathbb{R}^2\to\mathbb{R}$ defined as

$$q(x,y,\xi_1,\xi_2)=a_{11}(\xi_1^2+\xi_2^2)\xi_1$$

does not depend on $x,y\in\Omega$ and satisfies the Carathéodory conditions [4, p. 288] and also the growth condition

$$|q(x,y,\xi_1,\xi_2)|\leqslant g(x)+c\sum_{i=1}^2|\xi_i|^{p_i/r},$$

where $g\in L^r(\Omega)$, $c>0$, and $p_1,p_2,r\in[1,\infty)$. It is sufficient to set $g(x)=0$, $c=a_{\max,1}$, $p_1=2$, $p_2=0$, and $r=2$. Then the operator

$$\begin{aligned}H\colon\ L^2(\Omega)\times L^2(\Omega)&\to L^2(\Omega),\\ (v,w)&\mapsto a_{11}(v^2+w^2)v,\end{aligned}$$

is continuous, see [4, p. 288].

Let $\{u_n\}$ be a sequence in V such that $u_n \to u \in V$. Then $(u_n)_x \to u_x$ and $(u_n)_y \to u_y$ in $L^2(\Omega)$. Since the operator H is continuous, we have

$$a_{11}(|\nabla u_n|^2)(u_n)_x \to a_{11}(|\nabla u|^2)u_x \quad \text{in } L^2(\Omega). \tag{2.17}$$

We will show that $\|\mathcal{A}_1 u - \mathcal{A}_1 u_n\|_{V^*} \to 0$. We have

$$\|\mathcal{A}_1 u - \mathcal{A}_1 u_n\|_{V^*} = \sup_{\|v\|_V=1} \left| \iint_\Omega \big[a_{11}(|\nabla u|^2)u_x - a_{11}(|\nabla u_n|^2)(u_n)_x\big] v_x \,\mathrm{d}x\,\mathrm{d}y \right|.$$

From the Schwarz inequality and from the fact that $\|v_x\|_{L^2(\Omega)} \leqslant \|v\|_V = 1$, we obtain

$$\begin{aligned} &\left| \iint_\Omega \big[a_{11}(|\nabla u|^2)u_x - a_{11}(|\nabla u_n|^2)(u_n)_x\big] v_x \,\mathrm{d}x\,\mathrm{d}y \right| \\ &\qquad \leqslant \|a_{11}(|\nabla u|^2)u_x - a_{11}(|\nabla u_n|^2)(u_n)_x\|_{L^2(\Omega)} \|v_x\|_{L^2(\Omega)} \\ &\qquad \leqslant \|a_{11}(|\nabla u|^2)u_x - a_{11}(|\nabla u_n|^2)(u_n)_x\|_{L^2(\Omega)}. \end{aligned}$$

By (2.17), the last quantity tends to zero if $n \to \infty$. □

Lemma 2.3. *The operator $\mathcal{A}$ defined by* (2.3) *is coercive on V, that is,*

$$\lim_{\|u\|_V \to \infty} \frac{\langle \mathcal{A}u, u\rangle}{\|u\|_V} = \infty. \tag{2.18}$$

Proof. Let $a_{\min}$ be the constant defined in Lemma 2.1. We have

$$\begin{aligned} \langle \mathcal{A}u, u\rangle &:= \iint_\Omega [a_{11}(|\nabla u|^2)u_x^2 + a_{22}(|\nabla u|^2)u_y^2] \,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant a_{\min} \iint_\Omega (u_x^2 + u_y^2) \,\mathrm{d}x\,\mathrm{d}y \geqslant C_2\|u\|_V^2, \end{aligned} \tag{2.19}$$

where $C_2 > 0$. Consequently, (2.18) holds. □

Lemma 2.4. *Suppose that the condition* (2.7) *is fulfilled. Then the operator $\mathcal{A}$ defined by* (2.3) *is strictly monotone, that is,*

$$\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle > 0 \quad \textit{for all } u, v \in V,\ u \neq v. \tag{2.20}$$

Proof. Let $u \neq v$ in $H_0^1(\Omega)$. Since the seminorm $|\cdot|_{H^1(\Omega)}$ is a norm in $H_0^1(\Omega)$ equivalent to the norm $\|\cdot\|_{H_0^1(\Omega)}$, it holds $|u - v|_{H_0^1(\Omega)} > 0$. This means that $u_x \neq v_x$ in $L^2(\Omega)$ or $u_y \neq v_y$ in $L^2(\Omega)$. Consequently, there exists a set Ω_1 with positive measure and such that $u_x \neq v_x$ or $u_y \neq v_y$ in Ω_1. It is sufficient to prove the following statement: If

$$u_1 \neq v_1 \quad \text{or} \quad u_2 \neq v_2, \tag{2.21}$$

then the function z defined by (2.10) is positive. Again, we consider three cases:

1. Let
$$u_1^2 + u_2^2 \leqslant x_{\rm C} \quad \text{and} \quad v_1^2 + v_2^2 \leqslant x_{\rm C}.$$
If $u_1^2 + u_2^2 = 0$, then (2.21) implies that $v_1 \neq 0$ or $v_2 \neq 0$, and thus $z(u_1, u_2, v_1, v_2) > 0$, see (2.13). If $u_1^2 + u_2^2 > 0$, the condition (2.21) guarantees that at least one of the first two terms in (2.13) is positive. Furthermore, the sum of the remaining terms is non-negative (see the proof of Lemma 2.1). Consequently, $z(u_1, u_2, v_1, v_2) > 0$.
2. If
$$u_1^2 + u_2^2 \leqslant x_{\rm C} \quad \text{and} \quad v_1^2 + v_2^2 \geqslant x_{\rm C},$$
we can anologously prove that z is positive.
3. Let
$$u_1^2 + u_2^2 \geqslant x_{\rm C} \quad \text{and} \quad v_1^2 + v_2^2 \geqslant x_{\rm C}.$$
Since $|u - v|_V > 0$, it is obvious that (2.16) is positive.

□

Theorem 2.1. *Suppose that the inequality* (2.7) *is fulfilled. Then the problem* (2.1) *has a unique solution.*

Proof. The existence of a solution is guaranteed by [14, Theorem 2.K], see also [5]. It is sufficient to verify that $\mathcal{A}$ is monotone, continuous, and coercive on V, which, if we suppose that $4x_{\rm C}C_{\rm L}^{\max} \leqslant a_{\min}$, follows from Lemmas 2.1, 2.2, and 2.3. In addition, according to Lemma 2.4, the operator $\mathcal{A}$ is strictly monotone and the uniqueness follows from [14, p. 93, Corollary 1]. □

The last theorem means that the assumption (ii) (see Section 1) is fulfilled.

Remark 2.2. The existence of a weak solution to quasilinear elliptic equation of the type (2.2) is examined also in [12]. In that work, the crucial assumption for ensuring the existence of a weak solution is the so-called monotonicity in the main part, see [12, p. 47]. This assumption is equivalent to our condition (2.11).

R e m a r k 2.3. In our problem, a condition of the type (2.7) to ensure (2.11) cannot be omitted. Indeed, for example, let us consider the input parameters

$$a_{11}(x) = \begin{cases} \frac{1}{2}x + 2 & \text{for } 0 \leqslant x \leqslant 10, \\ 7 & \text{for } x > 10, \end{cases}$$

$$a_{22}(x) = \begin{cases} \frac{1}{10}x + 1.275 & \text{for } 0 \leqslant x \leqslant 7.25, \\ \frac{32}{3}x - \frac{226}{3} & \text{for } 7.25 < x \leqslant 8, \\ \frac{1}{10}x + 9.2 & \text{for } 8 < x \leqslant 10, \\ 10.2 & \text{for } x > 10, \end{cases}$$

and take $u_1 = 2$, $u_2 = 2$, $v_1 = 1$ and $v_2 = 2.5$. Then, by substitution into (2.10), we get $z(u_1, u_2, v_1, v_2) = -1.125$. In this case, the inequality (2.11) is not valid.

Now, we turn our attention to the approximation of the equation (2.1) and to the corresponding approximate worst scenario problem (1.3). We will define the set $\mathcal{U}_{\rm ad}^M \subset \mathcal{U}_{\rm ad}$ and a finite-dimensional space V_h. Let x_j, $j = 1, \ldots, M$, be equally spaced points in $[0, x_{\rm C}]$, $x_1 = 0$ and $x_M = x_{\rm C}$. For $i \in \{1, 2\}$, we define

$$\mathcal{U}_{\rm ad}^{M,i} := \{a \in \mathcal{U}_{\rm ad}^i \colon a|_{[x_j, x_{j+1}]} \in P_1([x_j, x_{j+1}]),\ j = 1, \ldots, M-1\},$$

where $P_1([x_j, x_{j+1}])$ denotes the linear polynomials on the interval $[x_j, x_{j+1}]$. The admissible set $\mathcal{U}^M$ is defined as the Cartesian product $\mathcal{U}_{\rm ad}^{M,1} \times \mathcal{U}_{\rm ad}^{M,2}$.

To approximate the space V, we introduce a triangulation $\mathcal{T}_h = \{T_1, \ldots, T_N\}$ of Ω. The finite-dimensional subspace V_h is defined as

$$V_h := \{v_h \in V \cap C(\overline{\Omega}) \colon v_h|_{T_j} \in P_1(T_j),\ j = 1, \ldots, N\}, \tag{2.22}$$

where $C(\overline{\Omega})$ denotes the space of continuous functions on $\overline{\Omega}$, and $P_1(T_j)$ are polynomials of degree less than or equal to one on the triangle T_j. We assume that the diameter of any triangle T_j, $j \in \{1, \ldots, N\}$, does not exceed h.

The Galerkin approximation $u_h(A) \in V_h$ of the solution to problem (2.1) is defined by the identity

$$\iint_\Omega [a_{11}(|\nabla u_h|^2)(u_h)_x v_x + a_{22}(|\nabla u_h|^2)(u_h)_y v_y] \,\mathrm{d}x\,\mathrm{d}y = \iint_\Omega f v \,\mathrm{d}x\,\mathrm{d}y \quad \forall\, v \in V_h. \tag{2.23}$$

Theorem 2.2. *Suppose that the condition* (2.7) *is fulfilled. Then there exists a unique Galerkin approximation* $u_h(A)$ *of the solution to the problem* (2.1).

P r o o f. The space V_h, as well as V, is a real, separable, and reflexive Banach space. Since the operator $\mathcal{A}$ is strictly monotone, continuous, and coercive on V and, consequently, on its subspace V_h, the existence of a unique Galerkin approximation follows from [14, Theorem 2.K] and [14, p. 93, Corollary 1] applied to (2.23). □

Thus, the assumption (v) of Section 1 is fulfilled.

We will show in Theorem 2.3 (see bellow) that the Galerkin approximation $u_h(A)$ of the nonlinear problem (2.1) can be determined as the limit of a sequence of solutions to linearized problems.

Let us introduce the following notation. We set

$$a(y;u,v) := \iint_\Omega [a_{11}(|\nabla y|^2)u_x v_x + a_{22}(|\nabla y|^2)u_y v_y]\,\mathrm{d}x\,\mathrm{d}y, \qquad y,u,v \in H_0^1(\Omega).$$

Let $y \in H_0^1(\Omega)$ be fixed. In view of (2.5) and (2.19), the expression $a(y;\cdot,\cdot)$ defines a bounded (continuous) and V_h-elliptic bilinear form.

In the proof of Theorem 2.3 we will use the equivalence of norms on finite-dimensional spaces. To this end, we fix a triangulation $\mathcal{T}_h$.

First, let $V_{h,c}$ be the space of functions on Ω that are constant on each triangle $T_j \in \mathcal{T}_h$, $j \in \{1,\ldots,N\}$. It follows from the equivalence of norms on $V_{h,c}$ that

$$\|u\|_{L^\infty(\Omega)} \leqslant C_3\|u\|_{L^2(\Omega)} \quad \forall\, u \in V_{h,c}, \tag{2.24}$$

where $C_3 \geqslant 0$.

Further, we consider the corresponding space V_h. We have

$$\|u_x - v_x\|_{L^2(\Omega)} + \|u_y - v_y\|_{L^2(\Omega)} \leqslant C_4\|u-v\|_V \quad \forall\, u,v \in V_h, \tag{2.25}$$

where $C_4 > 0$.

Theorem 2.3. *Suppose that $\mathcal{T}_h$ is the fixed triangulation considered above and that V_h is the corresponding finite-dimensional space. Let $C_{\mathrm{L}}^{\max}$ be the constant defined in Lemma* 2.1 *and let C_1, C_2, C_3, and C_4 be the constants defined in* (2.6), (2.19), (2.24), *and* (2.25), *respectively. Moreover, we assume that*

$$\frac{2C_1C_3C_4C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}}{C_2^2} < 1. \tag{2.26}$$

Under these assumptions, the Galerkin approximation $u_h \equiv u_h(A) \in V_h$ of the solution to the problem (2.1) *can be calculated by means of the Kachanov method:*

Let $u^0 \in V_h$ be arbitrary. If $u^k \in V_h$ is known, let $u^{k+1} \in V_h$ be defined by the relation

$$a(u^k; u^{k+1}, v) = \langle b, v\rangle \quad \forall\, v \in V_h.$$

Then

$$\|u_h - u^k\|_V \to 0 \quad \text{as } k \to \infty. \tag{2.27}$$

P r o o f. We will proceed similarly as the authors of [10]. We define a mapping $S\colon V_h \to V_h$ by the formula

$$a(u; Su, v) = \langle b, v\rangle \quad \forall\, v \in V_h.$$

Since the bilinear form $a(y;\cdot,\cdot)$ is continuous and V-elliptic, it follows from the Lax-Milgram theorem that the element Su is uniquely determined. Moreover,

$$C_2\|Su\|_V^2 \leqslant a(u; Su, Su) = \langle b, Su\rangle \leqslant C_1\|Su\|_V,$$

hence

$$\|Su\|_V \leqslant \frac{C_1}{C_2}, \tag{2.28}$$

independently of u. We will show that S is a contractive mapping on V_h. Let $u, v \in V_h$ be arbitrary. We set $w := Su - Sv$. Then

$$\begin{aligned} C_2\|w\|_V^2 &\leqslant a(u; w, w) = a(u; Su, w) - a(u; Sv, w) \\ &= \langle b, w\rangle - a(u; Sv, w) = a(v; Sv, w) - a(u; Sv, w) \\ &= \iint_\Omega \big[a_{11}(|\nabla v|^2)(Sv)_x w_x + a_{22}(|\nabla v|^2)(Sv)_y w_y\big]\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \iint_\Omega \big[a_{11}(|\nabla u|^2)(Sv)_x w_x + a_{22}(|\nabla u|^2)(Sv)_y w_y\big]\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_\Omega \big[(a_{11}(|\nabla v|^2) - a_{11}(|\nabla u|^2))(Sv)_x w_x \\ &\quad + (a_{22}(|\nabla v|^2) - a_{22}(|\nabla u|^2))(Sv)_y w_y\big]\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \|a_{11}(|\nabla v|^2) - a_{11}(|\nabla u|^2)\|_{L^\infty(\Omega)} \iint_\Omega |(Sv)_x w_x|\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \|a_{22}(|\nabla v|^2) - a_{22}(|\nabla u|^2)\|_{L^\infty(\Omega)} \iint_\Omega |(Sv)_y w_y|\,\mathrm{d}x\,\mathrm{d}y =: I. \end{aligned} \tag{2.29}$$

Since the partial derivatives of u and v belong to the space $V_{h,c}$ defined above, in other words they are constant on each triangle, also $a_{ii}(|\nabla v|^2) - a_{ii}(|\nabla u|^2) \in V_{h,c}$,

$i \in \{1,2\}$. First we will show that for each element $T_j \in \mathcal{T}_h$, $j \in \{1,\dots,N\}$, and for $i \in \{1,2\}$ the following estimate holds:

$$\begin{aligned} \|a_{ii}(|\nabla v|^2) - a_{ii}(|\nabla u|^2)\|_{L^\infty(T_j)} \\ \leqslant 2C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}(\|v_x - u_x\|_{L^\infty(T_j)} + \|v_y - u_y\|_{L^\infty(T_j)}). \end{aligned} \tag{2.30}$$

To this end, let us consider the three following cases:

1. Let $|\nabla v|^2 \leqslant x_{\mathrm{C}}$ and $|\nabla u|^2 \leqslant x_{\mathrm{C}}$. Then

$$\begin{aligned} \|a_{ii}(|\nabla v|^2) - a_{ii}(|\nabla u|^2)\|_{L^\infty(T_j)} &\leqslant C_{\mathrm{L}}^{\max}|v_x^2 + v_y^2 - u_x^2 - u_y^2| \\ &\leqslant C_{\mathrm{L}}^{\max}(|v_x + u_x||v_x - u_x| + |v_y + u_y||v_y - u_y|) \\ &\leqslant 2C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}(\|v_x - u_x\|_{L^\infty(T_j)} + \|v_y - u_y\|_{L^\infty(T_j)}). \end{aligned}$$

2. Let $|\nabla v|^2 \leqslant x_{\mathrm{C}}$ and $|\nabla u|^2 \geqslant x_{\mathrm{C}}$. Then

$$\begin{aligned} &\|a_{ii}(|\nabla v|^2) - a_{ii}(|\nabla u|^2)\|_{L^\infty(T_j)} \\ &\quad= |a_{ii}(x_{\mathrm{C}}) - a_{ii}(|\nabla v|^2)| \\ &\quad\leqslant C_{\mathrm{L}}^{\max}[x_{\mathrm{C}} - (v_x^2 + v_y^2)] \\ &\quad= C_{\mathrm{L}}^{\max}\Big(\sqrt{x_{\mathrm{C}}} + \sqrt{v_x^2 + v_y^2}\Big)\Big(\sqrt{x_{\mathrm{C}}} - \sqrt{v_x^2 + v_y^2}\Big) \\ &\quad\leqslant 2C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}\Big(\sqrt{u_x^2 + u_y^2} - \sqrt{v_x^2 + v_y^2}\Big) \\ &\quad\leqslant 2C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}\sqrt{(u_x - v_x)^2 + (u_y - v_y)^2} \\ &\quad\leqslant 2C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}(|u_x - v_x| + |u_y - v_y|) \\ &\quad= 2C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}(\|u_x - v_x\|_{L^\infty(T_j)} + \|u_y - v_y\|_{L^\infty(T_j)}). \end{aligned}$$

3. Let $|\nabla v|^2 \geqslant x_{\mathrm{C}}$ and $|\nabla u|^2 \geqslant x_{\mathrm{C}}$. In this case we have

$$\|a_{ii}(|\nabla v|^2) - a_{ii}(|\nabla u|^2)\|_{L^\infty(T_j)} = 0$$

   and the estimate (2.30) holds.

Hence,

$$\begin{aligned} &\|a_{ii}(|\nabla v|^2) - a_{ii}(|\nabla u|^2)\|_{L^\infty(\Omega)} \\ &\quad= \max_{T_j \in \mathcal{T}_h} \|a_{ii}(|\nabla v|^2) - a_{ii}(|\nabla u|^2)\|_{L^\infty(T_j)} \\ &\quad\leqslant 2C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}(\|v_x - u_x\|_{L^\infty(\Omega)} + \|v_y - u_y\|_{L^\infty(\Omega)}). \end{aligned} \tag{2.31}$$

By combining (2.24), (2.25), (2.28), (2.31), and

$$\iint_{\Omega}(|(Sv)_x w_x| + |(Sv)_y w_y|)\,\mathrm{d}x\,\mathrm{d}y \\ \leqslant \|(Sv)_x\|_{L^2(\Omega)}\|w_x\|_{L^2(\Omega)} + \|(Sv)_y\|_{L^2(\Omega)}\|w_y\|_{L^2(\Omega)} \leqslant \|Sv\|_V\|w\|_V,$$

we obtain

$$\begin{aligned} I &\leqslant 2C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}(\|u_x - v_x\|_{L^\infty(\Omega)} \\ &\quad + \|u_y - v_y\|_{L^\infty(\Omega)})\iint_{\Omega}(|(Sv)_x w_x| + |(Sv)_y w_y|)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant 2C_3 C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}(\|u_x - v_x\|_{L^2(\Omega)} + \|u_y - v_y\|_{L^2(\Omega)})\|Sv\|_V\|w\|_V \\ &\leqslant \frac{2C_1C_3C_4C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}}{C_2}\|u - v\|_V\|w\|_V. \end{aligned}$$

By using this result in (2.29), we infer that

$$\|Su - Sv\|_V \leqslant \frac{2C_1C_3C_4C_{\mathrm{L}}^{\max}\sqrt{x_{\mathrm{C}}}}{C_2^2}\|u - v\|_V.$$

By virtue of (2.26), the mapping S is contractive. Consequently, the Banach fixed-point theorem gives (2.27). □

By the Arzelà-Ascoli theorem [13, page 35], the sets $\mathcal{U}_{\mathrm{ad}}^i$, $\mathcal{U}_{\mathrm{ad}}^{M,i}$, $i \in \{1,2\}$, are compact in U. Since the Cartesian product of compact sets is compact, the admissible sets $\mathcal{U}_{\mathrm{ad}}, \mathcal{U}_{\mathrm{ad}}^M$ are compact, and the asssumptions (i) and (iv) of Section 1 are fulfilled.

Further, we show that the assumptions (vi)–(viii) from Section 1 are also fulfilled.

Theorem 2.4. *Let us assume that condition* (2.7) *from Lemma* 2.1 *is valid. If* $A_n \in \mathcal{U}_{\mathrm{ad}}$ *and* $A_n \to A$ *in* U^2 *as* $n \to \infty$, *then* $u_h(A_n) \to u_h(A)$ *in* V_h.

Proof. Let us fix the space V_h. Let us denote the Galerkin approximation $u_h(A_n) \in V_h$ by u_n. By using (2.1), (2.3), (2.4), (2.6), (2.19), and Friedrichs' inequality, we obtain

$$\|u_n\|_V \leqslant \frac{C\|f\|_{L^2(\Omega)}}{a_{\min}}$$

independently of n, where C is a positive constant. Since V_h is finite-dimensional, this sequence has a convergent subsequence $\{u_{n_k}\}$, we denote it simply by $\{u_k\}$. The corresponding subsequences of input parameters are $\{a_{ii,k}\}$, $i \in \{1,2\}$. Thus,

$$u_k \to w_h \quad \text{in } H^1(\Omega) \text{ as } k \to \infty, \tag{2.32}$$

where w_h is an element of V_h. We will show that $w_h = u_h(A)$. Let $v \in V_h$ be arbitrary. We can write:

$$
\begin{aligned}
(2.33)\quad \iint_\Omega fv \,\mathrm{d}x\,\mathrm{d}y &= \iint_\Omega \bigl[a_{11,k}(|\nabla u_k|^2)(u_k)_x v_x + a_{22,k}(|\nabla u_k|^2)(u_k)_y v_y\bigr]\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_\Omega \bigl[a_{11,k}(|\nabla u_k|^2)((u_k)_x - (w_h)_x)v_x \\
&\qquad + a_{22,k}(|\nabla u_k|^2)((u_k)_y - (w_h)_y)v_y\bigr]\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint_\Omega \bigl([a_{11,k}(|\nabla u_k|^2) - a_{11}(|\nabla u_k|^2)](w_h)_x v_x \\
&\qquad + [a_{22,k}(|\nabla u_k|^2) - a_{22}(|\nabla u_k|^2)](w_h)_y v_y\bigr)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint_\Omega \bigl([a_{11}(|\nabla u_k|^2) - a_{11}(|\nabla w_h|^2)](w_h)_x v_x \\
&\qquad + [a_{22}(|\nabla u_k|^2) - a_{22}(|\nabla w_h|^2)](w_h)_y v_y\bigr)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint_\Omega \bigl[a_{11}(|\nabla w_h|^2)(w_h)_x v_x + a_{22}(|\nabla w_h|^2)(w_h)_y v_y\bigr]\,\mathrm{d}x\,\mathrm{d}y \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

As $k \to \infty$, the integrals I_1, I_2, and I_3 tend to zero by virtue of (2.32), the boundedness and the uniform convergence of the sequences $\{a_{ii,k}\}$, $i \in \{1,2\}$, the boundedness of $\{u_k\}$, and the equivalence of norms on a finite dimensional space. Let us examine the convergence of I_3. We can estimate I_3 as follows:

$$
\begin{aligned}
I_3 &\leqslant \|(w_h)_x\|_{L^\infty(\Omega)}\|v_x\|_{L^\infty(\Omega)} \iint_\Omega |a_{11}(|\nabla u_k|^2) - a_{11}(|\nabla w_h|^2)|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \|(w_h)_y\|_{L^\infty(\Omega)}\|v_y\|_{L^\infty(\Omega)} \iint_\Omega |a_{22}(|\nabla u_k|^2) - a_{22}(|\nabla w_h|^2)|\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant K_1 C_{\mathrm{L}}^{\max} \iint_\Omega ||\nabla u_k|^2 - |\nabla w_h|^2|\,\mathrm{d}x\,\mathrm{d}y \\
&= K_1 C_{\mathrm{L}}^{\max} \iint_\Omega |(u_k)_x^2 - (w_h)_x^2 + (u_k)_y^2 - (w_h)_y^2|\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant K_1 C_{\mathrm{L}}^{\max} \Bigl(\iint_\Omega |(u_k)_x + (w_h)_x||(u_k)_x - (w_h)_x|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint_\Omega |(u_k)_y + (w_h)_y||(u_k)_y - (w_h)_y|\,\mathrm{d}x\,\mathrm{d}y\Bigr) \\
&\leqslant K_1 C_{\mathrm{L}}^{\max} \bigl[\|(u_k)_x + (w_h)_x\|_{L^2(\Omega)}\|(u_k)_x - (w_h)_x\|_{L^2(\Omega)} \\
&\quad + \|(u_k)_y + (w_h)_y\|_{L^2(\Omega)}\|(u_k)_y - (w_h)_y\|_{L^2(\Omega)}\bigr] \\
&\leqslant K_1 K_2 C_{\mathrm{L}}^{\max} \bigl[\|(u_k)_x - (w_h)_x\|_{L^2(\Omega)} + \|(u_k)_y - (w_h)_y\|_{L^2(\Omega)}\bigr],
\end{aligned}
$$

where we have set

$$K_1 := \|(w_h)_x\|_{L^\infty(\Omega)}\|v_x\|_{L^\infty(\Omega)} + \|(w_h)_y\|_{L^\infty(\Omega)}\|v_y\|_{L^\infty(\Omega)},$$

and where $K_2 > 0$ stems from the boundedness of $\{u_k\}$ in $H^1(\Omega)$. Thus, (2.32) implies that for $k \to \infty$ the integral I_3 tends to zero.

Consequently, the left-hand side of (2.33) equals I_4 for any $v \in V_h$, which means that $w_h = u_h(A)$. It follows from the uniqueness of the Galerkin approximation that the entire sequence $\{u_n\}$ converges to $u_h(A)$. □

To verify assumption (vii) from Section 1, we have to introduce an appropriate sequence of finite-dimensional subspaces of V. To this end, let $\{\mathcal{T}_h\}$, $h \to 0$, be a regular family of triangulations of Ω. Then $\bigcup\limits_h V_h$ is dense in V (this is a simple consequence of [3, Theorem 3.2.1]).

Theorem 2.5. *Suppose that condition* (2.7) *is fulfilled. Let* $\{A_n\}$, *where* $A_n \in \mathcal{U}_{\rm ad}$ *and* $A_n \to A$ *in* U^2 *as* $n \to \infty$, *be a sequence of parameters. Further, let* $\{\mathcal{T}_h\}$, $h \to 0$, *be a regular family of triangulations of* Ω, $\{\mathcal{T}_{h_n}\} \subset \{\mathcal{T}_h\}$, $h_n \to 0$ *as* $n \to \infty$, *be a sequence of these triangulations,* $\{V_{h_n}\}$ *be the corresponding sequence of the finite-dimenional spaces defined by* (2.22), *and let* $\{u_{h_n}(A_n)\}$, $u_{h_n}(A_n) \in V_{h_n}$, *be the corresponding sequence of the Galerkin approximations. Then*

$$u_{h_n}(A_n) \rightharpoonup u(A) \quad \textit{(weakly) in } V,$$

where $u(A)$ *is the solution of problem* (2.1) *for the parameter* A.

P r o o f. We can prove analogously to the proof of Theorem 2.4 that the sequence $\{u_{h_n}(A_n)\}$ is bounded in V.

Since V is a reflexive Banach space, the sequence $\{u_{h_n}(A_n)\}$ has a weakly convergent subsequence, we denote it simply by $\{u_k\}$, such that

$$u_k \rightharpoonup w \quad \text{as } k \to \infty, \tag{2.34}$$

where $w \in V$.

For any $u \in V$ let us define the operators $\mathcal{A}, \mathcal{A}_k\colon V \to V^*$ by

$$\langle \mathcal{A}u, v\rangle := \iint_\Omega [a_{11}(|\nabla u|^2)u_x v_x + a_{22}(|\nabla u|^2)u_y v_y]\,\mathrm{d}x\,\mathrm{d}y \quad \forall\, v \in V,$$
$$\langle \mathcal{A}_k u, v\rangle := \iint_\Omega [a_{11,k}(|\nabla u|^2)u_x v_x + a_{22,k}(|\nabla u|^2)u_y v_y]\,\mathrm{d}x\,\mathrm{d}y \quad \forall\, v \in V.$$

By virtue of [5, Lemma 4.4], a generalization of [14, p. 94, Lemma 3], we obtain $w = u(A)$. It is sufficient to verify the assumptions, that is:

(α) $\langle \mathcal{A}_k u_k, v\rangle \to \langle b, v\rangle$ as $k \to \infty$ $\forall\, v \in V$,
(β) $\langle \mathcal{A}_k u_k, u_k\rangle \to \langle b, w\rangle$ as $k \to \infty$,
(γ) $\langle \mathcal{A}_k v, u_k\rangle \to \langle \mathcal{A} v, w\rangle$ as $k \to \infty$ $\forall\, v \in V$,
(δ) $\langle \mathcal{A}_k v, v\rangle \to \langle \mathcal{A} v, v\rangle$ as $k \to \infty$ $\forall\, v \in V$,

where the functional b is defined by (2.4). Then w is a solution of the equation $\mathcal{A}w = b$. We can verify (α)–(δ) analogously as in the proof of [5, Theorem 4.4]. □

Lemma 2.5. *Let $A \in \mathcal{U}_{\rm ad}$ be arbitrary. Then there exists a sequence $\{A^M\}$, $A^M \in \mathcal{U}^M_{\rm ad}$, such that*

$$A^M \to A \quad \textit{in } U^2 \textit{ as } M \to \infty.$$

Proof. The assertion is a consequence of [5, Lemma 4.5]. □

We have shown that under condition (2.7), the assumptions from Section 1, if we replace the strong convergence $v_n \to v$ in (iii) and the strong convergence of the Galerkin approximations in (vii) by the weak convergence, are fulfilled. It is possible to show, analogously as in [9, Theorem 3.3], that the approximate worst scenario problem (1.3) has at least one solution. According to [5, Theorem 3.1 and Remark 3.1], there exists a sequence of approximate worst scenarios that converges to A^0, where $A^0 \in \mathcal{U}_{\rm ad}$ solves the problem (1.2). Furthermore, the corresponding sequence of state solutions weakly converges to $u(A^0) \in V$, where $u(A^0)$ is the state solution related to the parameter A^0, and the corresponding sequence of values of the criterion functional Φ converges to $\Phi(A^0, u(A^0))$.

In addition, we have shown that the Galerkin approximation $u_h(A)$ of the state solution $u(A)$ can be calculated as the limit of a sequence of solutions to linearized problems if the condition (2.26) is fulfilled.

Acknowledgment. The author would like to thank Dr. J. Chleboun for his important help during the work on this paper.

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