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A CANCELLATIVE AMENABLE ASCENDING UNION OF NONAMENABLE SEMIGROUPS

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Abstract. We construct an example of a cancellative amenable semigroup which is the ascending union of semigroups, none of which are amenable.

Keywords: amenability, semigroups, ascending union

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1. Introduction

The theory of amenable groups and semigroups began in the early twentieth century with the work of Banach, when he showed that there exists an invariant mean on the set of all bounded real valued functions on $\mathbb{R}$ [1]. Von Neumann then showed that for high enough dimension, the rotation group of the n-sphere has free subgroups on two generators [14]. This elaborated on a previous result of Hausdorff, which shows that no mean, which is invariant under all rotations of the 2-sphere, exists on the set of bounded functions on the 2-sphere [10]. Von Neumann proved that if a group G contains a free subgroup on two generators, then G is nonamenable [14]. The converse of this result has been shown to be false in general. In particular, Olshanskii has constructed an example of a nonamenable group which contains no free subgroup on two generators [15]. Examples of amenable groups and semigroups include all commutative semigroups (in particular all abelian groups), all finite groups, and all solvable groups. Examples of nonamenable groups and semigroups include any free group or semigroup of rank two or higher. For more on amenable groups and semigroups, see [2], [5], [7], [8], [9].

It is well known that if a semigroup S is the ascending union of right amenable semigroups, then S is right amenable [2]. In [4], the author constructs an example of a right amenable semigroup which is the ascending union of subsemigroups, none of which are right amenable. However, the semigroup constructed in [4] is not cancellative. It is known that any subgroup of an amenable group is also amenable [5]. Thus, if a group G is the ascending union of nonamenable subgroups, then G is nonamenable. There are many such results about amenability which are true for groups, but false for semigroups in general. Yet, while many of these results are false for semigroups in general, many of them are true not only for groups, but also for cancellative semigroups. For example:

- If S is a cancellative semigroup which contains a free subgroup on two generators, then S is nonamenable.
- A cancellative semigroup S is right amenable if and only if for each $\varepsilon \in (0,1)$, and for each finite, nonempty subset $H \subseteq S$, there exists a finite, nonempty subset $E \subseteq S$ such that for each $h \in H$, $|E \cap Eh|/|E| > \varepsilon$ [3], [6], [13].
- If S is a cancellative semigroup and $\mathcal{P}(S)$ is the power set of S, then S is right amenable if and only if there exists a finitely additive, right invariant measure $m\colon \mathcal{P}(S) \to [0,1]$ which is normed by S [12], [16].
- If every finitely generated subsemigroup of a cancellative semigroup S is right amenable, then S is right amenable [6].
- If S is a finite cancellative semigroup, then S is right amenable [2], [3], [6].

Thus, one can ask the question: If a cancellative semigroup S is the ascending union of subsemigroups, none of which are right amenable, then is it necessary for S not to be right amenable? In this paper, we show that the answer to this question is no by constructing an even stronger example. Namely, we construct an example of an amenable group which is the ascending union of submonoids, none of which are right amenable (note that by the above statement, none of the submonoids are themselves groups).

2. An amenable ascending union of nonamenable monoids

In this section we construct an example of an amenable group which is the ascending union of submonoids, none of which are right amenable. The following lemma is proved in [2].

Lemma 2.1. *Let S and Q be semigroups. If S is right amenable, and $f\colon S \to Q$ is a semigroup homomorphism onto Q, then Q is right amenable.*

In [11], Hochster constructs an example of an amenable group H which is generated by the free semigroup S_2 on two generators. Let S_2^* denote the free monoid on two generators. Let $\mathcal{D}$ be the weak product of infinitely many copies of H with itself.

That is, $\mathcal{D}$ is the group such that each element of $\mathcal{D}$ is an infinite sequence of elements from H such that all but finitely many terms in the sequence are the identity element of H.

Theorem 2.1. *The group $\mathcal{D}$ is the amenable ascending union of submonoids, none of which are right amenable.*

Proof. Let $n \in \mathbb{Z}$ with $n \geqslant 1$. For each $j = 1, 2, 3, \ldots, n$, let $B_j = H$, and for each $j \in \mathbb{Z}$ with $j \geqslant n+1$, let B_j be the identity element 1_H of the group H. Let $J_n = \prod_{j=1}^{\infty} B_j$. For each $j = 1, 2, 3, \ldots, n$, let $A_j = H$. Let $A_{n+1} = S_2^*$. For each $j \in \mathbb{Z}$ with $j \geqslant n+2$, let A_j be the identity element 1_H of the group H. Let $K_n = \prod_{j=1}^{\infty} A_j$. Clearly, $\mathcal{D}$ is the ascending union of the subgroups $J_1 \subseteq J_2 \subseteq J_3 \subseteq J_4 \subseteq \ldots \subseteq J_n \subseteq \ldots$.

Since $1_H \in S_2^*$, $J_n \subseteq K_n$ for each $n \geqslant 1$, which implies that $\bigcup_{n=1}^{\infty} J_n \subseteq \bigcup_{n=1}^{\infty} K_n$. Since $S_2^* \subseteq H$, $K_n \subseteq J_{n+1}$ for each $n \geqslant 1$, which implies that $\bigcup_{n=1}^{\infty} K_n \subseteq \bigcup_{n=1}^{\infty} J_n$. Thus, $\mathcal{D} = \bigcup_{n=1}^{\infty} J_n = \bigcup_{n=1}^{\infty} K_n$. In particular, $\mathcal{D}$ is the ascending union of submonoids $K_1 \subseteq K_2 \subseteq K_3 \subseteq K_4 \subseteq \ldots \subseteq K_n \subseteq \ldots$.

Since H is amenable, it follows that for each $n \geqslant 1$, J_n is amenable [2]. Since $\mathcal{D}$ is the ascending union of the amenable subgroups $J_1 \subseteq J_2 \subseteq J_3 \subseteq J_4 \subseteq \ldots \subseteq J_n \subseteq \ldots$, it follows that $\mathcal{D}$ is amenable [2].

Let $n \geqslant 1$. Let $\varphi_n \colon K_n \to S_2^*$ be the projection map from K_n onto its $n + 1^{st}$ coordinate. It is straitforward to check that φ_n is a semigroup homomorphism onto S_2^*. Since S_2^* is not right amenable, it follows by Lemma 2.1 that K_n is not right amenable. □

3. Concluding remarks

Note that the example $\mathcal{D}$ constructed above in this paper, as well as the example constructed in [4], contain a free subsemigroup on two generators. In [6], Frey proves that if S is a right amenable, cancellative semigroup which does not contain a free subsemigroup on two generators, then every subsemigroup of S is right amenable. Thus, if S is a cancellative semigroup which does not contain a free subsemigroup on two generators, and S is the ascending union of subsemigroups, none of which are right amenable, then S is not right amenable.

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