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A HYBRID MEAN VALUE RELATED TO CERTAIN HARDY SUMS
AND KLOOSTERMAN SUMS

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Abstract. The main purpose of this paper is using the mean value formula of Dirichlet L-functions and the analytic methods to study a hybrid mean value problem related to certain Hardy sums and Kloosterman sums, and give some interesting mean value formulae and identities for it.

Keywords: Hardy sums, the Kloosterman sums, hybrid mean value, asymptotic formula, identity

MSC 2010: 11M20

1. INTRODUCTION

Let c be a natural number and d an integer prime to c . The classical Dedekind sums

$$(1) \quad S(d, c) = \sum_{a=1}^c \left(\left(\frac{a}{c} \right) \right) \left(\left(\frac{ad}{c} \right) \right),$$

with

$$\left((x) \right) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer} \end{cases}$$

describes the behavior of the logarithm of the eta-function (cf. [5]) under modular transformations. B. C. Berndt [1] gave an analogous transformation formula for the logarithm of the classical theta-function

$$\theta(z) = \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 z), \quad \text{Im}(z) > 0.$$

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Put $Vz = (az + b)(cz + d)$ with $a, b, c, d \in \mathbb{Z}$, $c > 0$ and $ad - bc = 1$. Then

$$(2) \quad \log \theta(Vz) = \log \theta(z) + \frac{1}{2} \log(cz + d) - \frac{1}{4} \pi i + \frac{1}{4} \pi i S_1(d, c),$$

where

$$S_1(d, c) = \sum_{j=1}^{c-1} (-1)^{j+1+[dj/c]}.$$

The sums $S_1(d, c)$ and certain related ones are sometimes called the Hardy sums. They are closely connected with Dedekind sums. Many authors studied various properties of $S(d, c)$, and obtained a series of interesting results. For example, L. Carlitz [2] obtained a reciprocity theorem of $S(d, c)$. J.B. Conrey et al. [3] studied the mean value distribution of $S(d, c)$, and proved the following important and interesting asymptotic formula:

$$(3) \quad \sum'_{d=1}^c |S(d, c)|^{2m} = f_m(c) \left(\frac{c}{12}\right)^{2m} + O((c^{9/5} + c^{2m-1+1/(m+1)}) \log^3 c),$$

where \sum'_d denotes the summation over all d such that $(d, c) = 1$, and

$$\sum_{n=1}^{\infty} \frac{f_m(n)}{n^s} = 2 \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \zeta(s),$$

$\zeta(s)$ being the Riemann zeta-function.

Jia Chaohua [4] improved the error term in (3) to $O(c^{2m-1})$ provided $m \geq 2$. Zhang Wenpeng [7] improved the error term of (3) for $m = 1$, and proved the asymptotic formula

$$\sum'_{d=1}^c |S(d, c)|^2 = \frac{5}{144} c \varphi(c) \cdot \frac{\prod_{p^\alpha \parallel c} ((1+1/p)^2 - 1/p^{3\alpha+1})}{\prod_{p|c} (1+1/p+1/p^2)} + O\left(c \exp\left(\frac{4 \ln c}{\ln \ln c}\right)\right),$$

where $p^\alpha \parallel c$ denotes that $p^\alpha | c$ and $p^{\alpha+1} \nmid c$.

In this paper we find that there are some close relationships between $S_1(d, c)$ and the Kloosterman sums $S(m, n; c)$, which are defined as follows:

$$S(m, n; c) = \sum'_{b=1}^c e\left(\frac{mb + n\bar{b}}{c}\right),$$

where $e(y) = e^{2\pi i y}$ and \bar{b} denotes the solution of the congruent equation $x \cdot b \equiv 1 \pmod{c}$. In fact, for any positive integer k and some special integers c , we can use the

estimates for character sums and the mean value theorem of Dirichlet L-functions to get some interesting hybrid mean value formulae and identities for

$$(4) \quad \sum_{a=1}^c \sum_{b=1}^c S^2(m, a; c) S^2(m, b; c) S_1^k(2a\bar{b}, c).$$

We will prove the following several assertions:

Theorem 1. *Let p be a prime with $p \equiv 1 \pmod{4}$, then for any fixed positive integer k we have the identities*

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} S^2(a, 1; p) S^2(b, 1; p) S_1^{2k-1}(2a\bar{b}, p) = 0$$

and

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} S^2(a, 1; p) S^2(b, 1; p) S_1^{2k}(2a\bar{b}, p) \\ &= (p^3 - 2p^2 - 3p - 1) \cdot \sum_{a=1}^{p-1} S_1^{2k}(2a, p) - p^2 \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) S_1^{2k}(2a, p), \end{aligned}$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol.

Theorem 2. *Let p be a prime with $p \equiv 3 \pmod{4}$, then for any fixed positive integer k we have the identities*

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} S^2(a, 1; p) S^2(b, 1; p) S_1^{2k}(2a\bar{b}, p) \\ &= 4^k \cdot p^3 + (p^3 - 2p^2 - 3p - 1) \cdot \sum_{a=1}^{p-1} S_1^{2k}(2a, p) \end{aligned}$$

and

$$\begin{aligned} & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} S^2(a, 1; p) S^2(b, 1; p) S_1^{2k-1}(2a\bar{b}, p) \\ &= 2^{2k-1} \cdot p^3 - p^2 \cdot \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) S_1^{2k-1}(2a, p). \end{aligned}$$

It is clear that from our theorems we can establish an asymptotic formula for (2). Especially for $k = 1$, we have:

Corollary 1. *Let p be a prime, then we have the asymptotic formula*

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} S^2(a, 1; p) S^2(b, 1; p) S_1^2(2a\bar{b}, p) = 3 p^5 + O\left(p^4 \cdot \exp\left(\frac{3 \ln p}{\ln \ln p}\right)\right),$$

where $\exp(y) = e^y$.

Corollary 2. *Let p be a prime with $p \equiv 3 \pmod{4}$, then we have the identity*

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} S^2(a, 1; p) S^2(b, 1; p) S_1(2a\bar{b}, p) = 2 p^3 + \frac{16p^3}{\pi^2} \cdot \left(\frac{2}{p}\right) \cdot L^2(1, \chi_{2p}^2),$$

where χ_{2p}^2 denotes the product of the Legendre symbol and the principal character $\lambda_2 \pmod{2}$.

2. SOME LEMMAS

In this section, we give some lemmas which are necessary in the proof of our theorems.

Lemma 1. *Let $q > 2$ be an integer, then for any integer a with $(a, q) = 1$ we have the identity*

$$S(a, q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi \pmod{d} \\ \chi(-1) = -1}} \chi(a) |L(1, \chi)|^2,$$

where $\varphi(q)$ is the Euler function, $\sum_{\substack{\chi \pmod{d} \\ \chi(-1) = -1}}$ denotes the summation over all odd characters modulo d , $L(s, \chi)$ denotes the Dirichlet L -function corresponding to $\chi \pmod{d}$.

Proof. For the proof of Lemma 1 see [8]. □

Lemma 2. *Let p be an odd prime, then we have the identity*

$$S_1(2, p) = \sum_{j=1}^{p-1} (-1)^{j+1+[2j/p]} = \begin{cases} 2, & \text{if } p \equiv 3 \pmod{4}; \\ 0, & \text{if } p \equiv 1 \pmod{4}. \end{cases}$$

Proof. For all integers $1 \leq j \leq p-1$, it is clear that $[2j/p] = 0$ if $1 \leq j \leq \frac{1}{2}(p-1)$, and $[2j/p] = 1$ if $\frac{1}{2}(p+1) \leq j \leq p-1$. So we have

$$\begin{aligned} S_1(2, p) &= \sum_{j=1}^{p-1} (-1)^{j+1+[2j/p]} \\ &= \sum_{j=1}^{(p-1)/2} (-1)^{j+1} + \sum_{j=(p+1)/2}^{p-1} (-1)^{j+1+1} \\ &= \sum_{j=1}^{(p-1)/2} (-1)^{j+1} + \sum_{j=1}^{(p-1)/2} (-1)^{p-j} \\ &= 2 \sum_{j=1}^{(p-1)/2} (-1)^{j+1}. \end{aligned}$$

If $\frac{1}{2}(p-1)$ is an even number, then $\sum_{j=1}^{(p-1)/2} (-1)^{j+1} = 0$. If $\frac{1}{2}(p-1)$ is an odd number, then $\sum_{j=1}^{(p-1)/2} (-1)^{j+1} = 1$. This proves Lemma 2. \square

Lemma 3. Let $k > 0$ and $(h, k) = 1$. Then we have the identity

$$S_1(h, k) = -8S(h+k, 2k) + 4S(h, k).$$

Proof. This formula is an immediate consequence of (5.9) and (5.10) in [6]. \square

Lemma 4. Let p be an odd prime, then for any integer h with $(h, p) = 1$ we have the identity

$$S_1(h, p) = \begin{cases} -\frac{16p}{\pi^2(p-1)} \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \chi(h) |L(1, \chi\lambda_2)|^2, & \text{if } 2 \mid h, \\ 0, & \text{if } 2 \nmid h, \end{cases}$$

where λ_2 denotes the principal character mod 2.

Proof. It is clear that the divisors of $2p$ are $1, 2, p$ and $2p$. So from Lemma 1,

Lemma 3 and the fact that for any character $\chi \bmod p$, $\chi(h+p) = \chi(h)$, we have

$$\begin{aligned}
 (5) \quad S_1(h, p) &= -8S(h+p, 2p) + 4S(h, p) \\
 &= -\frac{4}{\pi^2 p} \sum_{d|2p} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h+p) |L(1, \chi)|^2 \\
 &\quad + \frac{4}{\pi^2 p} \sum_{d|p} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h) |L(1, \chi)|^2 \\
 &= -\frac{4}{\pi^2 p} \frac{(2p)^2}{\varphi(2p)} \sum_{\substack{\chi \bmod 2p \\ \chi(-1)=-1}} \chi(h+p) |L(1, \chi)|^2 \\
 &= -\frac{16p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(h+p) \lambda_2(h+p) |L(1, \chi \lambda_2)|^2.
 \end{aligned}$$

Note that $\chi(h+p) \lambda_2(h+p) = \chi(h)$ if $2 \mid h$; and $\chi(h+p) \lambda_2(h+p) = 0$ if $2 \nmid h$. From (5) we immediately deduce Lemma 4. \square

Lemma 5. *Let p be an odd prime, χ any non-real character modulo p , then we have the identities*

$$\begin{aligned}
 (a) \quad &\sum_{a=1}^{p-1} S^2(a, 1; p) = p^2 - p - 1, \\
 (b) \quad &\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) S^2(a, 1; p) = p, \\
 (c) \quad &\sum_{a=1}^{p-1} \chi(a) S^2(a, 1; p) = \frac{\tau^4(\chi)}{\tau(\chi^2)},
 \end{aligned}$$

where $\left(\frac{a}{p}\right)$ denotes the Legendre symbol mod p , $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a}{p}\right)$ is the Gauss sum, and $e(y) = e^{2\pi iy}$.

Proof. We only prove formula (c). Similarly, we can deduce (a) and (b). From the definition of Kloosterman sums and the properties of Gauss sums we have

$$(6) \quad \sum_{a=1}^{p-1} \chi(a) S^2(a, 1; p) = \sum_{a=1}^{p-1} \chi(a) \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} e\left(\frac{a(m+n) + \bar{m} + \bar{n}}{p}\right)$$

$$\begin{aligned}
&= \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a(m+n) + \bar{m} + \bar{n}}{p}\right) \\
&= \tau(\chi) \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \bar{\chi}(m+n) e\left(\frac{\bar{m} + \bar{n}}{p}\right) \\
&= \tau(\chi) \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \bar{\chi}(mn+n) e\left(\frac{\bar{m}\bar{n} + \bar{n}}{p}\right) \\
&= \tau(\chi) \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \chi(n) \bar{\chi}(m+1) e\left(\frac{n(\bar{m}+1)}{p}\right) \\
&= \tau^2(\chi) \sum_{m=1}^{p-1} \bar{\chi}((m+1)(\bar{m}+1)).
\end{aligned}$$

On the other hand, since χ is a non-real character modulo p , so $\chi^2 \neq \chi_p^0$, the principal character modulo p . Therefore, from the properties of the Gauss sums we also have

$$\begin{aligned}
(7) \quad \tau^2(\chi) &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a)\chi(b) \left(\frac{a+b}{p}\right) \\
&= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(ab)\chi(b) \left(\frac{ab+b}{p}\right) \\
&= \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \chi^2(b) e\left(\frac{b(a+1)}{p}\right) \\
&= \tau(\chi^2) \sum_{a=1}^{p-1} \chi(a) \bar{\chi}^2(a+1) \\
&= \tau(\chi^2) \sum_{a=1}^{p-1} \bar{\chi}((a+1)(\bar{a}+1)).
\end{aligned}$$

Combining (6) and (7) we immediately deduce the identity

$$\sum_{a=1}^{p-1} \chi(a) S^2(a, 1; p) = \frac{\tau^4(\chi)}{\tau(\chi^2)}.$$

This proves Lemma 5. □

3. PROOF OF THE THEOREMS

In this section we shall complete the proof of our theorems. First we prove Theorem 1. Note that if $\chi_1, \chi_2, \dots, \chi_{2k-1}$ are any $2k - 1$ odd characters mod p , then $\chi_1\chi_2 \dots \chi_{2k-1}$ is an odd character mod p . And if $p \equiv 1 \pmod{4}$, then $\chi_1\chi_2 \dots \chi_{2k-1}$ is not a real character mod p . So from Lemma 2, Lemma 4 and Lemma 5 we have

$$\begin{aligned}
 & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} S^2(a, 1; p) S^2(b, 1; p) S_1^{2k-1}(2a\bar{b}, p) \\
 &= \left(-\frac{16p}{\pi^2(p-1)} \right)^{2k-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1(-1)=-1}} \dots \sum_{\substack{\chi_{2k-1} \pmod{p} \\ \chi_{2k-1}(-1)=-1}} \left| \sum_{a=1}^{p-1} \chi_1\chi_2 \dots \chi_{2k-1}(a) S^2(a, 1; p) \right|^2 \\
 & \quad \times \chi_1\chi_2 \dots \chi_{2k-1}(2) |L(1, \chi_1\lambda_2)|^2 \cdot |L(1, \chi_2\lambda_2)|^2 \dots |L(1, \chi_{2k-1}\lambda_2)|^2 \\
 &= \left(-\frac{16p}{\pi^2(p-1)} \right)^{2k-1} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1(-1)=-1}} \dots \sum_{\substack{\chi_{2k-1} \pmod{p} \\ \chi_{2k-1}(-1)=-1}} \left| \frac{\tau^4(\chi_1\chi_2 \dots \chi_{2k-1})}{\tau(\chi_1^2\chi_2^2 \dots \chi_{2k-1}^2)} \right|^2 \\
 & \quad \times \chi_1\chi_2 \dots \chi_{2k-1}(2) |L(1, \chi_1\lambda_2)|^2 \cdot |L(1, \chi_2\lambda_2)|^2 \dots |L(1, \chi_{2k-1}\lambda_2)|^2 \\
 &= p^3 \left(-\frac{16p}{\pi^2(p-1)} \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \chi(2) |L(1, \chi\lambda_2)|^2 \right)^{2k-1} \\
 &= p^3 S_1^{2k-1}(2, p) = 0.
 \end{aligned}$$

This proves the first formula of Theorem 1.

If $\chi_1, \chi_2, \dots, \chi_{2k}$ are any $2k$ odd characters mod p , then $\chi_1\chi_2 \dots \chi_{2k}$ is an even character mod p . So if $\chi_1\chi_2 \dots \chi_{2k}$ is a real character mod p , then it must be the principal character or the Legendre symbol mod p . Let χ_p^2 denote the Legendre symbol, then from Lemma 2, Lemma 4 and Lemma 5 we have

$$\begin{aligned}
 & \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} S^2(a, 1; p) S^2(b, 1; p) S_1^{2k}(2a\bar{b}, p) \\
 &= \left(-\frac{16p}{\pi^2(p-1)} \right)^{2k} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1(-1)=-1}} \dots \sum_{\substack{\chi_{2k} \pmod{p} \\ \chi_{2k}(-1)=-1}} \left| \sum_{a=1}^{p-1} \chi_1\chi_2 \dots \chi_{2k}(a) S^2(a, 1; p) \right|^2 \\
 & \quad \times \chi_1\chi_2 \dots \chi_{2k}(2) |L(1, \chi_1\lambda_2)|^2 \cdot |L(1, \chi_2\lambda_2)|^2 \dots |L(1, \chi_{2k}\lambda_2)|^2
 \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{16p}{\pi^2(p-1)}\right)^{2k} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1\chi_2\cdots\chi_{2k}\neq\chi_0, \chi_p^2}} \left| \frac{\tau^4(\chi_1\chi_2\cdots\chi_{2k})}{\tau(\chi_1^2\chi_2^2\cdots\chi_{2k}^2)} \right|^2 \\
&\quad \times \chi_1\chi_2\cdots\chi_{2k}(2) |L(1, \chi_1\lambda_2)|^2 \cdot |L(1, \chi_2\lambda_2)|^2 \cdots |L(1, \chi_{2k}\lambda_2)|^2 \\
&+ \left(\frac{16p}{\pi^2(p-1)}\right)^{2k} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1\chi_2\cdots\chi_{2k}=\chi_0}} (p^2-p-1)^2 \\
&\quad \times |L(1, \chi_1\lambda_2)|^2 \cdot |L(1, \chi_2\lambda_2)|^2 \cdots |L(1, \chi_{2k}\lambda_2)|^2 \\
&+ \left(\frac{16p}{\pi^2(p-1)}\right)^{2k} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1\chi_2\cdots\chi_{2k}=\chi_p^2}} (p^2-p-1)^2 \\
&\quad \times \chi_p^2(2) |L(1, \chi_1\lambda_2)|^2 \cdot |L(1, \chi_2\lambda_2)|^2 \cdots |L(1, \chi_{2k}\lambda_2)|^2 \\
&= p^3 \left(\frac{16p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) |L(1, \chi\lambda_2)|^2 \right)^{2k} \\
&+ \left(\frac{16p}{\pi^2(p-1)}\right)^{2k} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1\chi_2\cdots\chi_{2k}=\chi_0}} [(p^2-p-1)^2 - p^3] \\
&\quad \times |L(1, \chi_1\lambda_2)|^2 \cdot |L(1, \chi_2\lambda_2)|^2 \cdots |L(1, \chi_{2k}\lambda_2)|^2 \\
&+ \left(\frac{p}{\pi^2(p-1)}\right)^{2k} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \bmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1\chi_2\cdots\chi_{2k}=\chi_p^2}} [p^2 - p^3] \\
&\quad \times \chi_p^2(2) |L(1, \chi_1\lambda_2)|^2 \cdot |L(1, \chi_2\lambda_2)|^2 \cdots |L(1, \chi_{2k}\lambda_2)|^2 \\
&= p^3 S_1^{2k}(2, p) + \frac{(p^2-p-1)^2 - p^3}{p-1} \sum_{a=1}^{p-1} S_1^{2k}(2a, p) - \frac{p^3-p^2}{p-1} \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) S_1^{2k}(2a, p) \\
&= (p^3 - 2p^2 - 3p - 1) \sum_{a=1}^{p-1} S_1^{2k}(2a, p) - p^2 \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) S_1^{2k}(2a, p).
\end{aligned}$$

This proves the second formula of Theorem 1.

Now we prove Theorem 2. If $\chi_1, \chi_2, \dots, \chi_{2k}$ are any $2k$ odd characters mod p , then $\chi_1\chi_2\cdots\chi_{2k}$ is an even character mod p . And if $p \equiv 3 \pmod{4}$, then $\chi_1\chi_2\cdots\chi_{2k}$

is not a real non-principal character mod p . So by the method of proving the second formula of Theorem 1 we can easily deduce that

$$\begin{aligned}
& \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} S^2(a, 1; p) S^2(b, 1; p) S_1^{2k}(2a\bar{b}, p) \\
&= \left(\frac{16p}{\pi^2(p-1)} \right)^{2k} \sum_{\substack{\chi_1 \pmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \pmod p \\ \chi_{2k}(-1)=-1}} \left| \sum_{a=1}^{p-1} \chi_1 \chi_2 \cdots \chi_{2k}(a) S^2(a, 1; p) \right|^2 \\
&\quad \times \chi_1 \chi_2 \cdots \chi_{2k}(2) |L(1, \chi_1 \lambda_2)|^2 \cdot |L(1, \chi_2 \lambda_2)|^2 \cdots |L(1, \chi_{2k} \lambda_2)|^2 \\
&= \left(\frac{16p}{\pi^2(p-1)} \right)^{2k} \sum_{\substack{\chi_1 \pmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \pmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1 \chi_2 \cdots \chi_{2k} \neq \chi_0}} \left| \frac{\tau^4(\chi_1 \chi_2 \cdots \chi_{2k})}{\tau(\chi_1^2 \chi_2^2 \cdots \chi_{2k}^2)} \right|^2 \\
&\quad \times \chi_1 \chi_2 \cdots \chi_{2k}(2) |L(1, \chi_1 \lambda_2)|^2 \cdot |L(1, \chi_2 \lambda_2)|^2 \cdots |L(1, \chi_{2k} \lambda_2)|^2 \\
&\quad + \left(\frac{16p}{\pi^2(p-1)} \right)^{2k} \sum_{\substack{\chi_1 \pmod p \\ \chi_1(-1)=-1}} \cdots \sum_{\substack{\chi_{2k} \pmod p \\ \chi_{2k}(-1)=-1 \\ \chi_1 \chi_2 \cdots \chi_{2k} = \chi_0}} (p^2 - p - 1)^2 \\
&\quad \times |L(1, \chi_1 \lambda_2)|^2 \cdot |L(1, \chi_2 \lambda_2)|^2 \cdots |L(1, \chi_{2k} \lambda_2)|^2 \\
&= p^3 S_1^{2k}(2, p) + (p^3 - 2p^2 - 3p - 1) \sum_{a=1}^{p-1} S_1^{2k}(2a, p) \\
&= 4^k \cdot p^3 + (p^3 - 2p^2 - 3p - 1) \sum_{a=1}^{p-1} S_1^{2k}(2a, p).
\end{aligned}$$

This proves the first formula of Theorem 2.

Similarly, we can deduce the second formula of Theorem 2.

Note that by virtue of the asymptotic formula (see [9])

$$\sum_{a=1}^{p-1} S_1^{2m}(a, p) = p^{2m} \frac{\zeta^2(2m)(1 - \frac{1}{4^m})}{\zeta(4m)(1 + \frac{1}{4^m})} + O\left(p^{2m-1} \cdot \exp\left(\frac{6 \ln p}{\ln \ln p}\right)\right),$$

where $\zeta(s)$ is the Riemann zeta-function and $\exp(y) = e^y$, from the second formula of Theorem 1 we may immediately deduce Corollary 1.

Corollary 2 follows from Lemma 4 and the second formula of Theorem 2.

This completes the proof of our theorems. \square

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