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AFFINE CONNECTIONS ON ALMOST PARA-COSYMPLECTIC MANIFOLDS

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Abstract. Identities for the curvature tensor of the Levi-Cività connection on an almost para-cosymplectic manifold are proved. Elements of harmonic theory for almost product structures are given and a Bochner-type formula for the leaves of the canonical foliation is established.

Keywords: para-cosymplectic manifold, harmonic product structure

MSC 2010: 53C15, 58A10, 70G45

1. Introduction

The almost para-cosymplectic manifolds contain the class of weakly para-cosymplectic manifolds which are almost para-cosymplectic manifolds satisfying an additional curvature property. The latter were studied (for dimension 3) by P. Dacko and Z. Olszak [2], who showed that if a 3-dimensional weakly para-cosymplectic manifold is locally homogeneous as a Riemannian manifold, then it is para-cosymplectic (which means that the 1- and 2-forms of the structure are parallel with respect to the Levi-Cività connection of the metric) or is locally flat. They also gave a classification for such manifolds.

In the present paper we deal with the almost para-contact hyperbolic metric structures and establish properties of the Levi-Cività connection associated to the pseudo-Riemannian structure (Proposition 2.1 and Theorem 2.2).

Let M be a $(2n+1)$-dimensional smooth manifold, φ a $(1,1)$-tensor field called the *structure endomorphism*, ξ a vector field called the *characteristic vector field*, η a 1-form called the *contact form* and g a pseudo-Riemannian metric on M. In this case, we say that (φ, ξ, η, g) defines an *almost para-contact hyperbolic metric structure* on M [3] if

(1) $\varphi^2 = I - \eta \otimes \xi$;
(2) $\eta(\xi) = 1$;
(3) $g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y)$ for any $X, Y \in \Gamma(TM)$.

The definition implies $\varphi\xi = 0$, $\eta(\varphi X) = 0$, $\eta(X) = g(X,\xi)$, $g(\xi,\xi) = 1$ and $g(\varphi X, Y) = -g(\varphi Y, X)$ for any $X, Y \in \Gamma(TM)$. The fundamental 2-form $\omega(X,Y) := g(\varphi X, Y)$, $X, Y \in \Gamma(TM)$, defined by φ and g, is skew-symmetric. The $2n$-dimensional distribution $\mathscr{D} := \ker\eta$ is called the *canonical distribution* associated with the almost para-contact hyperbolic metric structure (φ,ξ,η,g) and the foliation $\mathscr{F}$ generated by $\mathscr{D}$, the *canonical foliation* on M. Note that the canonical distribution is involutive and φ-invariant (as $\mathscr{D} = \operatorname{Im}\varphi$) and ξ is orthogonal to $\mathscr{D}$. The restrictions $\varphi_\alpha := \varphi|_{F_\alpha}$ of φ and $g_\alpha := g|_{F_\alpha}$ of g to the leaves $\{F_\alpha\}_{\alpha\in I}$ of the foliation $\mathscr{F}$ satisfy

$$\varphi_\alpha^2 X = X, \quad g_\alpha(\varphi_\alpha X, \varphi_\alpha Y) = -g_\alpha(X,Y)$$

for any $X, Y \in \Gamma(TM)$ and $\alpha \in I$, so they define an *almost para-Hermitian structure* $(\varphi_\alpha, g_\alpha)$ on each leaf F_α of $\mathscr{F}$.

If the 1-form η and the 2-form ω are closed, we say that M together with the almost para-contact hyperbolic metric structure (φ,ξ,η,g) is *almost para-cosymplectic manifold* [2]. In this case, for any $\alpha \in I$, $\eta_\alpha := \eta|_{F_\alpha}$ is closed. The fundamental 2-form $\omega_\alpha(X,Y) := g_\alpha(\varphi_\alpha X, Y)$, $X, Y \in \Gamma(\mathscr{D})$, defined by φ_α and g_α, is closed, too, so each leaf $(F_\alpha, \varphi_\alpha, g_\alpha)$ becomes an *almost para-Kähler manifold* for any $\alpha \in I$ [2]. Therefore, all almost product structures φ_α are integrable.

These properties yield the fact stated in the next proposition:

Proposition 1.1. *Let (M,φ,ξ,η,g) be an almost para-cosymplectic manifold. Assume that the Levi-Cività connection ∇_α associated to g_α is flat for any $\alpha \in I$. Then the leaves $(F_\alpha, \varphi_\alpha, \nabla_\alpha)$ are special para-complex manifolds.*

Proof. According to [8], $(F_\alpha, \varphi_\alpha, \nabla_\alpha)$ is a special para-complex manifold if φ_α is integrable, $\varphi_\alpha^2 = I$, $\varphi_\alpha \neq I$, ∇_α is a torsion free, flat affine connection and satisfies $(\nabla_{\alpha X}\eta_\alpha)Y = (\nabla_{\alpha Y}\eta_\alpha)X$ for any $X, Y \in \Gamma(TM)$. Taking into account that η_α is closed and $d\eta_\alpha(X,Y) = (\nabla_{\alpha X}\eta_\alpha)Y - (\nabla_{\alpha Y}\eta_\alpha)X$ for any $X, Y \in \Gamma(TM)$, we get the conclusion. □

2. Curvature properties

Let $(M, \varphi, \xi, \eta, g)$ be an almost para-cosymplectic manifold. Relations and curvature properties for the Levi-Cività connection ∇ associated with the pseudo-Riemannian metric g, similar to those in the almost contact metric case studied by Z. Olszak [6], can be found for almost para-cosymplectic manifolds.

From the condition $d\omega = 0$ we obtain

$$(\nabla_X \omega)(Y, Z) + (\nabla_Y \omega)(Z, X) + (\nabla_Z \omega)(X, Y) = 0 \tag{2.1}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proposition 2.1. *Let $(M, \varphi, \xi, \eta, g)$ be an almost para-cosymplectic manifold and ∇ the Levi-Cività connection associated with g. Then, for any $X, Y, Z \in \Gamma(TM)$,*

$$(\nabla_X \omega)(\varphi Y, \varphi Z) - (\nabla_X \omega)(Y, Z) = \eta(Z)(\nabla_X \eta)(\varphi Y) - \eta(Y)(\nabla_X \eta)(\varphi Z); \tag{2.2}$$

$$(\nabla_X \omega)(\varphi Y, Z) - (\nabla_X \omega)(Y, \varphi Z) = -\eta(Z)(\nabla_X \eta)Y - \eta(Y)(\nabla_X \eta)Z; \tag{2.3}$$

$$(\nabla_X \omega)(Z, Y) - (\nabla_{\varphi X} \omega)(\varphi Z, Y) = \frac{1}{2}\eta(Z)(L_\xi g)(Y, \varphi X). \tag{2.4}$$

Proof. The first two relations follow from direct computation. Writing the relation (2.1) for circular permutations $-(X, \varphi Z, \varphi Y) + (Y, \varphi X, \varphi Z) + (Z, \varphi Y, \varphi X) - (X, Z, Y)$ and taking into account that $(L_\xi g)(X, Y) = (\nabla_X \eta)Y + (\nabla_Y \eta)X$, we obtain the last relation. □

In particular, if we put $X = \xi$ in (2.4), we get $\nabla_\xi \omega = 0$. Moreover, $\nabla_\xi \varphi = 0$.

If we replace Z by φZ in the relation (2.3), we obtain

$$g(\varphi Y, \nabla_X \xi) = (\nabla_X \eta)(\varphi Y) \tag{2.5}$$

and

$$g(Y, \varphi(\nabla_X \xi)) = \eta(\nabla_X \varphi Y) \tag{2.6}$$

for any $X, Y, Z \in \Gamma(TM)$.

We also have

$$(\nabla_{\varphi X} \varphi)\varphi Y = -\varphi((\nabla_{\varphi X} \varphi)Y) - \eta(Y)\nabla_{\varphi X}\xi - (\nabla_{\varphi X}\eta)Y \cdot \xi \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$.

From

$$(\nabla_X \omega)(Z, Y) - (\nabla_{\varphi X} \omega)(\varphi Z, Y) = \eta(Z)(\nabla_{\varphi X}\eta)Y,$$

we get

$$(\nabla_X\varphi)Y-(\nabla_{\varphi X}\varphi)\varphi Y=\eta(Y)\nabla_{\varphi X}\xi \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$.

Replacing (2.7) in (2.8), we obtain

$$(\nabla_X\varphi)Y+\varphi((\nabla_{\varphi X}\varphi)Y)+(\nabla_{\varphi X}\eta)Y\cdot\xi=0 \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$.

Applying φ to (2.9), we have

$$\varphi((\nabla_X\varphi)Y)+(\nabla_{\varphi X}\varphi)Y+(\nabla_{\varphi X}\eta)\varphi Y\cdot\xi=0 \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$.

For $X = Y = \xi$ in the previous relation we deduce that $\varphi(\nabla_\xi\xi) = 0$. But $\nabla_\xi\xi = \eta(\nabla_\xi\xi)\xi$ and also $g(\nabla_\xi\xi, X) = (\nabla_\xi\eta)X$ for any $X \in \Gamma(TM)$. In particular, for $X = \xi$ we have $\eta(\nabla_\xi\xi) = 0$ and so $\nabla_\xi\xi = 0$.

From (2.8) we have $(\nabla_X\varphi)\xi = \nabla_{\varphi X}\xi$ and so

$$\varphi(\nabla_X\xi)=-\nabla_{\varphi X}\xi \tag{2.11}$$

for any $X \in \Gamma(TM)$. Then we obtain

$$(\nabla_{\varphi X}\eta)Y=(\nabla_X\eta)(\varphi Y) \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$.

We have

$$\eta(\nabla_X\xi)=0 \tag{2.13}$$

for any $X \in \Gamma(TM)$ and so

$$(\nabla_{\varphi X}\eta)\varphi Y=(\nabla_X\eta)Y \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$.

Theorem 2.2. *Let $(M, \varphi, \xi, \eta, g)$ be an almost para-cosymplectic manifold and ∇ the Levi-Cività connection associated with g. Then the following identity holds:*

$$\begin{aligned} R_{XY\varphi Z\varphi W} &- R_{\varphi XYZ\varphi W} + R_{\varphi X\varphi Y\varphi Z\varphi W} - R_{X\varphi YZ\varphi W} \\ &- R_{\varphi XY\varphi ZW} + R_{\varphi X\varphi YZW} + R_{XYZW} - R_{X\varphi Y\varphi ZW} \\ &+ \eta(W)[R_{\varphi XY\varphi Z\xi} - R_{\varphi X\varphi YZ\xi} - R_{XYZ\xi} + R_{X\varphi Y\varphi Z\xi}] \\ &+ g(\nabla_{[\varphi X,\varphi Y]+[X,Y]}\varphi Z + \varphi(\nabla_{[\varphi X,Y]+[X,\varphi Y]}\varphi Z), \varphi W) = 0 \end{aligned} \tag{2.15}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

P r o o f. The proof follows the same lines as in [6], taking into account the relations obtained above for the almost para-cosymplectic case. □

Proposition 2.3. *Under the hypotheses of Theorem* 2.2, *we have:*

$$R_{\varphi XY\varphi Z\xi} + R_{X\varphi Y\varphi Z\xi} - R_{\varphi X\varphi YZ\xi} - R_{XYZ\xi} = 0$$

for any $X, Y, Z \in \Gamma(TM)$.

P r o o f. Antisymmetrizing (2.15) with respect to Z and W and taking ($W \leftrightarrow Z$ and $W \to \xi$), we get the required relation. □

The leaves F_α of constant and quasi-constant φ_α-sectional curvature

Consider the $(0,4)$-tensor fields defined in [7]:

$$\begin{aligned} R_0^\alpha(X,Y,Z,W) := \frac{1}{4}[&g_\alpha(X,Z)g_\alpha(Y,W) - g_\alpha(X,W)g_\alpha(Y,Z) \\ &- g_\alpha(X,\varphi_\alpha Z)g_\alpha(Y,\varphi_\alpha W) + g_\alpha(X,\varphi_\alpha W)g_\alpha(Y,\varphi_\alpha Z) \\ &- 2g_\alpha(X,\varphi_\alpha Y)g_\alpha(Z,\varphi_\alpha W)] \end{aligned}$$

and, respectively, in [1]:

$$R_1^\alpha(X,Y,Z,W) := g_\alpha(S_\alpha(X,Y,Z),W) + g_\alpha(S_\alpha(\varphi_\alpha X,\varphi_\alpha Y,Z),W),$$

for

$$S_\alpha(X,Y,Z) := P_\alpha(X,Y,Z) - P_\alpha(Y,X,Z),$$

where

$$\begin{aligned} P_\alpha(X,Y,Z) := \frac{1}{8}\{&\eta_\alpha(Y)\eta_\alpha(Z)X + \eta_\alpha(X)\eta_\alpha(\varphi_\alpha Z)\varphi_\alpha Y \\ &+ \eta_\alpha(X)\eta_\alpha(\varphi_\alpha Y)\varphi_\alpha Z + g_\alpha(Y,Z)\eta_\alpha(X)\xi_\alpha \\ &+ g_\alpha(X,\varphi_\alpha Z)\eta_\alpha(Y)\varphi_\alpha\xi_\alpha \\ &+ \frac{1}{2}g_\alpha(X,\varphi_\alpha Y)[\eta_\alpha(\varphi_\alpha Z)\xi_\alpha + \eta_\alpha(Z)\varphi_\alpha\xi_\alpha]\} \end{aligned}$$

and

$$\begin{aligned} R_2^\alpha(X,Y,Z,W) := [&\eta_\alpha(X)\eta_\alpha(\varphi_\alpha Y) - \eta_\alpha(\varphi_\alpha X)\eta_\alpha(Y)] \\ &\times [\eta_\alpha(\varphi_\alpha Z)\eta_\alpha(W) - \eta_\alpha(Z)\eta_\alpha(\varphi_\alpha W)]. \end{aligned}$$

Definition 2.4 ([1]). A para-Kähler manifold (M,φ,g) endowed with a unit vector field ξ is said to be

(1) of constant φ-sectional curvature if the sectional curvature of $\operatorname{span}\{u, \varphi u\}$ is constant for any $x \in M$ and any u non-isotropic tangent vector in T_xM;
(2) of quasi-constant φ-sectional curvature if the sectional curvature of $\operatorname{span}\{u, \varphi u\}$ is constant for any $x \in M$, any $\theta \in \left[0, \frac{\pi}{2}\right]$ and any u non-isotropic tangent vector in T_xM making the angle θ with $\operatorname{span}\{\xi_x, \varphi\xi_x\}$.

According to Theorem 2.1 from [1], the following result holds:

Theorem 2.5. *Let $(M, \varphi, \xi, \eta, g)$ be an almost para-cosymplectic manifold. Then the leaf $(F_\alpha, \varphi_\alpha, g_\alpha)$*

(1) *is of constant φ_α-sectional curvature if and only if there exists a function $c_\alpha\colon F_\alpha \to \mathbb{R}$ such that the curvature tensor field R^α satisfies $R^\alpha = c_\alpha R_0^\alpha$;*
(2) *is of quasi-constant φ_α-sectional curvature if and only if there exists three functions $c_\alpha^0, c_\alpha^1, c_\alpha^2\colon F_\alpha \to \mathbb{R}$ such that the curvature tensor field R^α satisfies $R^\alpha = c_\alpha^0 R_0^\alpha + c_\alpha^1 R_1^\alpha + c_\alpha^2 R_2^\alpha$.*

For the complex case, S. Funabashi, H. S. Kim, Y.-M. Kim, J. S. Pak [4] gave necessary and sufficient conditions for a Kähler manifold to be of constant holomorphic sectional curvature, involving certain spectral properties of the Laplace operator.

In the next section we will determine a relation between the curvature of the leaves of the canonical foliation and the Hodge-Laplace operator (equation (3.3)).

3. Harmonic forms on the leaves of the canonical foliation

Let $(M, \varphi, \xi, \eta, g)$ be an almost para-cosymplectic manifold of dimension $2n+1$. Consider the exterior differential and codifferential operators defined for any tangent bundle-valued p-form $T \in \Gamma(\Lambda^p T^*M \otimes TM)$ by

$$dT(X_0, \dots, X_p) := \sum_{i=0}^{p} (-1)^i (\nabla_{X_i} T)(X_0, \dots, \widehat{X_i}, \dots, X_p)$$

and

$$\delta T(X_1, \dots, X_{p-1}) := -\sum_{i=0}^{2n} (\nabla_{E_i} T)(E_i, X_1, \dots, X_{p-1}),$$

for an orthonormal frame field $\{E_i\}_{0 \leqslant i \leqslant 2n}$ and the Hodge-Laplace operator on $\Gamma(\Lambda^p T^*M \otimes TM)$

$$\Delta := d \circ \delta + \delta \circ d. \tag{3.1}$$

W. Jianming studied in [5] some properties of harmonic complex structures. Similar results hold for the leaves of the canonical foliation of an almost para-cosymplectic manifold. In our case, the leaves being almost para-Kähler manifolds, we shall deal with harmonic almost product structures and give the following obvious definition:

Definition 3.1. An almost product structure E is called harmonic if $\Delta E = 0$.

From the definition we infer that E is harmonic if and only if $dE = 0$ and $\delta E = 0$ which is equivalent to $(\nabla_X E)Y = (\nabla_Y E)X$ for any $X, Y \in \Gamma(TM)$ and $\operatorname{trace}(\nabla E) = 0$ for ∇ the Levi-Cività connection associated with the pseudo-Riemannian structure g.

Proposition 3.2. *Any harmonic almost product structure E is integrable (that is, it is a product structure).*

Proof. Let $X, Y \in \Gamma(TM)$. Then

$$\begin{aligned}(dE)(X,Y) &:= (\nabla E)(X,Y) - (\nabla E)(Y,X)\\ &= [X, EY] + \nabla_{EY}X - [Y, EX] - \nabla_{EX}Y - E[X,Y].\end{aligned}$$

As $\Delta E = 0$ implies $dE = 0$, we get

$$\begin{aligned}0 &= (dE)(EX,Y) + (dE)(X,EY)\\ &= [EX, EY] + [X,Y] - E[EX,Y] - E[X,EY],\end{aligned}$$

which shows the integrability of E. □

In particular, for any $T \in \Gamma(\Lambda^1 T^*M \otimes TM)$ we have [9]

$$\Delta T = -\nabla^2 T - S, \tag{3.2}$$

where $\nabla^2 T := \sum_{i=0}^{2n} \nabla_{E_i}\nabla_{E_i} T - \nabla_{\nabla_{E_i}E_i} T$ and $S(X) := \sum_{i=0}^{2n} (R_{E_i X} T)E_i$, $X \in \Gamma(TM)$, for $\{E_i\}_{0 \leqslant i \leqslant 2n}$ an orthonormal frame field and $R_{XY} := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, $X, Y \in \Gamma(TM)$, the Riemann curvature tensor field. We shall also use the notation $R_{XYZ} =: R_{XY}Z$ and $R_{XYZW} =: g(R_{XYZ}, Z)$, $X, Y, Z, W \in \Gamma(TM)$. Then for T equal to E and for any vector field X,

$$\begin{aligned}S(X) := \sum_{i=0}^{2n} (R_{E_i X} E)E_i &= \sum_{i=0}^{2n} R_{E_i X E E_i} - \sum_{i=0}^{2n} E(R_{E_i X E_i})\\ &= \sum_{i=0}^{2n} [R_{E_i X E E_i} - E(R_{E_i X E_i})].\end{aligned}$$

Denote by $e(E) := \sum_{i=0}^{2n} \frac{1}{2} g(EE_i, EE_i) = \frac{1}{2}|E|^2$ the energy density of E (which does not depend on the orthonormal frame field $\{E_i\}_{0 \leqslant i \leqslant 2n}$). We can state the following theorem:

Theorem 3.3. *Let $(M, \varphi, \xi, \eta, g)$ be an almost para-cosymplectic manifold and assume that any φ_α is a harmonic product structure. Then on each leaf F_α, $\alpha \in I$, of the canonical foliation $\mathscr{F}$, a Bochner-type formula*

$$\Delta e(\varphi_\alpha) = |\nabla \varphi_\alpha|^2 - \sum_{0 \leqslant i,j \leqslant 2n} (R_{E_i^\alpha E_j^\alpha \varphi_\alpha E_i^\alpha \varphi_\alpha E_j^\alpha} + R_{E_i^\alpha E_j^\alpha E_i^\alpha E_j^\alpha}) \tag{3.3}$$

holds for an orthonormal frame field $\{E_i^\alpha\}_{0 \leqslant i \leqslant 2n}$ on F_α with $\nabla_{E_i} E_i = 0$, $0 \leqslant i \leqslant 2n$.

Proof. A computation similar to that in [5] leads to

$$\langle \nabla^2 \varphi_\alpha, \varphi_\alpha \rangle = \sum_{i=0}^{2n} \langle \nabla_{E_i^\alpha} \nabla_{E_i^\alpha} \varphi_\alpha, \varphi_\alpha \rangle = \Delta e(\varphi_\alpha) - |\nabla \varphi_\alpha|^2$$

and

$$\begin{aligned}
\langle S, \varphi_\alpha \rangle &= \sum_{j=0}^{2n} \langle S E_j^\alpha, \varphi_\alpha E_j^\alpha \rangle \\
&= \sum_{0 \leqslant i,j \leqslant 2n} g(R_{E_i^\alpha E_j^\alpha \varphi_\alpha E_i^\alpha}, \varphi_\alpha E_j^\alpha) - g(\varphi_\alpha (R_{E_i^\alpha E_j^\alpha E_i^\alpha}), \varphi_\alpha E_j^\alpha).
\end{aligned}$$

Therefore, as φ_α is harmonic if $\Delta \varphi_\alpha = 0$, from (3.2) we obtain

$$\begin{aligned}
0 &= \langle \Delta \varphi_\alpha, \varphi_\alpha \rangle \\
&= -\langle \nabla^2 \varphi_\alpha, \varphi_\alpha \rangle - \langle S, \varphi_\alpha \rangle \\
&= -\Delta e(\varphi_\alpha) + |\nabla \varphi_\alpha|^2 \\
&\quad - \sum_{0 \leqslant i,j \leqslant 2n} [g(R_{E_i^\alpha E_j^\alpha \varphi_\alpha E_i^\alpha}, \varphi_\alpha E_j^\alpha) - g(\varphi_\alpha (R_{E_i^\alpha E_j^\alpha E_i^\alpha}), \varphi_\alpha E_j^\alpha)] \\
&= -\Delta e(\varphi_\alpha) + |\nabla \varphi_\alpha|^2 - \sum_{0 \leqslant i,j \leqslant 2n} (R_{E_i^\alpha E_j^\alpha \varphi_\alpha E_i^\alpha \varphi_\alpha E_j^\alpha} + R_{E_i^\alpha E_j^\alpha E_i^\alpha E_j^\alpha}).
\end{aligned}$$

□

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