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CONGRUENCE KERNELS OF DISTRIBUTIVE PJP-SEMILATTICES

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Abstract. A meet semilattice with a partial join operation satisfying certain axioms is a JP-semilattice. A PJP-semilattice is a pseudocomplemented JP-semilattice. In this paper we describe the smallest PJP-congruence containing a kernel ideal as a class. Also we describe the largest PJP-congruence containing a filter as a class. Then we give several characterizations of congruence kernels and cokernels for distributive PJP-semilattices.

Keywords: semilattice, distributivity, pseudocomplementation, congruence, kernel ideal, cokernel

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1. Introduction

Partial lattices have been studied by many authors. We refer the reader to [6], [7], [9], [10], [11] for partial lattices. Cornish and Noor [8], [12] have studied partial lattices which they preferred to call near lattices. A near lattice \( \mathbf{N} \) is a meet semilattice such that for any \( a, b \in \mathbf{N} \), \( a \lor b \) exists whenever there is a common upper bound of \( a, b \). We also refer the reader to the recent publications [3], [4], [5] for near lattices. Throughout the paper by semilattice we mean the meet semilattice. First we introduce the notion of the JP-semilattice.

A meet semilattice \( \mathbf{S} = \langle \mathbf{S}; \wedge, \lor \rangle \) with a partial binary operation \( \lor \) is said to be a join partial semilattice (or JP-semilattice) if for all \( x, y, z \in S \),

(i) \( x \lor x \) exists and \( x \lor x = x \);
(ii) if \( x \lor y \) exists, then \( y \lor x \) exists and \( x \lor y = y \lor x \);
(iii) if \( x \lor y, y \lor z, (x \lor y) \lor z \) exist, then \( x \lor (y \lor z) \) exists and \( (x \lor y) \lor z = x \lor (y \lor z) \);
(iv) if \( x \lor y \) exists, then \( x = x \land (x \lor y) \);
(v) if \( y \lor z \) exists, then \( (x \land y) \lor (x \land z) \) exists for all \( x \in S \).
Observe that not every semilattice needs to be a JP-semilattice, for example, the semilattice $P$ given in Figure 1 is not a JP-semilattice. Here $b \lor c$ exists, but $(a \land b) \lor (a \land c)$ does not. Moreover, it is easy to see that every near lattice is a JP-semilattice but the converse is not necessarily true, for example, the semilattice $M$ given in Figure 1 is a JP-semilattice but not a near lattice.

A JP-semilattice $S$ is said to be JP-distributive iff for all $x, y, z \in S$ such that $y \lor z$ exists one has $x \land (y \lor z) = (x \land y) \lor (x \land z)$ (remember the right-hand side exists by condition (v) above). It is evident that every distributive semilattice, considered as a JP-semilattice, is distributive (see [13]). However, the converse is not true. Consider the JP-semilattice $M_\infty$ given by Figure 2. It is JP-distributive but not distributive as a semilattice. The authors have studied JP-distributive JP-semilattices in [1].

Let $S$ be a JP-semilattice with smallest element $0$ and let $a \in S$. An element $d \in S$ is called the pseudocomplement of $a \in S$ if $a \land d = 0$ and for all $x \in S$, $a \land x = 0$ implies $x \leq d$. Clearly the pseudocomplement of an element is unique whenever it exists. The pseudocomplement of an element $a \in S$ is denoted by $a^\ast$. A JP-semilattice is said to be a pseudocomplemented JP-semilattice (or simply a PJP-semilattice) if every element has a pseudocomplement.
Let $S$ be a pseudocomplemented JP-semilattice. The set

$$\text{Sk}(S) = \{a^*: a \in S\}$$

is called the skeleton of $S$. The elements of $\text{Sk}(S)$ are called skeletal. It is evident that $\sup\{a^*, b^*\}$ in $\text{Sk}(S)$ always exists and we denote it by $a^* \vee b^*$. That is, for any $a, b \in \text{Sk}(S)$ we have $a \vee b = \sup\{a, b\}$ in $\text{Sk}(S)$. The following identities hold for PJP-semilattices as they hold for pseudocomplemented semilattices (see [9]). We often use the identities in this paper.

**Lemma 1.1.**

(a) $a \leq a^{**}$;
(b) $a \leq b$ implies $b^* \leq a^*$;
(c) $a^* = a^{***}$;
(d) $0^* = 1$, the largest element of $S$;
(e) $a \wedge b^* = a \wedge (a \wedge b)^*$;
(f) $a \in \text{Sk}(S) \Leftrightarrow a = a^{**}$;
(g) $a, b \in \text{Sk}(S) \Rightarrow a \wedge b = (a \wedge b)^*$;
(h) $a, b \in \text{Sk}(S) \Rightarrow a \vee b = (a^* \wedge b^*)^*$.

As in the case of distributive lattices, JP-distributivity in JP-semilattices does not imply the existence of pseudocomplements. For example, consider the distributive JP-semilattice $M_\infty$ given in Figure 2. Clearly, $M_\infty$ is not pseudocomplemented. In this paper we concentrate our attention on distributive PJP-semilattices.

Let $S$ be a PJP-semilattice. A semilattice congruence $\theta$ on $S$ is called a JP-congruence on $S$, if $x_1 \equiv y_1(\theta)$ and $x_2 \equiv y_2(\theta)$ implies $x_1 \vee x_2 \equiv y_1 \vee y_2(\theta)$ whenever $x_1 \vee x_2$ and $y_1 \vee y_2$ exist. A JP-congruence $\theta$ on $S$ is called a PJP-congruence on $S$, if $x \equiv y(\theta) \Rightarrow x^* \equiv y^*(\theta)$. A non-empty subset $I$ of a JP-semilattice $S$ is called an ideal of $S$ if the following conditions hold:

(i) if $i \in I$, $j \in S$ and $j \leq i$, then $j \in I$, and
(ii) if $i, j \in I$ and $i \vee j$ exists, then $i \vee j \in I$.

In Section 2 we give a useful characterization of PJP-congruences. We also give a description of the smallest PJP-congruence containing a certain ideal as a class.

Let $\theta$ be a PJP-congruence on $S$. Then $\ker(\theta) = \{x \in S: x \equiv 0(\theta)\}$ is called the kernel of $\theta$. A subset $J$ of $S$ is said to be a congruence kernel if $J = \ker(\theta)$ for some PJP-congruence $\theta$ on $S$. Observe that in the PJP-semilattice $M$ given in Figure 1, the ideal $I = \{0, a, b\}$ is not a kernel of any PJP-congruence on $M$. If $0 \equiv a(\theta)$ for any PJP-congruence $\theta$ on $M$, then $1 \equiv a^* = b(\theta)$, that is, $0 \equiv 1(\theta)$. Thus $I$ is not a PJP-congruence kernel. An ideal $I$ of a PJP-semilattice $S$ is called a kernel...
ideal if \( I = \ker(\theta) \) for some PJP-congruence \( \theta \) on \( S \). The set of all kernel ideals will be denoted by \( \text{KI}(S) \). Congruence kernels have been studied by Cornish [6] for pseudocomplemented distributive lattices and by Blyth [2] for pseudocomplemented semilattices. In this paper we characterize congruence kernels of distributive PJP-semilattices. In Section 3 we give characterizations of kernel ideals of distributive PJP-semilattices.

Let \( S \) be a PJP-semilattice. Let \( \theta \) be a PJP-congruence on \( S \). Then

\[
\text{Coker}(\theta) = \{ x \in S : x \equiv 1(\theta) \}
\]

is called the cokernel of \( \theta \). A subset \( J \) of \( S \) is said to be a congruence cokernel if \( J = \text{Coker}(\theta) \) for some PJP-congruence \( \theta \) on \( S \). A filter \( F \) of \( S \) is called a \( * \)-filter if

\[
f^{**} \in F \Rightarrow f \in F.
\]

In Section 4, we study cokernel filters. Here we characterize \( * \)-filters as a cokernel filters.

2. PJP-CONGRUENCES

For the basic properties of pseudocomplementation we refer the reader to [9]. First we have the following useful characterization of PJP-congruences.

**Theorem 2.1.** Let \( S \) be a PJP-semilattice. Then a JP-congruence \( \theta \) on \( S \) is a PJP-congruence if and only if

\[
x \equiv 0(\theta) \Rightarrow x^* \equiv 1(\theta).
\]

**Proof.** If \( \theta \) is a PJP-congruence, then clearly the condition holds. Conversely, let \( \theta \) be a JP-congruence such that the condition holds. Let \( x \equiv y(\theta) \). Then \( x^* \land x = 0(\theta) \) and so \( (x^* \land y)^* = 1(\theta) \). This implies

\[
x^* = x^* \land 1 = x^* \land (x^* \land y)(\theta) = x^* \land y^* \quad \text{(by Lemma 1.1 (e))}.
\]

Similarly, we have \( y^* \equiv x^* \land y^*(\theta) \). Hence \( x^* \equiv y^*(\theta) \) and therefore \( \theta \) is a PJP-congruence. \( \square \)

The following theorem gives us a description of the smallest PJP-congruence containing a certain ideal as a class.

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Theorem 2.2. Let $S$ be a distributive PJP-semilattice and let $I$ be an ideal of $S$ such that $i, j \in I$ implies $(i^* \land j^*)^* \in I$. Define a binary relation $\Theta(I)$ on $S$ by

$$x \equiv y(\Theta(I)) \quad \text{if and only if} \quad x \land i^* = y \land i^* \text{ for some } i \in I.$$  

Then $\Theta(I)$ is the smallest PJP-congruence containing $I$ as a class.

Proof. Clearly, $\Theta(I)$ is both reflexive and symmetric. To prove that it is transitive, let $x \equiv y(\Theta(I))$ and $y \equiv z(\Theta(I))$. Then $x \land i^* = y \land i^*$ and $y \land j^* = z \land j^*$ for some $i, j \in I$. Then by the assumption $k = (i^* \land j^*)^* \in I$. We have

$$x \land k^* = x \land (i^* \land j^*)^* = x \land (i^* \land j^*) \quad \text{(by Lemma 1.1 (b))}$$

$$= (x \land i^*) \land j^* = (y \land i^*) \land j^* = (y \land j^*) \land i^*$$

$$= (z \land j^*) \land i^* = z \land (i^* \land j^*) = z \land (i^* \land j^*)^*$$

$$= z \land k^*.$$  

Hence $x \equiv z(\Theta(I))$. Thus $\Theta(I)$ is transitive.

Let $x \equiv y(\Theta(I))$ and $s \equiv t(\Theta(I))$. Then there are $i, j \in I$ with $k = (i^* \land j^*)^* \in I$ such that $x \land i^* = y \land i^*$ and $s \land j^* = t \land j^*$. Hence

$$(x \land s) \land k^* = (x \land s) \land (i^* \land j^*)^*$$

$$= (x \land s) \land (i^* \land j^*) \quad \text{(by Lemma 1.1 (b))}$$

$$= (x \land i^*) \land (s \land j^*) = (y \land i^*) \land (t \land j^*)$$

$$= (y \land t) \land (i^* \land j^*)^*$$

$$= (y \land t) \land k^*.$$  

Also, if $x \lor s$ and $y \lor t$ exist, then

$$(x \lor s) \land k^* = (x \land k^*) \lor (s \land k^*) \text{ as } S \text{ is a distributive JP-semilattice}$$

$$= (x \land i^* \land j^*) \lor (s \land i^* \land j^*) \quad \text{(by Lemma 1.1 (b))}$$

$$= (y \land i^* \land j^*) \lor (t \land i^* \land j^*)$$

$$= (y \land k^*) \lor (t \land k^*)$$

$$= (y \lor t) \land k^* \text{ as } S \text{ is a distributive JP-semilattice}$$

$$= (y \lor t) \land k^*.$$  

Hence $\Theta(I)$ is a JP-congruence. To prove that $\Theta(I)$ is a PJP-congruence, let $x \equiv 0(\Theta(I))$. Then $x \land i^* = 0 \land i^* = 0$. This implies $i^* \leq x^*$. Hence $x^* \land i^* = i^* = 1 \land i^*$. This implies $x^* \equiv 1(\Theta(I))$. Hence by Theorem 2.1, $\Theta(I)$ is a PJP-congruence.
Finally, let $\theta$ be a PJP-congruence containing $I$ as a class and let $x \equiv y(\Theta(I))$. Then $x \wedge i^* = y \wedge i^*$ for some $i \in I$. Since $\theta$ is a PJP-congruence containing $I$ as a class, we have $i \equiv 0(\theta)$. This implies $i^* \equiv 1(\theta)$. Hence

$$x = x \wedge 1 \equiv x \wedge i^*(\theta) = y \wedge i^* \equiv y \wedge 1(\theta) = y.$$ 

Therefore $\Theta(I)$ is the smallest congruence containing $I$ as a class. \hfill \Box

3. Kernel ideals

Not every ideal of a JP-distributive PJP-semilattice is a kernel ideal. For a counterexample, consider the distributive PJP-semilattice $M$ given in Figure 1. Let $I = \{0, a, b\}$. Then $I$ is an ideal of $M$ but not a kernel ideal, since $0 \equiv a(\theta)$ for some PJP-congruence $\theta$ on $M$ implies $1 \equiv b(\theta)$.

We have the following characterization of kernel ideals.

**Theorem 3.1.** An ideal $I$ of a distributive PJP-semilattice $S$ is a kernel ideal of $S$ if and only if

$$i, j \in I \Rightarrow (i^* \wedge j^*)^* \in I.$$ 

**Proof.** Let $I$ be a kernel ideal of $S$. Then $I = \ker \theta$ for some PJP-congruence $\theta$. If $i, j \in I$, then $i \equiv 0(\theta)$ and $j \equiv 0(\theta)$. This implies immediately that $i^* \equiv 1(\theta)$ and $j^* \equiv 1(\theta)$. Hence $i^* \wedge j^* \equiv 1(\theta)$. This implies $(i^* \wedge j^*)^* \equiv 0(\theta)$. Thus $(i^* \wedge j^*)^* \in I$.

Conversely, let $I$ be an ideal of $S$ and suppose the condition holds. Then by Theorem 2.2, the binary relation $\Theta(I)$ on $S$ defined by

$$x \equiv y(\Theta(I)) \quad \text{if and only if} \quad x \wedge i^* = y \wedge i^* \quad \text{for some } i \in I$$ 

is a PJP-congruence containing the ideal $I$ as a class. So it is enough to show that $I$ is a kernel ideal of $\Theta(I)$. For all $i \in I$, by taking $i = j$ in the condition we have $i^{**} \in I$. Hence

$$x \equiv 0(\Theta(I)) \iff x \wedge i^* = 0 \quad \text{for some } i \in I$$ 

$$\iff x \leq i^{**} \quad \text{for some } i \in I$$ 

$$\iff x \in I.$$ 

Thus $I$ is a kernel ideal. \hfill \Box
**Theorem 3.2.** Let \( S \) be a distributive PJP-semilattice. An ideal \( I \) of \( S \) is a kernel ideal if and only if

(i) \( i \in I \) implies \( i^{**} \in I \);

(ii) for every \( i, j \in I \) there is \( k \in I \) such that \( i^* \land j^* = k^* \).

**Proof.** Let \( I \) be a kernel ideal. Then by taking \( i = j \) in Theorem 3.1 we
have \( i \in I \Rightarrow i^{**} \in I \). Thus (i) holds. Let \( i, j \in I \). Put \( k = (i^* \land j^*)^* \), then by
Theorem 3.1, \( k \in I \). Also \( k^* = i^* \land j^* \). Thus (ii) holds.

Conversely, let \( I \) be an ideal and \( i, j \in I \). Then by (iii), there is \( k \in I \) such that
\( k^* = i^* \land j^* \). Thus by (i), \( k^{**} = (i^* \land j^*)^* \in I \). Hence by Theorem 3.1, \( I \) is a kernel ideal. □

**Theorem 3.3.** Let \( S \) be a distributive PJP-semilattice. A principal ideal \( I = (x) \)
of \( S \) is a kernel ideal if and only if \( x \in Sk(S) \).

**Proof.** Suppose \( I = (x) \) is a kernel ideal, then \( x^{**} \in I \). This implies \( x^{**} \leq x \).
But \( x \leq x^{**} \). Hence \( x = x^{**} \in Sk(S) \).

Conversely, let \( I = (x) \) be a principal ideal and \( x \in Sk(S) \). Then by Lemma 1.1
(i), we have \( x = x^{**} \). Let \( i, j \in I \). Then \( i, j \leq x \). This implies \( x^* \leq i^* \land j^* \). Thus
\( (i^* \land j^*)^* \leq x^{**} = x \). This implies \( (i^* \land j^*)^* \in I \). Hence by Theorem 3.1, \( I \) is a kernel ideal. □

It is well known that the binary relation \( \psi(I) \) on a semilattice \( S \) defined by

\[
x \equiv y(\psi(I)) \text{ if and only if } x \land a \in I \iff y \land a \in I \text{ for any } a \in S
\]

is the largest semilattice congruence containing the ideal \( I \) as a class.

Now we have the following result for distributive JP-semilattices.

**Theorem 3.4.** Let \( S \) be a distributive JP-semilattice and let \( I \) be an ideal of \( S \).
Then \( \psi(I) \) is the largest JP-congruence containing \( I \) as a class.

**Proof.** It is enough to show that \( \psi(I) \) has the substitution property for the
partial operation \( \lor \). Let \( x \equiv y(\psi(I)) \), \( s \equiv t(\psi(I)) \) and let \( x \lor s \), \( y \lor t \) exist. Since \( S \)
is a distributive JP-semilattice, for any \( a \in S \) we have that \( (x \land a) \lor (s \land a) \), \( (y \land a) \lor (t \land a) \)
exist and \( (x \lor s) \land a = (x \land a) \lor (s \land a) \), \( (y \lor t) \land a = (y \land a) \lor (t \land a) \). Thus

\[
(x \lor s) \land a \in I \iff (x \land a) \lor (s \land a) \in I
\]

\[
\iff x \land a \in I \text{ and } s \land a \in I
\]

\[
\iff y \land a \in I \text{ and } t \land a \in I
\]

\[
\iff (y \land a) \lor (t \land a) \in I \iff (y \lor t) \land a \in I.
\]

Thus \( x \lor s \equiv y \lor t(\psi(I)) \). Hence \( \psi(I) \) is the largest JP-congruence. □

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The following result is a description of the largest PJP-congruence containing a kernel ideal as a class.

**Theorem 3.5.** Let $S$ be a distributive PJP-semilattice. If $I$ is a kernel ideal of $S$, then $\psi(I)$ is the largest PJP-congruence containing $I$ as a class.

**Proof.** By Theorem 3.4, $\psi(I)$ is a largest JP-congruence. Let $x \equiv 0(\psi(I))$. Then $x \in I$. Now for any $a \in S$,

$$x^* \land a \in I \Rightarrow (x^* \land (x^* \land a)^*)^* \in I,$$

by Theorem 3.1

$$\Rightarrow (x^* \land a)^* \in I,$$

by Lemma 1.1 (e)

$$\Rightarrow a \in I,$$

since $a \leq a^{**} \leq (x^* \land a)^*$

$$\Rightarrow 1 \land a \in I.$$

Also

$$1 \land a = a \in I \Rightarrow x^* \land a \in I.$$

Thus $x^* \equiv 1(\psi(I))$. Hence by Theorem 2.1, $\psi(I)$ is a PJP-congruence. □

**-ideal.** An ideal $I$ of a JP-semilattice is called a $*$-ideal if it satisfies condition (i) of Theorem 3.2, that is,

$$i \in I \text{ implies } i^{**} \in I.$$

Clearly, every kernel ideal of a distributive PJP-semilattice is a $*$-ideal. Consider the distributive PJP-semilattice $M$ given in Figure 1. Here the ideal $I = \{0, a, b\}$ is a $*$-ideal but not a kernel ideal.

**Theorem 3.6.** Let $S$ be a distributive PJP-semilattice. Every principal $*$-ideal $I$ of $S$ can be written as $(a^{**})$ for some $a \in I$. Moreover, for any $a \in S$ the principal ideal $I = (a^{**})$ is a kernel ideal.

**Proof.** Let $I$ be a principal $*$-ideal of $S$. Then $I = (a)$ for some $a \in S$. Since $I$ is a $*$-ideal, for $a \in I$ we have $a^{**} \in I$. Thus $a^{**} \leq a$. But $a \leq a^{**}$. Hence $I = (a^{**})$ for some $a \in S$.

Moreover, for any $a \in S$, since $a^{**} \in \text{Sk}(S)$, so by Theorem 3.3, $I = (a^{**})$ is a kernel ideal. □

**Theorem 3.7.** A $*$-ideal $I$ of a distributive PJP-semilattice is a kernel ideal if and only if $i^{**} \lor j^{**} \in I$ for all $i, j \in I$. 232
Proof. For any \(i, j \in I\) we have
\[
(i^* \land j^*)^* = (i^{**} \land j^{**})^* \text{ by Lemma 1.1 (c)}
\]
\[
= i^{**} \lor j^{**} \text{ by Lemma 1.1 (h)}.
\]
By Theorem 3.1, \(I\) is a kernel ideal if and only if \(i, j \in I\) implies \(i^{**} \lor j^{**} \in I\). \(\square\)

Glivenko congruence. Let \(S\) be a distributive PJP-semilattice. The binary relation \(G\) on \(S\) defined by
\[
x \equiv y(G) \iff x^* = y^*
\]
is a semilattice congruence called the Glivenko congruence. It is evident that \(G\) is compatible with \(*\). We shall show that \(G\) is a PJP-congruence.

Let \(I\) be an ideal. Define
\[
I^0 = \{x \in S: x \land i = 0 \text{ for all } i \in I\}.
\]

Theorem 3.8. \(I^0\) is a kernel ideal.

Proof. Let \(x, y \in I^0\). Then \(x \land i = y \land i = 0\) for all \(i \in I\). Hence \(i \leq x^*, y^*\) and consequently, \((x^* \land y^*)^* \leq i^*\). This implies \((x^* \land y^*)^* \land i \leq i^* \land i = 0\). Hence \((x^* \land y^*)^* \in I^0\). Thus by Theorem 3.1, \(I^0\) is a kernel ideal. \(\square\)

Lemma 3.9. If \(x \equiv y(\psi(I))\), then \([(x \land y^*)^* \land (x^* \land y)^*)^* \in I\).

Proof. Let \(x \equiv y(\psi(I))\). Then \(x \land x^* = 0 \equiv y \land x^*(\psi(I))\). Therefore \(y \land x^* \in I\). Similarly, \(x \land y^* \in I\). Hence \([(x \land y^*)^* \land (x^* \land y)^*)^* \in I\) as \(I\) is a kernel ideal. \(\square\)

Theorem 3.10. Let \(I\) be a kernel ideal of a distributive PJP-semilattice \(S\). Then \(\psi(I) \land \psi(I^0) = G\).

Proof. Let \(x \equiv y(\psi(I) \land \psi(I^0))\). Then by Lemma 3.9, we have \([(x \land y^*)^* \land (x^* \land y)^*)^* \in I\) and \([(x \land y^*)^* \land (x^* \land y)^*)^* \in I^0\) whence \([(x \land y^*)^* \land (x^* \land y)^*)^* = 0\). This implies
\[
x \land y^* \leq (x \land y^*)^{**} \leq [(x \land y^*)^* \land (x^* \land y)^*)^* = 0.
\]
Thus \(x \land y^* = 0\). Hence \(y^* \leq x^*\). Similarly, \(x^* \leq y^*\). This implies \(x^* = y^*\) and consequently, \(x^{**} = y^{**}\). Hence \(x \equiv y(G)\).
Conversely, let \( x \equiv y(G) \). Since \( a \equiv a**(G) \) for any \( a \in S \), we have \( x \wedge a \equiv x \wedge a**(G) \), \( y \wedge a \equiv y \wedge a**(G) \) and \( x \wedge a \equiv y \wedge a**(G) \). Hence \( (x \wedge a)** = (x \wedge a**)**, \( (y \wedge a)** = (y \wedge a**)** \) and \( (x \wedge a)** = (y \wedge a**)** \). Now for any \( a \in S \),

\[
x \wedge a \in I \iff (x \wedge a)** \in I \text{ as } I \text{ is a kernel ideal of } S \\
\iff (y \wedge a**)** \in I \\
\iff (y \wedge a)** \in I \\
\iff y \wedge a \in I.
\]

Also, for all \( i \in I \),

\[
x \wedge a \in I^0 \iff (x \wedge a) \wedge i = 0 \\
\iff x \wedge (a \wedge i) = 0 \\
\iff x \leq (a \wedge i)* \\
\iff x** \leq (a \wedge i)* \\
\iff y** \leq (a \wedge i)* \\
\iff y \leq (a \wedge i)* \\
\iff y \wedge (a \wedge i) = 0 \\
\iff y \wedge a \in I^0.
\]

Hence \( x \equiv y(\psi(I) \wedge \psi(I^0)) \). Therefore \( G = \psi(I) \wedge \psi(I^0) \).

\[\square\]

**Corollary 3.11.** \( G \) is a PJP-congruence.

**Proof.** This is immediate from the fact that \( \psi(I) \wedge \psi(I^0) \) is a PJP-congruence. \[\square\]

### 4. Congruence cokernels

Let \( S \) be a JP-semilattice. A non-empty subset \( F \) of \( S \) is called a filter of \( S \) if

(i) \( a \in F \) and \( b \in S \) with \( a \leq b \) implies \( b \in F \), and

(ii) \( a, b \in F \) implies \( a \wedge b \in F \).

Now we have the following lemma.

**Lemma 4.1.** Let \( S \) be a JP-semilattice. Then every cokernel of \( S \) is a filter.
Let $F = \text{Coker}(\theta)$ for some PJP-congruence $\theta$. If $x, y \in F$, then $x \equiv 1(\theta)$ and $y \equiv 1(\theta)$. Hence $x \wedge y \equiv 1(\theta)$. Thus $x \wedge y \in F$. Now let $x \in F$ and $x \leq y$. Then $x = x \wedge y \equiv 1 \wedge y(\theta) = y$. Thus $y \equiv 1(\theta)$. Hence $y \in F$. Therefore $F$ is a filter.

Let $S$ be a JP-semilattice and let $F$ be a filter of $S$. Define a binary relation $\Theta(F)$ on $S$ by

$$x \equiv y(\Theta(F)) \quad \text{if and only if} \quad x \wedge f = y \wedge f \quad \text{for some} \quad f \in F.$$  

**Theorem 4.2.** Let $F$ be a filter of a distributive JP-semilattice $S$. Then the relation $\Theta(F)$ on $S$ is a JP-congruence containing $F$ as a class. Moreover, if $S$ has a largest element 1, then $\Theta(F)$ is the smallest JP-congruence containing $F$ as a class.

**Proof.** Clearly $\Theta(F)$ is an equivalence relation. Let $x \equiv y(\Theta(F))$ and $s \equiv t(\Theta(F))$. Then $x \wedge f_1 = y \wedge f_1$ and $s \wedge f_2 = t \wedge f_2$ for some $f_1, f_2 \in F$. This implies

$$(x \wedge s) \wedge (f_1 \wedge f_2) = (x \wedge f_1) \wedge (s \wedge f_2) = (y \wedge f_1) \wedge (t \wedge f_2) = (y \wedge t) \wedge (f_1 \wedge f_2).$$

Since $f_1 \wedge f_2 \in F$, we have $x \wedge s \equiv y \wedge t(\Theta(F))$.

Also, if $x \vee s$ and $y \vee t$ exist, then

$$(x \vee s) \wedge (f_1 \wedge f_2) = (x \wedge f_1 \wedge f_2) \vee (s \wedge f_1 \wedge f_2)$$

$$= (y \wedge f_1 \wedge f_2) \vee (t \wedge f_1 \wedge f_2) = (y \vee t) \wedge (f_1 \wedge f_2).$$

Thus $\Theta(F)$ is a JP-congruence. Clearly, $\Theta(F)$ contains $F$ as a class.

Moreover, assume that $S$ has a largest element 1. Let $\theta$ be any congruence on $S$ containing $F$ as a class. Assume $x \equiv y(\Theta(F))$. Then $x \wedge f = y \wedge f$ for some $f \in F$. This implies $x = x \wedge 1 \equiv x \wedge f(\theta)$. Similarly, $y \equiv y \wedge f(\theta)$. Hence $x \equiv y(\theta)$. Thus $\Theta(F)$ is the smallest JP-congruence containing $F$ as a class.  

The following result is the description of the smallest PJP-congruence containing a filter as a class.

**Theorem 4.3.** Let $S$ be a PJP-semilattice and let $F$ be a filter of $S$. Then $\Theta(F)$ is the smallest PJP-congruence containing $F$ as a class.

**Proof.** By Theorem 4.2, $\Theta(F)$ is a JP-congruence containing $F$ as a class. Let $x \equiv 0(\Theta(F))$. Then $x \wedge f = 0$ for some $f \in F$. This implies $f \leq x^*$. Thus $x^* \in F$. Hence $x^* \equiv 1(\Theta(F))$. Hence by Theorem 2.1, we have that $\Theta(F)$ is a PJP-congruence.
**Corollary 4.4.** Every filter of a PJP-semilattice is a cokernel.

**Proof.** It is clear from the fact that for any filter $F$ of $S$ we have

$$x \in F \Leftrightarrow x \equiv 1(\Theta(F)).$$

\[\Box\]

**-filters.** First we prove the following useful result.

**Lemma 4.5.** Let $S$ be a distributive PJP-semilattice. If $a \lor b$ exists, then

$$(a \lor b)^* = a^* \land b^*.$$  

**Proof.** We have $$(a \lor b) \land (a^* \land b^*) = (a \land a^* \land b^*) \lor (b \land a^* \land b^*) = 0 \lor 0 = 0.$$  

Let $(a \lor b) \land x = 0$. Then $(a \land x) \lor (b \land x) = 0$. Hence $a \land x = 0$ and $b \land x = 0$. This implies $x \leq a^*, b^*$. Hence $x \leq a^* \land b^*$. Therefore $(a \lor b)^* = a^* \land b^*$.  

For every filter $F$ of $S$ define

$$F^* = \{x \in S : x^* \in F\}.$$  

**Lemma 4.6.** Let $S$ be a distributive PJP-semilattice and $F$ a filter of $S$. Then $F^*$ is a kernel ideal of $S$.

**Proof.** Let $x, y \in F^*$.

Then $x, y^* \in F$. If $x \lor y$ exists, then by Lemma 4.5 we have $(x \lor y)^* = x^* \land y^* \in F$ as $F$ is a filter. Hence $x \lor y \in F^*$. Let $x \in F^*$ and $y \leq x$. Then $y^* \geq x^* \in F$.

This implies $y^* \in F$. Thus $y \in F^*$. Hence $F^*$ is an ideal.

To prove that $F^*$ is a kernel ideal, let $x, y \in F^*$. Then $x^*, y^* \in F$ so that $(x^* \land y^*)^* = x^* \land y^* \in F$ and consequently $(x^* \land y^*)^* \in F^*$. Hence by Theorem 3.1, $F^*$ is a kernel ideal.  

For every $I \in KI(S)$ define

$$I^* = \{x \in S : x^* \in I\}.$$  

**Lemma 4.7.** Let $S$ be a distributive PJP-semilattice and $I$ a kernel ideal of $S$. Then $I^*$ is a *-filter of $S$.

**Proof.** Let $x, y \in I^*$. Then $x, y^* \in I$. So by Theorem 3.1, we have $(x \land y)^* = (x \land y)^** = (x^* \land y^*)^* \in I$. Hence $x \land y \in I^*$. Now let $x \in I^*$ and $y \geq x$. Then $y^* \leq x^* \in I$ so that $y^* \in I$ and consequently, $y \in I^*$. Hence $I^*$ is a filter. Let $x^* \in I^*$. Then $x^* = x^** \in I$ and hence $x \in I^*$. Therefore $I^*$ is a *-filter.  

The following theorem is a characterization of *-filters.
Theorem 4.8. A filter \( F \) of a JP-distributive PJP-semilattice is a \(*\)-filter if and only if \((F_*)_*= F\).

Proof. Let \((F_*)_*= F\) and let \(x^{**} \in F\). Since \(F\) is a filter, \(F_*\) is a kernel ideal. Hence \(x^* \in F_*\) and so \(x \in (F_*)_*= F\). Thus \(F\) is a \(*\)-filter.

Conversely, let \(F\) be a \(*\)-filter. Then

\[
x \in (F_*)_* \iff x^* \in F_* \\
\iff x^{**} \in F \\
\iff x \in F \quad (\Rightarrow \text{ as } F \text{ is a } *\text{-filter and } \Leftarrow \text{ as } F \text{ is a filter}).
\]

\(\square\)

D-filter. A filter \( F \) of a PJP-semilattice \( S \) is called a D-filter if it contains the dense filter \( D = \{ x \in S : x^* = 0 \} \).

Theorem 4.9. Every \(*\)-filter is a D-filter but the converse is not true.

Proof. Let \( F \) be a \(*\)-filter and let \( d \in D \). Then \( d^{**} = 1 \in F \) which implies that \( d \in F \). Hence \( F \) contains \( D \). Thus \( F \) is a D-filter.

To prove the converse is not true, consider the distributive PJP-semilattice \( N \) given in Figure 3. The filter \([c]\) is a D-filter but not a \(*\)-filter.

Let \( S \) be a PJP-semilattice. A PJP-congruence \( \theta \) on \( S \) is called a boolean congruence if the factor PJP-semilattice \( S/\theta \) is a Boolean lattice.

Theorem 4.10. A PJP-congruence \( \theta \) is a boolean congruence if and only if \( x \equiv x^{**}(\theta) \) for all \( x \in X \).

Proof. This is immediate from the fact that \(([x](\theta))^* = [x^*](\theta)\). \(\square\)
Theorem 4.11. Let $S$ be a distributive PJP-semilattice. Then the following conditions are equivalent:

(i) every $D$-filter is a $*$-filter;
(ii) $\Theta(D)$ is a boolean congruence.

Proof. (i) $\Rightarrow$ (ii). For each $x \in S$ we have that $F = [x^{**}] \lor D$ is a $D$-filter and hence $F$ is a $*$-filter. Since $x^{**} \in F$, we have $x \in F$. Thus $x = x^{**} \land d$ for some $d \in D$. This implies $x \land d = x^{**} \land d$. Hence $x \equiv x^{**} \Theta(D)$. Therefore, by Theorem 4.10, $\Theta(D)$ is a boolean congruence.

(ii) $\Rightarrow$ (i). Let $F$ be a $D$-filter. By (ii), $\Theta(D)$ is a boolean congruence. Hence by Theorem 4.10 $x \equiv x^{**}(\Theta(D))$. Thus $x \land d = x^{**} \land d$ for some $d \in D$. If $x^{**} \in F$, then $x^{**} \land d \in F$ as $D \subseteq F$. Hence $x \land d \in F$ and consequently, $x \in F$. Thus $F$ is a $*$-filter. $\square$

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References


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