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THE $\overline{\partial}$-NEUMANN OPERATOR ON LIPSCHITZ q-PSEUDOCONVEX DOMAINS

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Abstract. On a bounded q-pseudoconvex domain Ω in $\mathbb{C}^n$ with a Lipschitz boundary, we prove that the $\overline{\partial}$-Neumann operator N satisfies a subelliptic $(1/2)$-estimate on Ω and N can be extended as a bounded operator from Sobolev $(-1/2)$-spaces to Sobolev $(1/2)$-spaces.

Keywords: Sobolev estimate, $\overline{\partial}$ and $\overline{\partial}$-Neumann operator, q-pseudoconvex domains, Lipschitz domains

MSC 2010: 32F10, 32W05

0. Introduction and results

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with the standard Hermitian metric. The first major step towards subelliptic estimates for the $\overline{\partial}$-Neumann problem was achieved by Kohn [19], [20]. In these papers he considered smoothly bounded, strictly pseudoconvex domains and showed that on such domains a subelliptic $(1/2)$-estimate holds. A complete characterization of subelliptic $(1/2)$-estimate for (r,q)-forms at a boundary point z_0 was obtained by Hörmander [17]. That is, a subelliptic $(1/2)$-estimate holds for (r,q)-forms in a neighborhood of z_0 if and only if the domain satisfies the $Z(q)$ condition. If Ω is a smooth bounded domain admitting a defining function that is plurisubharmonic on the boundary of Ω, Boas-Straube [5] showed that the $\overline{\partial}$-Neumann operator N is bounded on Sobolev spaces $W^l(\Omega)$ for all $l \geqslant 0$. On a smoothly bounded, pseudoconvex domain Ω, Bonami-Charpentier [6] proved that the operator $\overline{\partial}^{\star} N$ is bounded on $W^{1/2}_{r,s}(\Omega)$. When the pseudoconvex domain Ω is only Lipschitz with a plurisubharmonic defining function, they also proved that the operator $\overline{\partial}^{\star} N$ is bounded from $W^{1/2+\varepsilon}_{r,s}(\Omega)$ to $W^{1/2}_{r,s}(\Omega)$ for any $\varepsilon > 0$. Henkin-Iordan-Kohn [15] and Michel-Shaw [21] obtained subelliptic $(1/2)$-estimates for N on piecewise smooth intersections of strictly pseudoconvex domains. Straube [24] obtained subelliptic ε-estimates ($\varepsilon < \frac{1}{2}$) for piecewise smooth intersections of finite 1-D'Angelo type domains. When the domain is bounded pseudoconvex with a plurisubharmonic Lipschitz defining function, Michel-Shaw [22] proved that N is bounded on $W^{1/2}_{r,s}(\Omega)$. Engliš [11] pointed out that Hörmander's [17] (in Folland-Kohn [13]) and Catlin's [7] arguments are local in obtaining the subelliptic estimate. Abdelkader-Saber [2], [1], showed that N can be extended as a bounded operator from $W^{-1/2}_{r,s}(\Omega)$ to $W^{1/2}_{r,s}(\Omega)$ on piecewise smooth (or Lipschitz) strictly pseudoconvex domains. Other results in this direction belong to Ehsani [9], [10]. The key ingredient in the proof of all of the results above is an exhaustion of the piecewise smooth domain by smooth (or uniformly Lipschitz) strictly pseudoconvex domains Ω on which the $\overline{\partial}$-Neumann operator exists and satisfies uniform L^2 or subelliptic ε-estimates. Ho [16] introduced the notion of weak q-convexity for domains with smooth boundaries. Ahn-Dieu [3] investigated a natural extension of these notions to the class of q-pseudoconvex domains with non-smooth boundaries. The aim of this paper is to extend the estimates for the $\overline{\partial}$-Neumann operator for some classes of bounded pseudoconvex domains to the situation in which the boundaries are assumed Lipschitz and q-pseudoconvex. More precisely, we prove the following result:

Theorem 1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded q-pseudoconvex domain, $1 \leqslant q \leqslant n$, with Lipschitz boundary. For each $q \leqslant s \leqslant n-1$, the $\overline{\partial}$-Neumann operator $N\colon L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega)$ satisfies the following estimate: for any $\alpha \in L^2_{r,s}(\Omega)$, there exists a constant $C > 0$ such that*

$$\|N\alpha\|_{1/2(\Omega)} \leqslant C\|\alpha\|_{\Omega}, \tag{0.1}$$

where $C = C(\Omega)$ is independent of α.

Corollary 1. *For any $\alpha \in L^2_{r,s}(\Omega) \cap \operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}^\star$, there exists a constant $C > 0$ such that*

$$\|\overline{\partial}N\alpha\|_{1/2(\Omega)} \leqslant C\|\alpha\|_{\Omega}, \quad q \leqslant s \leqslant n-1, \tag{0.2}$$

$$\|\overline{\partial}^\star N\alpha\|_{1/2(\Omega)} \leqslant C\|\alpha\|_{\Omega}, \quad q \leqslant s \leqslant n,\ s \geqslant 2. \tag{0.3}$$

Theorem 2. *Let $\Omega \subset\subset \mathbb{C}^n$ be a q-pseudoconvex domain, $1 \leqslant q \leqslant n$, with Lipschitz boundary. For each $q \leqslant s \leqslant n-1$, the $\overline{\partial}$-Neumann operator $N\colon L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega)$ satisfies the following estimate: for any $\alpha \in L^2_{r,s}(\Omega)$, there exists a constant $C > 0$ such that*

$$\|N\alpha\|_{1/2(\Omega)} \leqslant C\|\alpha\|_{-1/2(\Omega)}, \tag{0.4}$$

i.e., N can be extended as a bounded operator from $W^{-1/2}_{r,s}(\Omega)$ to $W^{1/2}_{r,s}(\Omega)$.

The plan of this paper is as follows. In Section 1 we first recall some definitions and facts on Lipschitz q-pseudoconvex domains. In Section 2 we discuss the L^2 existence theorems for $\overline{\partial}$ and the $\overline{\partial}$-Neumann operator on q-pseudoconvex domains. Theorems 1, 2 are proved in Section 3. These results extend to domains which are not necessarily pseudoconvex. The techniques of this paper come back to Michel and Shaw and others. Our results also hold on q-pseudoconvex domains with piecewise smooth boundary.

1. Lipschitz q-pseudoconvex domains

Let $\psi\colon \mathbb{R}^{2n-1} \to \mathbb{R}$ be a function that satisfies the Lipschitz condition

$$|\psi(x) - \psi(x')| \leqslant M|x - x'|, \quad \text{for all } x, x' \in \mathbb{R}^{2n-1}. \tag{1.1}$$

The smallest M for which (1.1) holds will be called the bound of the Lipschitz constant. A domain $\Omega \subset\subset \mathbb{R}^{2n}$ is called Lipschitz domain or a domain with Lipschitz boundary, if near every boundary point $p \in b\Omega$ there exists a neighborhood V of p such that, after a rotation,

$$\Omega \cap V = \{(x, x_{2n}) \in V \mid x_{2n} > \psi(x)\},$$

for some Lipschitz function ψ. By choosing finitely many balls $\{V_i\}$ covering $b\Omega$, the Lipschitz constant for a Lipschitz domain is the smallest M such that the Lipschitz constant is bounded by M in every ball V_i. A Lipschitz function is almost everywhere differentiable (see Evans-Gariepy [12] for a proof of this fact).

Definition 1.1. Let φ be an upper semi-continuous function on $U \subset \mathbb{C}^n$. Then we say that φ is q-subharmonic on U if for every q-complex dimension space H and for every compact set $K \subset H \cap U$, the following holds: if h is a continuous harmonic function on K and $h \leqslant \varphi$ on bK, then $h \leqslant \varphi$ on K.

Proposition 1.2 [3]. *Let φ be a real valued C^2-function defined on $U \subset \mathbb{C}^n$ and $1 \leqslant q \leqslant n$. Then the q-subharmonicity of φ is equivalent to*

$$\sum_{|K|=q-1} \sum_{j,k} \frac{\partial^2 \varphi}{\partial z^j \partial \overline{z}^k}\, \alpha_{jK}\, \overline{\alpha}_{kK} \geqslant 0 \quad \textit{for all } q\textit{-forms } \alpha = \sum_{|J|=q} \alpha_J \,\mathrm{d}\overline{z}^J. \tag{1.2}$$

One of the most typical examples of q-subharmonic function which is not plurisubharmonic is

$$\varphi(z) = -\sum_{j=1}^{q-1} |z_j|^2 + (q-1)\sum_{j=q}^{n} |z_j|^2.$$

Definition 1.3. A bounded (Lipschitz) domain Ω in $\mathbb{C}^n$ is called q-pseudoconvex if there is a q-subharmonic exhaustion (Lipschitz) function for Ω.

A C^2 smooth function u on $U \subset \mathbb{C}^n$ is called q-plurisubharmonic if its complex Hessian has at least $n-q$ non-negative eigenvalues at each point of U.

A domain $\Omega \subset \mathbb{C}^n$ is pseudoconvex if and only if it is 1-pseudoconvex, since 1-subharmonic function is just plurisubharmonic.

An n-subharmonic function is just subharmonic function in usual sense. An upper semicontinuous function on U is plurisubharmonic exactly when it is 1-subharmonic.

Remark 1.4 [3]**.** If $\Omega \subset \mathbb{C}^n$ is a q-pseudoconvex domain, $1 \leqslant q \leqslant n$, then the following hold

(1) If $b\Omega$ is of class C^2, then by (1.2), Ω is weakly q-convex in the sense of Ho [16].

(2) If $q \leqslant q'$, then q-pseudoconvexity implies q'-pseudoconvexity.

We say that $\varphi \in C^2(U)$ is strictly q-subharmonic if φ satisfies (1.2) with strict inequality. Also we say that Ω is strictly q-pseudoconvex if the boundary of Ω, is of class C^2 and its defining function is strictly q-subharmonic.

Example 1.5. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain satisfying the $Z(q)$ condition, that is, the Levi form of a smooth defining function of Ω has, at every boundary point of Ω, at least $n-q$ positive or at least $q+1$ negative eigenvalues. Then Ω is strictly q-pseudoconvex.

Denote by $C^\infty_{r,s}(\mathbb{C}^n)$ the space of complex-valued differential forms of class C^∞ and of type (r,s) on $\mathbb{C}^n$. Each such form can be written uniquely as a sum

$$\alpha = {\sum_{I,J}}' \alpha_{I,J}\, \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J,$$

where I and J are strictly increasing multi-indices with lengths r and s, respectively. Denote by $C^\infty_{r,s}(\bar{\Omega}) = \left\{\alpha|_{\overline{\Omega}};\ \alpha \in C^\infty_{r,s}(\mathbb{C}^n)\right\}$ the subspace of $C^\infty_{r,s}(\Omega)$ whose elements can be extended smoothly up to the boundary $b\Omega$. For $\alpha, u \in C^\infty_{r,s}(\mathbb{C}^n)$, we define a pointwise Hermitian structure (α, u) by

$$(\alpha, u) = \sum_{I,J} \alpha_{I,J}\, \bar{u}_{I,J}.$$

Let $L^2_{r,s}(\Omega)$ be the space of (r,s)-forms on Ω with square-integrable coefficients. The L^2-inner product and norms on Ω are defined by

$$\langle \alpha, u \rangle_\Omega = \int_\Omega (\alpha, u)\, \mathrm{d}V \quad \text{and} \quad \|\alpha\|^2_\Omega = \langle \alpha, \alpha \rangle_\Omega\,,$$

where $\mathrm{d}V$ is the volume element. Let $\overline{\partial}$ be the maximal closure of the Cauchy-Riemann operator and $\overline{\partial}^\star$ be its Hilbert space adjoint. For $1 \leqslant s \leqslant n$, we denote by $\square = \overline{\partial}\,\overline{\partial}^\star + \overline{\partial}^\star\overline{\partial}\colon \operatorname{dom}\square \to L^2_{r,s}(\Omega)$ the complex Laplacian operator, where $\operatorname{dom}\square = \{\alpha \in L^2_{r,s}(\Omega)\colon\ \alpha \in \operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}^\star; \overline{\partial}\alpha \in \operatorname{dom}\overline{\partial}^\star \text{ and } \overline{\partial}^\star\alpha \in \operatorname{dom}\overline{\partial}\}$. Denote by $\ker\square = \{\alpha \in \operatorname{dom}\square;\ \overline{\partial}\alpha = \overline{\partial}^\star\alpha = 0\}$ the kernel of $\square$. One defines the $\overline{\partial}$-Neumann operator $N\colon L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega)$ as the inverse of the restriction of $\square$ to $(\ker\square)^\perp$, i.e.,

$$N\alpha = \begin{cases} 0 & \text{if } \alpha \in \ker\square, \\ u & \text{if } \alpha = \square\, u \text{ and } u \perp \ker\square. \end{cases}$$

2. L^2 THEORY FOR $\overline{\partial}$ ON A q-PSEUDOCONVEX DOMAIN

In this section we establish that the $\overline{\partial}$-Neumann operator N exists for square-integrable forms on q-pseudoconvex domains Ω in $\mathbb{C}^n$. We define a mollifier $\varrho_\varepsilon(z) = \varrho(z/\varepsilon)/|\varepsilon|^{2n}$, where ϱ is a non-negative smooth radial function in $\mathbb{C}^n$ vanishing outside the unit ball and satisfying $\int_{\mathbb{C}^n} \varrho\,\mathrm{d}v = 1$. Here $\mathrm{d}v$ stands for the standard Lebesgue measure. By $W^l(\Omega)$ for $l \geqslant 0$ we denote the L^2-Sobolev space of order l on Ω, i.e., the restrictions of functions in $W^l(\mathbb{C}^n)$ to Ω with the quotient norm.

Lemma 2.1 ([8]; Lemma 5.1.6). *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary. Then, for $\alpha \in C^\infty_{r,s}(\overline{\Omega}) \cap \operatorname{dom}\overline{\partial}^\star$ with $1 \leqslant s \leqslant n-1$, we have*

$$\|\alpha\|^2_{1/2(\Omega)} \leqslant C\Big(\int_{b\Omega} |\alpha|^2\,\mathrm{d}S + \|\overline{\partial}\alpha\|^2_\Omega + \|\overline{\partial}^\star\alpha\|^2_\Omega + \|\alpha\|^2_\Omega\Big),$$

where $C > 0$ is a constant independent of α.

Lemma 2.2 [4]. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary and φ be its C^2 defining function. Let a be a real function that is twice continuously differentiable on $\overline{\Omega}$, with $a \geqslant 0$. Then, for $\alpha \in C^\infty_{r,s}(\overline{\Omega}) \cap \operatorname{dom}\overline{\partial}^\star$ with $1 \leqslant s \leqslant n-1$, we have*

$$\begin{aligned}(2.1)\qquad \|\sqrt{a}\,\overline{\partial}\alpha\|^2_\Omega + \|\sqrt{a}\,\overline{\partial}^\star\alpha\|^2_\Omega &= \sum_{I,K}{}' \sum_{j,k=1}^n \int_{b\Omega} a\frac{\partial^2\varphi}{\partial z^j\partial\overline{z}^k}\,\alpha_{I,jK}\,\overline{\alpha}_{I,kK}\,\mathrm{d}S \\ &+ \sum_{I,J}{}' \sum_{k=1}^n \int_\Omega a\Big|\frac{\partial\alpha_{I,J}}{\partial\overline{z}^k}\Big|^2\,\mathrm{d}V + 2\operatorname{Re}\Big(\sum_{I,K}{}'\sum_{j=1}^n \frac{\partial a}{\partial z^j}\,\alpha_{I,jK}\,\mathrm{d}\overline{z}_K, \overline{\partial}^\star\alpha\Big) \\ &- \sum_{I,K}{}'\sum_{j,k=1}^n \int_\Omega \frac{\partial^2 a}{\partial z^j\partial\overline{z}^k}\,\alpha_{I,jK}\,\overline{\alpha}_{I,kK}\,\mathrm{d}V.\end{aligned}$$

The case of $a \equiv 1$ is the classical Kohn-Morrey formula, see [19], [17].

Lemma 2.3 [3]. *Let $\Omega \subset \mathbb{C}^n$ be a q-pseudoconvex domain, $1 \leqslant q \leqslant n$. Then Ω has a C^∞-smooth strictly q-subharmonic exhaustion function. More precisely, there are strictly q-pseudoconvex domains Ω_ν, $\nu = 1, 2, \ldots$, satisfying*

$$\Omega = \bigcup_{\nu=1}^{\infty} \Omega_\nu, \quad \Omega_\nu \subset\subset \Omega_{\nu+1} \subset\subset \Omega.$$

Proof. Let φ be a q-subharmonic exhaustion function for Ω and $U_j = \{\varphi(z) < j\}$. Note that $U_j \nearrow \Omega$ as $j \to \infty$. By Sard's theorem, we can find a decreasing sequence $\{\varepsilon_j\}$ with $\lim\limits_{j\to\infty} \varepsilon_j = 0$ and two increasing sequences $\{a_j\}$, $\{b_j\}$ with $\lim\limits_{j\to\infty} a_j = \infty$, $\lim\limits_{j\to\infty} b_j = \infty$ such that for every j,

(a) $U_j \subset D_j := \{z \in D\colon\ u * \varrho_\varepsilon(z) + |z|^2/a_j < b_j\}$;
(b) $U_j \cup D_j \subset\subset D_{j+1}$;
(c) each D_j has smooth boundary.

Thus the proof follows. $\square$

From now, we assume that Ω and $\{\Omega_\nu\}$ are the same as in Lemma 2.3. Let N_μ and $\overline{\partial}^\star_\mu$ be the $\overline{\partial}$-Neumann operator and the adjoint of $\overline{\partial}$ on $L^2_{r,s}(\Omega_\mu)$, respectively. We prove the following theorem as in Boas-Straube [4].

Theorem 2.4. *Let $\Omega \subset \mathbb{C}^n$ be a q-pseudoconvex domain, $1 \leqslant q \leqslant n$. Then, for any $q \leqslant s \leqslant n$, there exists a bounded linear operator $N\colon L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega)$ with the following properties:*

(i) $\mathcal{R}(N) \subset \operatorname{dom}\Box$, $N\Box = I$ *on* $\operatorname{dom}\Box$,
(ii) *for* $\alpha \in L^2_{r,s}(\Omega)$, *we have* $\alpha = \overline{\partial}\,\overline{\partial}^\star N\alpha \oplus \overline{\partial}^\star\overline{\partial}N\alpha$,
(iii) $\overline{\partial}N = N\overline{\partial}$ *on* $\operatorname{dom}\overline{\partial}$, $q \leqslant s \leqslant n-1$,
(iv) $\overline{\partial}^\star N = N\overline{\partial}^\star$ *on* $\operatorname{dom}\overline{\partial}^\star$, $q \leqslant s \leqslant n$, $s \geqslant 2$,
(v) *if* $\overline{\partial}\alpha = 0$, *then* $u = \overline{\partial}^\star N\alpha$ *solves the equation* $\overline{\partial}u = \alpha$,
(vi) N, $\overline{\partial}N$ *and* $\overline{\partial}^\star N$ *are bounded operators with respect to the* L^2*-norms.*

Proof. Since $\Omega \subset \mathbb{C}^n$ is a q-pseudoconvex domain, we can choose strictly q-pseudoconvex domains Ω_ν with smooth boundary such that

$$\Omega = \bigcup_{\nu=1}^{\infty} \Omega_\nu, \quad \Omega_\nu \subset\subset \Omega_{\nu+1} \subset\subset \Omega \quad \text{for all } \nu.$$

Since Ω_ν is a strictly q-pseudoconvex domain with smooth boundary, then for every $\alpha \in C^\infty_{r,s}(\overline{\Omega}_\nu) \cap \operatorname{dom}\overline{\partial}^\star_\nu$ with $s \geqslant q$, we have

$$\sum_{I,K}{}' \sum_{j,\,k=1}^{n} \int_{b\Omega_\nu} \frac{\partial^2\varphi}{\partial z^j \partial\overline{z}^k}\,\alpha_{I,jK}\,\overline{\alpha}_{I,kK}\,\mathrm{d}S \geqslant C \int_{b\Omega_\nu} |\alpha|^2\,\mathrm{d}S,$$

where C is independent of ν. If we replace a by $1-e^b$, where b is an arbitrary twice continuously differentiable non-positive function, and apply the Cauchy-Schwarz inequality to the term in (2.1) involving first derivatives of a, we find

$$\|\sqrt{a}\,\overline{\partial}\alpha\|^2_{\Omega_\nu} + \|\sqrt{a}\,\overline{\partial}^\star\alpha\|^2_{\Omega_\nu} \geqslant \sum_{I,K}{}' \sum_{j,k=1}^{n} \int_{\Omega_\nu} e^b \frac{\partial^2 b}{\partial z^j \partial \overline{z}^k}\,\alpha_{I,jK}\,\overline{\alpha}_{I,kK}\,\mathrm{d}V - \|e^{b/2}\,\overline{\partial}^\star\alpha\|_{\Omega_\nu}.$$

Since $a+e^b=1$ and $a\leqslant 1$, it follows that

$$\|\overline{\partial}\,\alpha\|^2_{\Omega_\nu} + \|\overline{\partial}^\star\alpha\|^2_{\Omega_\nu} \geqslant \sum_{I,K}{}' \sum_{j,k=1}^{n} \int_{\Omega_\nu} e^b \frac{\partial^2 b}{\partial z^j \partial \overline{z}^k}\,\alpha_{I,jK}\,\overline{\alpha}_{I,kK}\,\mathrm{d}V$$

for every twice continuously differentiable non-positive function b. If p is a point of Ω_ν, and $b(z)=-1+|z-p|^2/d^2$, where $d=\sup\limits_{z,z'\in\Omega_\nu}|z-z'|$ is the diameter of the bounded domain Ω_ν, then the preceding inequality then implies the fundamental estimate

$$\|\alpha\|^2_{\Omega_\nu} \leqslant \left(\frac{d^2 e}{s}\right)\left(\|\overline{\partial}\alpha\|^2_{\Omega_\nu} + \|\overline{\partial}^\star\alpha\|^2_{\Omega_\nu}\right). \tag{2.2}$$

Although this estimate was derived under the assumption that α is continuously differentiable on the closure $\overline{\Omega}_\nu$, it holds by density for all square-integrable forms α that are in the intersection of the domains of $\overline{\partial}$ and $\overline{\partial}^\star$. Estimate (2.2) is equivalent to every form in $L^2_{r,s}(\Omega_\nu)$ admitting a representation as $\overline{\partial}\,v+\overline{\partial}^\star w$ with

$$\|v\|^2_{\Omega_\nu} + \|w\|^2_{\Omega_\nu} \leqslant \left(\frac{d^2 e}{s}\right)\|\alpha\|^2_{\Omega_\nu}.$$

The latter property carries over to arbitrary bounded q-pseudoconvex domains by exhausting a nonsmooth domain by smooth ones, and therefore so does the inequality (2.2). Thus, for $s\geqslant q$, we obtain

$$\|\alpha\|^2_{\Omega} \leqslant \left(\frac{d^2 e}{s}\right)\left(\|\overline{\partial}\alpha\|^2_{\Omega} + \|\overline{\partial}^\star\alpha\|^2_{\Omega}\right) \leqslant \left(\frac{d^2 e}{s}\right)\|\Box\alpha\|_{\Omega}\|\alpha\|_{\Omega},$$

i.e.,

$$\|\alpha\|_{\Omega} \leqslant \left(\frac{d^2 e}{s}\right)\|\Box\,\alpha\|_{\Omega}. \tag{2.3}$$

Since $\Box$ is a linear closed densely defined operator, then, from [17], Theorem 1.1.1, $\mathcal{R}(\Box)$ is closed. Thus, from (1.1.1) in [17] and the fact that $\Box$ is self adjoint, we have the Hodge decomposition

$$L^2_{r,s}(\Omega) = \overline{\partial}\,\overline{\partial}^\star\,\mathrm{dom}\,\Box \oplus \overline{\partial}^\star\overline{\partial}\,\mathrm{dom}\,\Box.$$

Since $\Box\colon \operatorname{dom}\Box \to \mathcal{R}(\Box) = L^2_{r,s}(\Omega)$ is one to one on $\operatorname{dom}\Box$ from (2.3), there exists a unique bounded inverse operator $N\colon \mathcal{R}(\Box) \to \operatorname{dom}\Box$ such that $N\Box\alpha = \alpha$ on $\operatorname{dom}\Box$. Also, from the definition of N, we obtain $\Box N = I$ on $L^2_{r,s}(\Omega)$. Thus (i) and (ii) are satisfied. To show that $\overline{\partial}^\star N = N\overline{\partial}^\star$ on $\operatorname{dom}\overline{\partial}^\star$, by using (ii), we have $\overline{\partial}^\star\alpha = \overline{\partial}^\star\overline{\partial}\,\overline{\partial}^\star N\,\alpha$, for $\alpha \in \operatorname{dom}\overline{\partial}^\star$. Thus

$$N\overline{\partial}^\star\alpha = N\overline{\partial}^\star\,\overline{\partial}\,\overline{\partial}^\star N\alpha = N(\overline{\partial}^\star\,\overline{\partial} + \overline{\partial}\,\overline{\partial}^\star)\,\overline{\partial}^\star N\alpha = \overline{\partial}^\star N\alpha.$$

A similar argument shows that $\overline{\partial}N = N\overline{\partial}$ on $\operatorname{dom}\overline{\partial}$. By using (iii) and the condition on α, $\overline{\partial}\alpha = 0$, we have $\overline{\partial}N\alpha = N\overline{\partial}\alpha = 0$. Then, by using (ii), we obtain $\alpha = \overline{\partial}\,\overline{\partial}^\star N\alpha$. Thus the form $u = \overline{\partial}^\star N\alpha$ satisfies the equation $\overline{\partial}u = \alpha$. Since $\mathcal{R}(N) \subset \operatorname{dom}\Box$, then by applying (2.3) to $N\alpha$ instead of α, we obtain

$$\|N\alpha\|_\Omega \leqslant \Big(\frac{e\,\delta^2}{s}\Big)\|\alpha\|_\Omega,$$

$$\|\overline{\partial}N\alpha\|_\Omega + \|\overline{\partial}^\star N\alpha\|_\Omega \leqslant 2\sqrt{\frac{e\,\delta^2}{s}}\;\|\alpha\|_\Omega.$$

Thus the proof follows. $\Box$

3. Proof of theorems 1, 2

If Ω is a Lipschitz domain, then $C^\infty(\overline{\Omega})$ is dense in $W^l(\Omega)$ with respect to the $W^l(\Omega)$-norm. Thus to prove Theorem 1, it suffices to prove (0.1) for any $\alpha \in C^\infty_{r,s}(\overline{\Omega})$. Since Ω_ν is a smooth domain, hence $C^\infty(\overline{\Omega}_\nu)$ is dense in $W^s(\Omega_\nu)$ in the $W^s(\Omega_\nu)$-norm. By using the boundary regularity for N_ν which was established by Zampieri [25] and Kohn [18], we have $N_\nu\alpha \in C^\infty(\overline{\Omega}_\nu) \cap \operatorname{dom}\Box_\nu$. By setting $a = 1$ in (2.1), there exists a constant $C > 0$ such that for any $\alpha \in L^2_{r,s}(\overline{\Omega}_\nu) \cap \operatorname{dom}\Box_\nu$, $q \leqslant s \leqslant n-1$,

$$\int_{b\Omega_\nu} |\alpha|^2\,\mathrm{d}S_\nu \leqslant C\left(\|\overline{\partial}\alpha\|^2_{\Omega_\nu} + \|\overline{\partial}^\star\alpha\|^2_{\Omega_\nu}\right).$$

Thus, by using (2.2) and by applying Lemma 2.1 on Ω_ν, we obtain

$$\|\alpha\|^2_{1/2(\Omega_\nu)} \leqslant C\left(\|\overline{\partial}\alpha\|^2_{\Omega_\nu} + \|\overline{\partial}^\star\alpha\|^2_{\Omega_\nu}\right) \leqslant C\|\alpha\|_{\Omega_\nu}\|\Box_\nu\alpha\|_{\Omega_\nu}, \tag{3.1}$$

where C is independent of α and ν. By using (vi) and (ii) in Theorem 2.4, the operators $N_\nu\alpha$, $\overline{\partial}N_\nu\alpha$, $\overline{\partial}^\star_\nu N_\nu\alpha$, $\overline{\partial}^\star_\nu\overline{\partial}N_\nu\alpha$, and $\overline{\partial}\,\overline{\partial}^\star_\nu N_\nu\alpha$ satisfy the following estimates:

$$\|N_\mu\alpha\|_{\Omega_\mu} \leqslant \Big(\frac{e\,\delta^2}{s}\Big)\|\alpha\|_{\Omega_\mu} \leqslant \Big(\frac{e\,\delta^2}{s}\Big)\|\alpha\|_\Omega, \tag{3.2}$$

$$\|\overline{\partial}N_\mu\alpha\|_{\Omega_\mu} + \|\overline{\partial}^\star_\mu N_\mu\alpha\|_{\Omega_\mu} \leqslant 2\sqrt{\frac{e\,\delta^2}{s}}\;\|\alpha\|_{\Omega_\mu} \leqslant 2\sqrt{\frac{e\,\delta^2}{s}}\;\|\alpha\|_\Omega\,, \tag{3.3}$$

$$\|\overline{\partial}\,\overline{\partial}^\star_\mu N_\mu\alpha\|^2_{\Omega_\mu} + \|\overline{\partial}^\star_\mu\overline{\partial}N_\mu\alpha\|^2_{\Omega_\mu} = \|\alpha\|^2_{\Omega_\mu} \leqslant \|\alpha\|^2_\Omega. \tag{3.4}$$

In view of (3.2), let us extend $N_\nu\alpha$ to all of Ω by setting $N_\nu\alpha = 0$ in $\Omega \setminus \Omega_\nu$, thus by the Rellich and Sobolev lemmas we can choose a subsequence (still denoted by $N_\nu\alpha$) converging weakly to some element $g \in L^2_{r,s}(\Omega)$ and $\overline{\partial} g \in L^2_{r,s+1}(\Omega)$.

Again in view of (3.3) and (3.4), we can assume that $\overline{\partial} N_\nu\alpha$, $\overline{\partial}^\star_\nu N_\nu\alpha$, $\overline{\partial}^\star_\nu \overline{\partial} N_\nu\alpha$, and $\overline{\partial}\,\overline{\partial}^\star_\nu N_\nu\alpha$ converge weakly to some elements g_1, g_2, g_3 and g_4 of $L^2_{r,s}(\Omega)$, respectively (here we again extending $\overline{\partial} N_\nu\alpha$ etc. by zero on $\Omega \setminus \Omega_\nu$).

We claim that $g \in \operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}^\star$ and $\overline{\partial} g = g_1$, $\overline{\partial}^\star g = g_2$. Indeed, for any $u \in \operatorname{dom}\overline{\partial} \cap L^2_{r,s-1}(\Omega)$,

$$|\langle g, \overline{\partial} u\rangle_\Omega| = \lim_{\nu\to\infty} |\langle \overline{\partial}^\star_\nu N_\nu\alpha, u\rangle_{\Omega_\nu}| \leqslant 2\sqrt{\frac{e\delta^2}{s}}\,\|\alpha\|_\Omega\,\|u\|_\Omega.$$

Thus $g \in \operatorname{dom}\overline{\partial}^\star$. The proof for $\overline{\partial}$ is the same. A similar argument shows that $g_1 \in \operatorname{dom}\overline{\partial}^\star$, $g \in \operatorname{dom}\overline{\partial}$ and $\overline{\partial}^\star g_1 = g_3$, $\overline{\partial} g_3 = g_4$. Thus $g \in \operatorname{dom}\Box$ and $\Box g$ is the weak limit of $\Box_\nu N_\nu\alpha = \alpha$, that is, $g = N\alpha$ and $N_\nu\alpha \to N\alpha$ weakly in L^2.

Following [23], Chapter VI, there exists a continuous linear extension operator $E_\nu\colon W^{1/2}(\Omega_\nu) \to W^{1/2}(\mathbb{C}^n)$ such that for each $\alpha \in W^{1/2}(\Omega_\nu)$, $E_\nu\alpha = \alpha$ on Ω_ν and

$$\|E_\nu\alpha\|_{1/2(\mathbb{C}^n)} \leqslant C\|\alpha\|_{1/2(\Omega_\nu)}, \tag{3.5}$$

for some positive constant C. The constant C in (3.5) can be chosen independent of ν since extension exists for any Lipschitz domain (cf. Theorem 1.4.3.1 in [14]). By applying (3.1) on $N_\nu\alpha$, we obtain

$$\|N_\nu\alpha\|_{1/2(\Omega_\nu)} \leqslant C\|\alpha\|_{\Omega_\nu}.$$

Thus, by using (3.5) and by applying E_ν to $N_\nu\alpha$ componentwise, we obtain

$$\|E_\nu N_\nu\alpha\|_{1/2(\Omega)} \leqslant \|E_\nu N_\nu\alpha\|_{1/2(\mathbb{C}^n)} \leqslant C\|N_\nu\alpha\|_{1/2(\Omega_\nu)} \leqslant C\|\alpha\|_{\Omega_\nu},$$

where $C > 0$ is independent of ν. Since $W^{1/2}_{r,s}(\Omega)$ is a Hilbert space, then from the weak compactness for Hilbert space, there exists a subsequence of $E_\nu N_\nu\alpha$ which converges weakly in $W^{1/2}_{r,s}(\Omega)$. Since $E_\nu N_\nu\alpha$ converges weakly to $N\alpha$ in $L^2_{r,s}(\Omega)$, we conclude that $N\alpha \in W^{1/2}_{r,s}(\Omega)$ and

$$\|N\alpha\|_{1/2(\Omega)} \leqslant \lim_{\nu\to\infty} \|E_\nu N_\nu\alpha\|_{1/2(\Omega_\nu)} \leqslant C\|\alpha\|_\Omega.$$

Thus (0.1) follows.

To prove the estimate (0.2), we note that from the above arguments, for any $\alpha \in L^2_{r,s}(\Omega)$, $\overline{\partial} N_\nu\alpha$ converges weakly to $\overline{\partial} N\alpha$ in $L^2_{r,s}(\Omega)$. If $q \leqslant s \leqslant n-1$, $\overline{\partial} N_\nu\alpha$ is in $\operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}^\star_\nu$, and substituting $\overline{\partial} N_\nu\alpha$ into (3.1), we have

$$\|\overline{\partial} N_\nu\alpha\|^2_{1/2(\Omega_\nu)} \leqslant C\left(\|\overline{\partial}\,\overline{\partial} N_\nu\alpha\|^2_{\Omega_\nu} + \|\overline{\partial}^\star_\nu \overline{\partial} N_\nu\alpha\|^2_{\Omega_\nu}\right) \leqslant C\|\alpha\|_{\Omega_\nu}.$$

Passing to the limit as before, we prove (0.2) when $q \leqslant s \leqslant n-1$. Similar arguments also give the estimate (0.3) when $q \leqslant s \leqslant n$, $s \geqslant 2$ since in this case, we also have $\overline{\partial}_\nu^\star N\alpha$ is in $\operatorname{dom}\overline{\partial} \cap \operatorname{dom}\overline{\partial}_\nu^\star$.

To prove Theorem 2, since $C^\infty(\overline{\Omega})$ is dense in $W^l(\Omega)$ for all $l < 0$ (cf. [14]), it suffices to prove (0.4) for any $\alpha \in C^\infty_{r,s}(\overline{\Omega})$. By using the Generalized Schwartz inequality (cf. Proposition (A.1.1) in [13]) and by using (2.4), there exists a constant $C > 0$ such that for any $\alpha \in L^2_{r,s}(\overline{\Omega}_\nu) \cap \operatorname{dom}\Box_\nu$, $0 \leqslant p \leqslant n$ and $q \leqslant s \leqslant n-1$,

$$\|\alpha\|^2_{1/2(\Omega_\nu)} \leqslant C\,\langle\alpha, \Box_\nu\,\alpha\rangle_{\Omega_\nu} \leqslant C\|\alpha\|_{1/2(\Omega_\nu)}\|\Box_\nu\alpha\|_{-1/2(\Omega_\nu)},$$

where C is independent of α and ν. Passing to the limit as before, we prove that

$$\|N\alpha\|_{1/2(\Omega)} \leqslant C\|\alpha\|_{-1/2(\Omega)},$$

when $q \leqslant s \leqslant n-1$. Thus (0.4) follows, i.e., N can be extended as a bounded operator from $W^{-1/2}_{r,s}(\Omega)$ to $W^{1/2}_{r,s}(\Omega)$. $\square$

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