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CHARACTERIZATIONS OF SOME RINGS WITH $\mathcal{C}$-PROJECTIVE, $\mathcal{C}$-(FP)-INJECTIVE AND $\mathcal{C}$-FLAT MODULES

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Abstract. Let R be a commutative ring and $\mathcal{C}$ a semidualizing R-module. We investigate the relations between $\mathcal{C}$-flat modules and $\mathcal{C}$-FP-injective modules and use these modules and their character modules to characterize some rings, including artinian, noetherian and coherent rings.

Keywords: semidualizing module, $\mathcal{C}$-projective module, $\mathcal{C}$-(FP)-injective module, $\mathcal{C}$-flat module, noetherian ring, coherent ring

MSC 2010: 13C11, 13D02, 13D05, 13E05, 18G25

Introduction

Foxby [10], Vasconcelos [21] and Golod [12] independently initiated the study of semidualizing modules, which are common generalizations of dualizing modules and finitely generated projective modules of rank one. Christensen [6] defined semidualizing complexes, and studied them in the context of derived categories. Recently, Holm and White [15] extended the definition of the semidualizing module to a pair of arbitrary associative rings. Especially, they defined the so-called $\mathcal{C}$-projective, $\mathcal{C}$-injective and $\mathcal{C}$-flat modules (Definition 1.4), to characterize the *Auslander class* $\mathcal{A}_{\mathcal{C}}(R)$ and the *Bass class* $\mathcal{B}_{\mathcal{C}}(R)$ (Definition 1.5), with respect to a semidualizing module $\mathcal{C}$. The notion of $\mathcal{C}$-projective ($\mathcal{C}$-injective, $\mathcal{C}$-flat) modules is important for the study of the relative homological algebra with respect to semidualizing modules. For example, Holm and Jørgensen [14] used these modules to define $\mathcal{C}$-Gorenstein injective (projective, flat) modules, introduced the notions of $\mathcal{C}$-Gorenstein projective,

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$\mathcal{C}$-Gorenstein injective, and $\mathcal{C}$-Gorenstein flat dimensions, and investigated the properties of these dimensions. Takahashi and White [20] investigated the $\mathcal{C}$-projective dimension of a module, and their results showed that three natural definitions of the finite $\mathcal{C}$-projective dimension agree.

This work is focused on the classes of $\mathcal{C}$-flat modules and $\mathcal{C}$-(FP)-injective modules. We do some preliminary work in Section 1 and give our main results in Section 2.

Lambek [17] proved that an R-module is flat if and only if its character module $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is injective, while Wagstaff [23, Lemma 4.1(2)] showed that when R is a commutative ring, an R-module N is $\mathcal{C}$-injective if and only if its character module N^* is $\mathcal{C}$-flat. In Section 2, we extend these results by replacing $\mathcal{C}$-injective with $\mathcal{C}$-FP-injective. The following result is from Proposition 2.5, see 1.4 for the definition of the $\mathcal{C}$-FP-injective module.

Theorem A. *If R is a commutative ring and $\mathcal{C}$ is a semidualizing module, then an R-module M is $\mathcal{C}$-flat if and only if its character module M^* is $\mathcal{C}$-*FP*-injective.*

It is well-known that we can characterize some rings with some special properties of projective, (FP)injective and flat modules. For instance, Megibben proved that a ring R is noetherian if and only if every FP-injective module is injective ([18, Theorem 3]). Cheatham and Stone showed that a ring R is coherent is equivalent to the condition that an R-module M is FP-injective if and only if its character module $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is flat ([4, Theorem 1]). Zhu proved that a ring R is coherent if and only if for every injective module E, its character module E^* is flat ([24, Theorem 6]). Continuing in this manner, we show that we can also characterize some rings with some special properties of $\mathcal{C}$-projective, $\mathcal{C}$-(FP)-injective and $\mathcal{C}$-flat modules. The following result is from Theorems 2.7 and 2.8.

Theorem B. *Commutative noetherian and coherent rings can be characterized by special properties of $\mathcal{C}$-projective, $\mathcal{C}$-*(FP)*-injective, and $\mathcal{C}$-flat modules.*

Throughout this work, R is a commutative ring with an identity and all modules are unitary. So when we say a ring is coherent we mean that it is a commutative coherent ring. For an R-module M, its character module is denoted by $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. $\mathcal{C}$ is always a semidualizing module for R. A subcategory or a class of modules always means a full subcategory of the category of R-modules, which is closed under isomorphisms. For unexplained concepts and notation, we refer readers to [1], [8] and [23].

1. Preliminaries

In this section we give some terminology for use throughout this work, and recall some definitions and some known results that we need in the sequel. Among them are semidualizing modules, Auslander class, Bass class, $\mathcal{C}$-projective, $\mathcal{C}$-(FP)-injective, and $\mathcal{C}$-flat modules with respect to a semidualizing module $\mathcal{C}$.

Definition 1.1 [23]. An R-complex is a sequence of R-module homomorphisms

$$X = \dots \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\partial_{n-1}^X} \dots$$

such that $\partial_{n-1}^X \partial_n^X = 0$ for each integer n, the n-th homology module of X is $\mathrm{H}_n(X) = \mathrm{Ker}(\partial_n^X)/\mathrm{Im}(\partial_{n+1}^X)$. An R-complex X is said to be $\mathrm{Hom}_R(\mathcal{X}, -)$-exact, if $\mathrm{Hom}_R(\mathcal{X}, X)$ is an exact sequence, where $\mathcal{X}$ is a class of modules. Similarly we can define $\mathrm{Hom}_R(-, \mathcal{X})$-exact and $- \otimes_R \mathcal{X}$-exact complexes.

Note that we always identify R-modules with complexes concentrated in degree 0.

Definition 1.2 [23]. For a given R-module M and a class of R-modules $\mathcal{X}$, an augmented $\mathcal{X}$-resolution of M is an exact sequence $X^+ \colon \dots \longrightarrow X_n \xrightarrow{\partial_n^X} X_{n-1} \longrightarrow \dots \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} M \longrightarrow 0$ with $X_i \in \mathcal{X}$, for all i. The truncated complex $X\colon \dots \longrightarrow X_n \xrightarrow{\partial_n^X} X_{n-1} \longrightarrow \dots \xrightarrow{\partial_1^X} X_0 \longrightarrow 0$ is called an $\mathcal{X}$-resolution of M. An $\mathcal{X}$-resolution of M is said to be proper, if the corresponding augmented resolution X^+ is $\mathrm{Hom}_R(\mathcal{X}, -)$-exact. The $\mathcal{X}$-projective dimension of M is defined as

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \geqslant 0\colon\ X_n \neq 0\}\colon\ X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

Dually, we can define a $\mathcal{Y}$-coresolution of an R-module N with a given class of R-modules $\mathcal{Y}$ and the $\mathcal{Y}$-injective dimension of N, denoted by $\mathcal{Y}\text{-id}_R(N)$. We denote an augmented $\mathcal{Y}$-coresolution by ${}^+Y$ and the corresponding $\mathcal{Y}$-coresolution Y is said to be proper if ${}^+Y$ is $\mathrm{Hom}_R(-, \mathcal{Y})$-exact.

As usual, for an R-module M, we denote projective, injective and flat dimension of M by $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ respectively.

The definition of semidualizing modules has already been extended to arbitrary associative rings (see [15, Definition 2.1]). Note here that the ground ring is commutative.

Definition 1.3 [22]. An R-module $\mathcal{C}$ is semidualizing if it satisfies the following conditions:

(1) $\mathcal{C}$ admits a (possibly unbounded) resolution by finitely generated projective modules.
(2) The natural homothety map $R \to \mathrm{Hom}_R(\mathcal{C}, \mathcal{C})$ is an isomorphism, and
(3) $\mathrm{Ext}_R^{\geqslant 1}(\mathcal{C}, \mathcal{C}) = 0$.

A free R-module of rank one is semidualizing. If R is noetherian and admits a dualizing module D, then D is semidualizing. More examples of semidualizing modules can be found in [6], [15].

In [15], Holm and White defined $\mathcal{C}$-projective, $\mathcal{C}$-injective and $\mathcal{C}$-flat modules in order to study the *Auslander class* $\mathcal{A}_{\mathcal{C}}(R)$ and the *Bass class* $\mathcal{B}_{\mathcal{C}}(R)$ with respect to a semidualizing module $\mathcal{C}$. These are the objects of our main interest.

Definition 1.4 [15]**.** Let $\mathcal{C}$ be a semidualizing module for a ring R. An R-module is $\mathcal{C}$-projective ($\mathcal{C}$-flat) if it is of the form $\mathcal{C} \otimes_R P(\mathcal{C} \otimes_R F)$ for some projective (flat) module $P(F)$. An R-module is $\mathcal{C}$-(FP)-injective if it is of the form $\mathrm{Hom}_R(\mathcal{C}, E)$ for some (FP)injective module E. We set

$$\begin{aligned} \mathcal{P}_{\mathcal{C}} &= \text{the category of } \mathcal{C}\text{-projective modules},\\ \mathcal{I}_{\mathcal{C}} &= \text{the category of } \mathcal{C}\text{-injective modules},\\ \mathcal{F}_{\mathcal{C}} &= \text{the category of } \mathcal{C}\text{-flat modules}. \end{aligned}$$

Over a commutative noetherian ring, Avramov and Foxby [2], [10] and Enochs, Jenda and Xu [7] connected the study of (semi)dualizing modules to the associated *Auslander class* and *Bass class* for (semi)dualizing modules, $\mathcal{A}_{\mathcal{C}}(R)$ and $\mathcal{B}_{\mathcal{C}}(R)$, which are subcategories of the category of R-modules.

Definition 1.5 [22]**.** The *Auslander class* of R with respect to $\mathcal{C}$, denoted $\mathcal{A}_{\mathcal{C}}(R)$, consists of modules M satisfying

(a1) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0$,

(a2) $\mathrm{Ext}^{\geqslant 1}_R(C, C \otimes_R M) = 0$, and

(a3) the canonical map $\mu_M \colon M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* of R with respect to C, denoted $\mathcal{B}_{\mathcal{C}}(R)$, consists of modules N satisfying

(b1) $\mathrm{Ext}^{\geqslant 1}_R(C, N) = 0$,

(b2) $\mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_R(C, N) = 0)$, and

(b3) the canonical map $\nu_N \colon C \otimes_R \mathrm{Hom}_R(C, N) \to N$ is an isomorphism.

Remark 1.6. Let $\mathcal{C}$ be a semidualizing R-module. Holm and White defined faithfully semidualizing modules over non-commutative rings, see [15, Definition 3.1]. And they showed that if R is commutative, then a semidualizing module is always faithfully semidualizing ([15, Proposition 3.6]). If two of the three modules in a short exact sequence are in $\mathcal{A}_{\mathcal{C}}(R)(\mathcal{B}_{\mathcal{C}}(R))$, then so is the third (see [15, Corollary 6.7.]). The category $\mathcal{A}_{\mathcal{C}}(R)$ contains modules of finite flat dimension and modules of finite $\mathcal{I}_{\mathcal{C}}$-injective dimension, and the category $\mathcal{B}_{\mathcal{C}}(R)$ contains modules of finite injective dimension and modules of finite $\mathcal{F}_{\mathcal{C}}$-projective, hence $\mathcal{P}_{\mathcal{C}}$-projective dimension (see [15, Corollaries 6.4 and 6.6]).

The next lemma is used frequently, and it goes back to [5, Lemma 3.2.9] and [20, Theorem 2.8]

Lemma 1.7. *Let $\mathcal{C}$ be a semidualizing module. Then for a given R-module M we have*
(1) *$M \in \mathcal{A}_{\mathcal{C}}(R)$ if and only if $M^* \in \mathcal{B}_{\mathcal{C}}(R)$, and $M \in \mathcal{B}_{\mathcal{C}}(R)$ if and only if $M^* \in \mathcal{A}_{\mathcal{C}}(R)$,*
(2) *$M \in \mathcal{B}_{\mathcal{C}}(R)$ if and only if $\operatorname{Hom}_R(\mathcal{C}, M) \in \mathcal{A}_{\mathcal{C}}(R)$,*
(3) *$M \in \mathcal{A}_{\mathcal{C}}(R)$ if and only if $\mathcal{C} \otimes_R M \in \mathcal{B}_{\mathcal{C}}(R)$.*

2. Characterizations of some rings

The main result of this work is to characterize coherent and noetherian rings in terms of $\mathcal{C}$-(FP)-injective and $\mathcal{C}$-flat modules. Before giving the main theorems we discuss the relationship between $\mathcal{C}$-flat and $\mathcal{C}$-FP-injective modules. We will show that [23, Lemma 4.1(b)] is true when $\mathcal{C}$-*injective* is replaced by $\mathcal{C}$-FP-*injective*. Note that Lambek [17] proved that an R-module is flat if and only if its character module M^* is injective. In fact, if we replace *injective* by FP-*injective*, we have the following result:

Lemma 2.1. *Let M be an R-module. Then M is flat if and only if M^* is FP-injective.*

Proof. If M is flat, then M^* is injective, hence FP-injective.

Now suppose that M^* is FP-injective. Let $\mathfrak{I}$ be any finitely generated ideal of R. We have the following commutative diagram with vertical isomorphisms, where the top exact sequence is obtained by applying the functor $\operatorname{Hom}_R(-, M^*)$ to the exact sequence $0 \to \mathfrak{I} \to R \to R/\mathfrak{I} \to 0$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_R(R/\mathfrak{I}, M^*) & \longrightarrow & \operatorname{Hom}_R(R, M^*) & \longrightarrow & \operatorname{Hom}_R(\mathfrak{I}, M^*) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & ((R/\mathfrak{I}) \otimes_R M)^* & \longrightarrow & (R \otimes_R M)^* & \longrightarrow & (\mathfrak{I} \otimes_R M)^* & \longrightarrow & 0
\end{array}$$

The exactness of the top row gives the exactness of the bottom one, especially, $(R \otimes_R M)^* \to (\mathfrak{I} \otimes_R M)^* \to 0$ is exact. Therefore, $0 \to \mathfrak{I} \otimes_R M \to R \otimes_R M$ is exact. So M is flat by [19, Theorem 3.53]. □

The following lemma is taken form [16], and is used for the proof of Lemma 2.3.

Lemma 2.2 [16, Lemma 4.54]. *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of modules.*

(1) *If* M *is finitely presented and* M' *is finitely generated, then* M'' *is finitely presented.*

(2) *If* M' *and* M'' *are finitely presented, then* M *is finitely presented.*

(3) *A direct sum* $M_1 \oplus \ldots \oplus M_n$ *is finitely presented if and only if each* M_i *is finitely presented.*

We know that injective modules are in the *Bass class* $\mathcal{B}_{\mathcal{C}}(R)$ (Remark 1.6.). Now we want to show that the same is true for FP-injective modules.

Lemma 2.3. *The class of* FP-*injective modules is contained in* $\mathcal{B}_{\mathcal{C}}(R)$. *Moreover, the class of modules of finite* FP-*injective dimension is contained in* $\mathcal{B}_{\mathcal{C}}(R)$.

P r o o f. It is easy to see that $\operatorname{Hom}_R(P, K)$ is finitely presented if P is a finitely generated projective module and K is finitely presented. Assume that I is an FP-injective module. Choose a degreewise finite projective resolution $\mathbb{P}_{\mathcal{C}}$ of $\mathcal{C}$:

$$\mathbb{P}_{\mathcal{C}} = \ldots \to P_n \xrightarrow{\partial_n^P} \ldots \longrightarrow P_0 \xrightarrow{\partial_0^P} \mathcal{C} \longrightarrow 0.$$

Since P_i is finitely generated projective for each i, hence every syzygy $K_i = \operatorname{Ker} \partial_{i-1}^P$ ($K_0 = \mathcal{C}$) is finitely presented, hence $\operatorname{Ext}_R^1(K_i, I) = 0$ for each $i \geqslant 0$. Therefore we get an exact sequence when we apply the functor $\operatorname{Hom}_R(-, I)$ to $\mathbb{P}_{\mathcal{C}}$, and this just means $\operatorname{Ext}_R^{\geqslant 1}(\mathcal{C}, I) = 0$. So (b1) is true for I. Next we show that (b2) is also true for I. Since $\operatorname{Ext}_R^{\geqslant 1}(\mathcal{C}, \mathcal{C}) = 0$, the sequence

$$\operatorname*{Hom}_R(\mathbb{P}_{\mathcal{C}}, \mathcal{C})\colon\ 0 \to \operatorname*{Hom}_R(\mathcal{C}, \mathcal{C}) \xrightarrow{(\partial_0^P)^*} \operatorname*{Hom}_R(P_0, \mathcal{C}) \xrightarrow{(\partial_1^P)^*} \operatorname*{Hom}_R(P_1, \mathcal{C}) \longrightarrow \ldots$$

is exact. The equality in the following sequence holds by definition and the isomorphism is the Hom-evaluation [5, Appendix].

$$\operatorname{Tor}_{i \geqslant 1}^R(\mathcal{C}, \operatorname*{Hom}_R(\mathcal{C}, I)) = \mathrm{H}_i(\mathbb{P}_{\mathcal{C}} \otimes_R \operatorname*{Hom}_R(\mathcal{C}, I)) \cong \mathrm{H}_i(\operatorname*{Hom}_R(\operatorname*{Hom}_R(\mathbb{P}_{\mathcal{C}}, \mathcal{C}), I).$$

According to Lemma 2.2, $\operatorname{Coker}(\partial_i^P)^*$ is finitely presented for each $i \geqslant 0$. So we get an exact sequence when we apply the functor $\operatorname{Hom}_R(-, I)$ to the sequence $\operatorname{Hom}_R(\mathbb{P}_{\mathcal{C}}, \mathcal{C})$. This is to say the sequence $\operatorname{Hom}_R(\operatorname{Hom}_R(\mathbb{P}_{\mathcal{C}}, \mathcal{C}), I)$ is homological trivial, so $\operatorname{Tor}_{i \geqslant 1}^R(\mathcal{C}, \operatorname{Hom}_R(\mathcal{C}, I)) = 0$. At last, we claim that the canonical map $\nu_I\colon \mathcal{C} \otimes_R \operatorname{Hom}_R(\mathcal{C}, I) \to I$ is an isomorphism. Taking the truncated resolution

$P_1 \to P_0 \to \mathcal{C} \to 0$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
P_1 \otimes_R \operatorname{Hom}_R(\mathcal{C}, I) & \longrightarrow & P_0 \otimes_R \operatorname{Hom}_R(C, I) & \longrightarrow & \mathcal{C} \otimes_R \operatorname{Hom}_R(\mathcal{C}, I) & \longrightarrow & 0 \\
\cong \downarrow & & \cong \downarrow & & \beta \downarrow & & \\
\operatorname{Hom}_R(\operatorname{Hom}_R(P_1, \mathcal{C}), I) & \to & \operatorname{Hom}_R(\operatorname{Hom}_R(P_0, \mathcal{C}), I) & \to & \operatorname{Hom}_R(\operatorname{Hom}_R(\mathcal{C}, \mathcal{C}), I) & \to & 0
\end{array}$$

The morphism β is an isomorphism by the *Five lemma*. So our claim holds, since $\operatorname{Hom}_R(\mathcal{C}, \mathcal{C}) \cong R$.

The lemma is proved if we apply Remark 1.6. □

As a consequence of the above lemma and Lemma 1.7(2), we have

Corollary 2.4. *If $\mathcal{C}$ is a semidualizing R-module, then the Auslander class $\mathcal{A}_{\mathcal{C}}(R)$ contains the class of $\mathcal{C}$-*FP*-injective modules.*

Now we can give the following proposition:

Proposition 2.5. *Let M be an R-module and $\mathcal{C}$ a semidualizing module. Then M is $\mathcal{C}$-flat if and only if M^* is $\mathcal{C}$-*FP*-injective.*

Proof. By [23, Lemma 4.1(b)], if M is $\mathcal{C}$-flat, then M^* is $\mathcal{C}$-injective, hence, $\mathcal{C}$-FP-injective.

Suppose that M^* is $\mathcal{C}$-FP-injective. Then there exists an FP-injective module I such that $M^* = \operatorname{Hom}_R(\mathcal{C}, I)$. The first isomorphism of the following sequence is from [8, Theorem 3.2.11] and the second is from Lemma 2.3:

$$\begin{aligned}
\operatorname*{Hom}_{\mathbb{Z}}(\operatorname*{Hom}_{R}(\mathcal{C}, M), \mathbb{Q}/\mathbb{Z}) &\cong \mathcal{C} \otimes_R \operatorname*{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \mathcal{C} \otimes_R M^* \\
&= \mathcal{C} \otimes_R \operatorname*{Hom}_{R}(\mathcal{C}, I) \cong I.
\end{aligned}$$

This means that $(\operatorname{Hom}_R(\mathcal{C}, M))^*$ is FP-injective. So $\operatorname{Hom}_R(\mathcal{C}, M)$ is flat by Lemma 2.1. Note that $M^* \in \mathcal{A}_{\mathcal{C}}(R)$ by Corollary 2.4, hence $M \in \mathcal{B}_{\mathcal{C}}(R)$ by Lemma 1.7(1). Thus $M \cong \mathcal{C} \otimes_R \operatorname{Hom}_R(\mathcal{C}, M)$ is $\mathcal{C}$-flat. □

The next lemma is used frequently throughout this work. We only give the proof of (1), the proof of (2) is similar to (1), using Lemma 2.3. And the proofs of (3) and (4) are dual to that of (1).

Lemma 2.6. *Let $\mathcal{C}$ be a semidualizing R-module. Then the following implications hold:*

(1) *If* $\operatorname{Hom}_R(\mathcal{C}, E)$ *is* $\mathcal{C}$*-injective, then* E *is an injective module.*

(2) *If* $\operatorname{Hom}_R(\mathcal{C}, I)$ *is* $\mathcal{C}$-FP*-injective, then* I *is an* FP*-injective module.*

(3) *If* $\mathcal{C} \otimes_R F$ *is* $\mathcal{C}$*-flat, then* F *is a flat module.*

(4) *If* $\mathcal{C} \otimes_R P$ *is* $\mathcal{C}$*-projective, then* P *is a projective module.*

Proof. (1) Since $\operatorname{Hom}_R(\mathcal{C}, E)$ is $\mathcal{C}$-injective, then $E \in \mathcal{B}_{\mathcal{C}}(R)$ (Remark 1.6 and Lemma 1.7), and there is an injective module E' such that $\operatorname{Hom}_R(\mathcal{C}, E) \cong \operatorname{Hom}_R(\mathcal{C}, E')$. Applying the functor $\mathcal{C} \otimes_R$ to the isomorphism, we have:

$$E \cong \mathcal{C} \otimes_R \operatorname*{Hom}_R(\mathcal{C}, E) \cong \mathcal{C} \otimes_R \operatorname*{Hom}_R(\mathcal{C}, E') \cong E'.$$

Thus E is injective. $\square$

We know that coherent rings can be characterized by various conditions. Cheatham and Stone [4, Theorem 1] proved that the following four conditions are equivalent for a ring R: (1) R is coherent, (2) an R-module I is FP-injective if and only if its character module I^* is flat, (3) an R-module I is FP-injective if and only if I^{**} is injective, (4) an R-module F is flat if and only if F^{**} is flat. Zhu [25, Theorem 2.11] showed that the condition that a ring R is coherent is equivalent to the following two conditions: (1) for every injective module E, its character module E^* is flat, (2) for every R-module M, $\operatorname{fd}_R M^* \leqslant \operatorname{id}_R M$. Moreover, Enochs and Jenda proved that a ring R is coherent if and only if the class of flat modules is preenveloping [8, Theorem 6.5.1], see [8, Definition 6.1.1] for the definition of the preenvelope. Now we are in the position to give one of our main results:

Theorem 2.7. *Let $\mathcal{C}$ be a semidualizing R-module. The following conditions are equivalent for a commutative ring R.*

(1) *R is coherent.*

(2) *An R-module N is* $\mathcal{C}$-FP*-injective if and only if its character module N^* is $\mathcal{C}$-flat.*

(3) *An R-module N is* $\mathcal{C}$-FP*-injective if and only if N^{**} is $\mathcal{C}$-injective.*

(4) *An R-module F is $\mathcal{C}$-flat if and only if F^{**} is $\mathcal{C}$-flat.*

(5) *For every $\mathcal{C}$-injective module E, its character module E^* is $\mathcal{C}$-flat.*

(6) *For every R-module M, $\mathcal{F}_{\mathcal{C}}\text{-}\operatorname{pd}_R M^* \leqslant \mathcal{I}_{\mathcal{C}}\text{-}\operatorname{id}_R M$.*

(7) *The class of $\mathcal{C}$-flat modules $\mathcal{F}_{\mathcal{C}}$ is preenveloping.*

Proof. We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$(4) and (4) $\Rightarrow$ (1). Note that (2) $\Rightarrow$ (5) is obvious. Then we show that (5) $\Rightarrow$ (6), (6) $\Rightarrow$ (1). Finally we prove that (1) $\Leftrightarrow$ (7). Thus we establish the desired conclusion.

(1) $\Rightarrow$ (2). Let R be a coherent ring.

Assume that N is $\mathcal{C}$-FP-injective, then we have $N \cong \mathrm{Hom}_R(\mathcal{C}, I)$ with I FP-injective. Thus $N^* = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(\mathcal{C}, I), \mathbb{Q}/\mathbb{Z}) \cong \mathcal{C} \otimes_R \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$. $\mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is flat, since R is coherent. Thus N^* is $\mathcal{C}$-flat.

Assume that N^* is $\mathcal{C}$-flat. By Remark 1.6, $N^* \in \mathcal{B}_{\mathcal{C}}(R)$. This gives the isomorphism $\mathcal{C} \otimes_R \mathrm{Hom}_R(\mathcal{C}, N^*) \cong N^*$, so $\mathcal{C} \otimes_R \mathrm{Hom}_R(\mathcal{C}, N^*)$ is $\mathcal{C}$-flat. Thus $(\mathcal{C} \otimes_R N)^* \cong \mathrm{Hom}_R(\mathcal{C}, N^*)$ is flat by Lemma 2.6(3), and $\mathcal{C} \otimes_R N$ is FP-injective by [4, Theorem 1(2)], hence $\mathcal{C} \otimes_R N \in \mathcal{B}_{\mathcal{C}}(R)$ by Lemma 2.3. From Lemma 1.7(3) we know that $N \in \mathcal{A}_{\mathcal{C}}(R)$. Therefore $N \cong \mathrm{Hom}_R(\mathcal{C}, \mathcal{C} \otimes_R N)$ is $\mathcal{C}$-FP-injective.

(2) $\Rightarrow$ (3). N is $\mathcal{C}$-FP-injective if and only if N^* is $\mathcal{C}$-flat by (2), if and only if N^{**} is $\mathcal{C}$-injective by [23, Lemma 4.1(b)]. Thus (3) holds.

(3) $\Rightarrow$ (4). Suppose that F is $\mathcal{C}$-flat. Then F^* is $\mathcal{C}$-injective, hence $\mathcal{C}$-FP-injective, so F^{***} is $\mathcal{C}$-injective by (3). [23, Lemma 4.1] yields that F^{**} is $\mathcal{C}$-flat. Conversely, assume that F^{**} is $\mathcal{C}$-flat, then [23, Lemma 4.1] yields that F^{***} is $\mathcal{C}$-injective. By (3) F^* is $\mathcal{C}$-FP-injective, hence Proposition 2.5 yields that F is $\mathcal{C}$-flat. Thus (4) holds.

(4) $\Rightarrow$ (1). By [19, Lemma 3.60] and the *Hom-tensor adjointness* we have the following sequence (2.1) for any R-module F:

$$\mathcal{C} \otimes_R F^{**} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(\mathcal{C}, F^*), \mathbb{Q}/\mathbb{Z}) \cong (\mathcal{C} \otimes_R F)^{**}. \tag{2.1}$$

Then F is flat if and only if $\mathcal{C} \otimes_R F$ is $\mathcal{C}$-flat by Lemma 2.6(3), if and only if $(\mathcal{C} \otimes_R F)^{**}$ is $\mathcal{C}$-flat by (4), if and only if $\mathcal{C} \otimes_R F^{**}$ is $\mathcal{C}$-flat by (2.1), if and only if F^{**} is flat by Lemma 2.6(3). Thus R is coherent by Lemma [4, Theorem 1(4)].

(5) $\Rightarrow$ (6). For every module M, we apply the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to an $\mathcal{I}_{\mathcal{C}}$-injective coresolution of M, then we get an $\mathcal{F}_{\mathcal{C}}$-projective resolution of M^* by (5). Therefore $\mathcal{F}_{\mathcal{C}}\text{-pd}_R M^* \leqslant \mathcal{I}_{\mathcal{C}}\text{-id}_R M$.

(6) $\Rightarrow$ (1). By [25, Theorem 2.11(6)], we only need to show that for every R-module M, $\mathrm{fd}_R M^* \leqslant \mathrm{id}_R M$. If $\mathrm{id}_R M = \infty$, we are done. Now assume that $\mathrm{id}_R M = n < \infty$, then $M \in \mathcal{B}_{\mathcal{C}}(R)$ by Remark 1.6. Choose an injective resolution $\mathbb{E}_M$ of M: $\mathbb{E}_M = 0 \to M \to E_0 \to \ldots \to E_{-n} \to 0$. This sequence is $\mathrm{Hom}_R(\mathcal{C}, -)$-exact, so $\mathcal{I}_{\mathcal{C}}\text{-id}_R \mathrm{Hom}_R(\mathcal{C}, M) \leqslant n$. By (6), we have $\mathcal{F}_{\mathcal{C}}\text{-pd}_R(\mathrm{Hom}_R(\mathcal{C}, M)^*) \leqslant n$. Suppose that $\mathbb{F}^{\mathcal{C}}_{\mathrm{Hom}_R(\mathcal{C},M)^*} = 0 \to \mathcal{C} \otimes_R F_n \to \ldots \to \mathcal{C} \otimes_R F_0 \to \mathrm{Hom}_R(\mathcal{C}, M)^*$ is an $\mathcal{F}_{\mathcal{C}}$-projective resolution of $\mathrm{Hom}_R(\mathcal{C}, M)^*$. By Lemma 1.7 $\mathrm{Hom}_R(\mathcal{C}, M) \in \mathcal{B}_{\mathcal{C}}(R)$, so $\mathbb{F}^{\mathcal{C}}_{\mathrm{Hom}_R(\mathcal{C},M)^*}$ is $\mathrm{Hom}_R(\mathcal{C}, -)$-exact. This implies that $\mathrm{fd}_R(\mathrm{Hom}_R(\mathcal{C}, \mathrm{Hom}_R(\mathcal{C}, M))^*) \leqslant n$. But $\mathrm{Hom}_R(\mathcal{C}, \mathrm{Hom}_R(\mathcal{C}, M)^*) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C} \otimes_R \mathrm{Hom}_R(\mathcal{C}, M), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, since $M \in \mathcal{B}_{\mathcal{C}}(R)$. Thus $\mathrm{fd}_R M^* \leqslant n$.

(1) $\Rightarrow$ (7). This follows from [15, Proposition 5.10.(d)].

(7) $\Rightarrow$ (1). Assume that $\{L_\lambda\}_\Lambda$ is a family of $\mathcal{C}$-flat modules. Since the class of $\mathcal{C}$-flat modules $\mathcal{F}_{\mathcal{C}}$ is preenveloping, we have the following commutative diagram, in

which $\mathcal{C} \otimes F$ is an $\mathcal{F}_\mathcal{C}$-preenvelope of $\prod L_\lambda$, p_λ is the canonical projection, and f_λ is the induced morphism.

$$\begin{array}{ccc} \prod L_\lambda & \xrightarrow{\sigma} & \mathcal{C} \otimes_R F \\ \downarrow{\scriptstyle p_\lambda} & \swarrow{\scriptstyle f_\lambda} & \\ L_\lambda & & \end{array}$$

It is easy to check that $\prod f_\lambda \circ \sigma = \mathrm{id}_{\prod L_\lambda}$, so $\prod L_\lambda$ is a direct summand of $\mathcal{C} \otimes_R F$, hence a $\mathcal{C}$-flat module by [15, Proposition 5.5.(a)]. Now we show that the direct product of flat modules is still flat; then by [11, Theorem 2.3.2] R is coherent. Suppose that $\{F_\lambda\}_\Lambda$ is a family of flat modules, then $\prod\{\mathcal{C} \otimes_R F_\lambda\}$ is a $\mathcal{C}$-flat module. Thus $\prod\{\mathcal{C} \otimes_R F_\lambda\} \cong \mathcal{C} \otimes_R F$ for some flat module F. Therefore we have the following isomorphisms, which shows that $\prod F_\lambda$ is flat:

$$\prod F_\lambda \cong \prod \operatorname*{Hom}_R\Big(\mathcal{C}, \prod\{\mathcal{C} \otimes_R F_\lambda\}\Big) \cong \operatorname*{Hom}_R(\mathcal{C}, \mathcal{C} \otimes_R F) \cong F.$$

□

Cheatham and Stone [4, Theorem 2] proved that a ring R is noetherian is equivalent to the condition that an R-module M is injective if and only if M^{**} is injective, Enochs and Jenda [8, Theorem 5.4.1] showed that a ring R is noetherian if and only if the class of injective modules $\mathcal{E}$ is precovering if and only if $\mathcal{E}$ is covering, Fieldhouse [9, Theorem 2.2] showed that R is noetherian is equivalent to the condition that for any R-module we have $\mathrm{fd}_R(M^*) = \mathrm{id}_R M$, while Megibben [18, Theorem 3] proved that R is noetherian if and only if FP-injective modules are injective. Recently, Takahashi and White [20, Proposition 5.3] proved that a commutative ring R is noetherian if and only if every direct sum of $\mathcal{C}$-injective modules is $\mathcal{C}$-injective. We include it in our theorem.

Note that the proof of the following theorem is similar to that of Theorem 2.7, with necessary but obvious modifications. So we just state it without any proof.

Theorem 2.8. *Let $\mathcal{C}$ be a semidualizing R-module. The following conditions are equivalent for a commutative ring R.*

(1) *R is noetherian.*
(2) *For an given R-module M, $\mathcal{I}_\mathcal{C}\text{-}\mathrm{id}_R M = \mathcal{F}_\mathcal{C}\text{-}\mathrm{pd}_R M^*$.*
(3) *An R-module E is $\mathcal{C}$-injective if and only if E^* is $\mathcal{C}$-flat.*
(4) *Every direct sum of $\mathcal{C}$-injective modules is $\mathcal{C}$-injective.*
(5) *The class of $\mathcal{C}$-injective modules $\mathcal{I}_\mathcal{C}$ is covering.*
(6) *The class of $\mathcal{C}$-injective modules $\mathcal{I}_\mathcal{C}$ is precovering.*
(7) *Every $\mathcal{C}$-FP-injective module is $\mathcal{C}$-injective.*

As more applications of the techniques developed in the proof of Theorem 2.7, we give the following theorem, which generalizes Theorem 3 and Theorem 4 of [4].

Theorem 2.9. *The following three statements are equivalent for a commutative ring R:*

(1) *R is artinian,*

(2) *R is coherent and perfect,*

(3) *an R-module N is $\mathcal{C}$-FP-injective if and only if its character module N^* is $\mathcal{C}$-projective.*

Proof. The equivalence between (1) and (2) follows from [3, Theorems 3.3 and 3.4].

(2) $\Rightarrow$ (3). Assume that R is coherent and perfect, then an R-module I is FP-injective if and only if I^* is projective by [4, Theorem 3].

If a module N is $\mathcal{C}$-FP-injective, then there is an FP-injective module I such that $N = \mathrm{Hom}_R(\mathcal{C}, I)$. Then $N^* = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(\mathcal{C}, I), \mathbb{Q}/\mathbb{Z}) \cong \mathcal{C} \otimes_R \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z}) = \mathcal{C} \otimes_R I^*$ is $\mathcal{C}$-projective.

If N^* is $\mathcal{C}$-projective, there is a projective module P such that $N^* = \mathcal{C} \otimes_R P$. Then the following sequence shows that $(\mathcal{C} \otimes_R N)^*$ is projective, so $\mathcal{C} \otimes_R N$ is FP-injective:

$$(\mathcal{C} \otimes_R N)^* \cong \mathop{\mathrm{Hom}}_R(\mathcal{C}, \mathop{\mathrm{Hom}}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})) \cong \mathop{\mathrm{Hom}}_R(\mathcal{C}, \mathcal{C} \otimes_R P) \cong P$$

Therefore $N \cong \mathrm{Hom}_R(\mathcal{C}, \mathcal{C} \otimes_R N)$ is $\mathcal{C}$-FP-injective.

(3) $\Rightarrow$ (2). We show Theorem 3(1) of [4] holds when (2) is true.

If I is an FP-injective module, then $\mathrm{Hom}_R(\mathcal{C}, I)$ is $\mathcal{C}$-FP-injective. By assumption, $\mathcal{C} \otimes_R \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(\mathcal{C}, I), \mathbb{Q}/\mathbb{Z}) = (\mathrm{Hom}_R(\mathcal{C}, I))^*$ is $\mathcal{C}$-projective, which implies that I^* is projective.

If I^* is projective, then $(\mathrm{Hom}_R(\mathcal{C}, I))^* = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(\mathcal{C}, I), \mathbb{Q}/\mathbb{Z}) \cong \mathcal{C} \otimes_R I^*$ is $\mathcal{C}$-projective. By assumption, $\mathrm{Hom}_R(\mathcal{C}, I)$ is $\mathcal{C}$-FP-injective, which implies that I is FP-injective by Lemma 2.6(2). □

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