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SUBORDINATION RESULTS FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. We introduce two classes of analytic functions related to conic domains, using a new linear multiplier Dziok-Srivastava operator $D_{\lambda,l}^{n,q,s}$ ($n \in \mathbb{N}_0 = \{0, 1, \ldots\}$, $q \leqslant s+1$; $q, s \in \mathbb{N}_0$, $0 \leqslant \alpha < 1$, $\lambda \geqslant 0$, $l \geqslant 0$). Basic properties of these classes are studied, such as coefficient bounds. Various known or new special cases of our results are also pointed out. For these new function classes, we establish subordination theorems and also deduce some corollaries of these results.

Keywords: uniformly convex function, subordination, conic domain, Hadamard product

MSC 2010: 30C45

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z\colon\ z \in \mathbb{C} \text{ and } |z| < 1\}$.

For functions $f(z) \in A$ given by (1.1) and $g(z) \in A$ defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \quad (z \in U).$$

Given γ $(0 \leqslant \gamma < 1)$, a function $f \in A$ is said to be in the class of starlike functions of order γ in U, denoted by $\mathrm{ST}(\gamma)$, if

$$\mathrm{Re}\Big\{\frac{zf'(z)}{f(z)}\Big\} > \gamma \quad (z \in U,\ 0 \leqslant \gamma < 1).$$

On the other hand, a function $f \in A$ is said to be in the class $\mathrm{CV}(\gamma)$ of convex functions of order γ in U if

$$\mathrm{Re}\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} > \gamma \quad (z \in U,\ 0 \leqslant \gamma < 1).$$

In particular, the classes $\mathrm{CV} \equiv \mathrm{CV}(0)$ and $\mathrm{ST} \equiv \mathrm{ST}(0)$ are, respectively, the familiar classes of convex and starlike functions in U.

A function $f \in A$ is said to be in the class of uniformly convex functions of order γ and type β, denoted by $\mathrm{UCV}(\beta,\gamma)$, see [10], if

$$\mathrm{Re}\Big\{1 + \frac{zf''(z)}{f'(z)} - \gamma\Big\} > \beta\Big|\frac{zf''(z)}{f'(z)}\Big|,$$

where $\beta \geqslant 0$, $\gamma \in [-1,1)$ and $\beta + \gamma \geqslant 0$, and is said to be in the corresponding class denoted by $\mathrm{SP}(\beta,\gamma)$ if

$$\mathrm{Re}\Big\{\frac{zf'(z)}{f(z)} - \gamma\Big\} > \beta\Big|\frac{zf'(z)}{f(z)} - 1\Big|,$$

where $\beta \geqslant 0$, $\gamma \in [-1,1)$ and $\beta + \gamma \geqslant 0$.

It is obvious that $f(z) \in \mathrm{UCV}(\beta,\gamma)$ if and only if $zf'(z) \in \mathrm{SP}(\beta,\gamma)$. These classes generalize various other classes. For $\beta = 0$, we get, respectively, the classes $\mathrm{CV}(\gamma)$ and $\mathrm{ST}(\gamma)$. The functions of the class $\mathrm{UCV}(1,0) \equiv \mathrm{UCV}$ are called uniformly convex functions and were introduced by Goodman with geometric interpretation in [15]. The class $\mathrm{SP}(1,0) \equiv \mathrm{SP}$ is defined by Ronning in [29]. The classes $\mathrm{UCV}(1,\gamma) \equiv \mathrm{UCV}(\gamma)$ and $\mathrm{SP}(1,\gamma) \equiv \mathrm{SP}(\gamma)$ are investigated by Ronning in [28]. For $\gamma = 0$, the classes $\mathrm{UCV}(\beta,0) \equiv \beta - \mathrm{UCV}$ and $\mathrm{SP}(\beta,0) \equiv \beta - \mathrm{SP}$ are defined, respectively, by Kanas and Wisniowska in [16] and [17].

Geometric interpretation ([1]). $f \in \mathrm{UCV}(\beta,\gamma)$ and $f \in \mathrm{SP}(\beta,\gamma)$ if and only if $1 + zf''(z)/f'(z)$ and $zf'(z)/f(z)$, respectively, take all the values in the conic domain $R_{\beta,\gamma}$ which is included in the right half plane, such that

$$R_{\beta,\gamma} = \{u + \mathrm{i}v\colon\ u > \beta\sqrt{(u-1)^2 + v^2} + \gamma\}. \tag{1.2}$$

Denote by $\varrho(P_{\beta,\gamma})$ $(\beta \geqslant 0,\ -1 \leqslant \gamma < 1)$ the family of functions p such that $p \in \varrho$ and $p \prec P_{\beta,\gamma}$ in U, where ϱ denotes the well-known class of Carathléodory functions and the function $P_{\beta,\gamma}$ maps the unit disc conformally onto the domain $R_{\beta,\gamma}$ such that $1 \in R_{\beta,\gamma}$ and $\partial R_{\beta,\gamma}$ is a curve defined by the equality

$$\partial R_{\beta,\gamma} = \{u + \mathrm{i}v\colon\ u^2 = (\beta\sqrt{(u-1)^2 + v^2} + \gamma)^2\}.$$

From elementary computations we see that $\partial R_{\beta,\gamma}$ represents the conic sections symmetric about the real axis. Thus $R_{\beta,\gamma}$ is an elliptic domain for $\beta > 1$, a parabolic domain for $\beta = 1$, a hyperbolic domain for $0 < \beta < 1$ and a right half plane $u > \gamma$ for $\beta = 0$.

The functions which play the role of extremal functions of the class $\varrho(P_{\beta,\gamma})$ were obtained in [1] as follows:

$$(1.3)\quad P_{\beta,\gamma}(z) = \begin{cases} \dfrac{1+(1-2\gamma)z}{1-z}, & \beta = 0,\\[2mm] 1 + \dfrac{2(1-\gamma)}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \beta = 1,\\[2mm] \dfrac{1-\gamma}{1-\beta^2}\cos\left\{\left(\dfrac{2}{\pi}\cos^{-1}\beta\right)\mathrm{i}\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \dfrac{\beta^2-\gamma}{1-\beta^2}, & 0 < \beta < 1,\\[2mm] \dfrac{1-\gamma}{\beta^2-1}\sin\dfrac{\pi}{2K(t)}\displaystyle\int_0^{u(z)/\sqrt{t}}\dfrac{1}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\,\mathrm{d}x & \\ \qquad\qquad + \dfrac{\beta^2-\gamma}{\beta^2-1}, & \beta > 1, \end{cases}$$

where $u(z) = (z-\sqrt{t})/(1-\sqrt{tz})$, $t \in (0,1)$, $z \in U$ and t is chosen such that $\beta = \cosh(\pi K'(t)/4K(t))$, $K(t)$ is Legendre's complete elliptic integral of the first kind and $K'(t)$ is the complementary integral of $K(t)$.

For $\beta = 0$ obviously $P_{0,\gamma}(z) = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + \ldots$, for $\beta = 1$ (compare [21] and [29]) $P_{1,\gamma}(z) = 1 + 8\pi^{-2}(1-\gamma)z + \frac{16}{3}\pi^{-2}(1-\gamma)z^2 + \ldots$, by comparing the Taylor series expansion in [18] we get for $0 < \beta < 1$

$$P_{\beta,\gamma}(z) = 1 + \frac{1-\gamma}{1-\beta^2}\sum_{k=1}^{\infty}\left[\sum_{l=1}^{2k} 2^l \binom{B}{l}\binom{2k-1}{2k-l}\right] z^k,$$

where $B = 2\pi^{-1}\cos^{-1}\beta$, and for $\beta > 1$

$$P_{\beta,\gamma}(z) = 1 + \frac{\pi^2(1-\gamma)}{4\sqrt{t}(\beta^2-1)K^2(t)(1+t)} \times \left\{z + \frac{4K^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}K^2(t)(1+t)}z^2 + \ldots\right\}.$$

For complex parameters

$$\alpha_1, \ldots, \alpha_q \ \text{ and } \ \beta_1, \ldots, \beta_s \quad (\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\};\ j = 1, 2, \ldots, s)$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s;z)$ by (see, for example, [36, p. 30])

$$(1.4)\qquad {}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k}.\frac{z^k}{k!}$$
$$(q\leqslant s+1;\ q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \mathbb{N}=\{1,2,\ldots\};\ z\in U),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined in terms of the Gamma function Γ by

$$(1.5)\qquad (\theta)_\nu=\frac{\Gamma(\theta+\nu)}{\Gamma(\theta)}=\begin{cases}1 & (\nu=0;\ \theta\in C\setminus\{0\}),\\ \theta(\theta-1)\ldots(\theta+\nu-1) & (\nu\in\mathbb{N};\ \theta\in C).\end{cases}$$

Corresponding to the function $h(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ defined by

$$(1.6)\qquad h(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=z\ {}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z),$$

we consider a linear operator $H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)\colon\ A\to A$ which is defined by the following Hadamard product (or convolution):

$$(1.7)\qquad H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)f(z)=h(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*f(z).$$

We observe that for a function $f(z)$ of the form (1.1) we have

$$(1.8)\quad H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=z+\sum_{k=2}^{\infty}\frac{(\alpha_1)_{k-1}\ldots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ldots(\beta_s)_{k-1}(1)_{k-1}}a_kz^k$$
$$=z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)a_kz^k,$$

where

$$\Gamma_k(\alpha_1)=\frac{(\alpha_1)_{k-1}\ldots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ldots(\beta_s)_{k-1}(1)_{k-1}}.$$

For convenience, we write

$$(1.9)\qquad H_{q,s}(\alpha_1)=H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s).$$

We define the linear extended multiplier Dizok-Srivastava operator $D_{\lambda,l}^{n,q,s}$ as follows:

$$(1.10)\qquad \begin{aligned} D_{\lambda,l}^{0,q,s}f(z)&=f(z),\\ D_{\lambda,l}^{1,q,s}f(z)&=(1-\lambda)H_{q,s}(\alpha_1)f(z)\\ &\quad+\frac{\lambda}{(l+1)z^{l-1}}(z^lH_{q,s}(\alpha_1)f(z))'\ (\lambda\geqslant 0;l\geqslant 0),\\ D_{\lambda,l}^{2,q,s}f(z)&=D_{\lambda,l}^{q,s}(D_{\lambda,l}^{1,q,s}f(z)),\end{aligned}$$

and (in general)

$$D_{\lambda,l}^{n,q,s}f(z)=D_{\lambda,l}^{q,s}(D_{\lambda,l}^{n-1,q,s}f(z))\quad(n\in\mathbb{N}). \tag{1.11}$$

If f is given by (1.1), then from (1.8) and (1.11) we see that

$$D_{\lambda,l}^{n,q,s}f(z)=z+\sum_{k=2}^{\infty}\Phi_{k,n}(\alpha_1,\lambda,l)a_kz^k\quad(n\in\mathbb{N}_0), \tag{1.12}$$

where

$$\Phi_{k,n}(\alpha_1,\lambda,l)=\Big[\frac{l+1+\lambda(k-1)}{l+1}\Gamma_k(\alpha_1)\Big]^n. \tag{1.13}$$

By virtue of (1.7) and (1.13), $D_{\lambda,l}^{n,q,s}f(z)$ can be written in terms of convolution as

$$\begin{aligned}D_{\lambda,l}^{n,q,s}&f(z)\\&=\underbrace{\big[(h(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)*g_{\lambda,l}(z))*\dots *(h(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)*g_{\lambda,l}(z))\big]}_{n\text{-times}}*f(z),\end{aligned} \tag{1.14}$$

where

$$g_{\lambda,l}(z)=\frac{(l+1)z-(l+1-\lambda)z^2}{(l+1)(1-z)^2}=\frac{z-(1-\lambda/(l+1))z^2}{(1-z)^2}.$$

By specializing the parameters q,s,α_1,β_1,l and λ, we obtain the following operators studied by various authors:

(i) For $q=2$, $s=1$, $\alpha_1=2$, $\alpha_2=1$, $\beta_1=2-\alpha$ ($\alpha\neq 2,3,4,\dots$) and $l=0$, we have $D_{\lambda,0}^{n,2,1}f(z)=D_{\lambda}^{n,\alpha}f(z)$ (see Al-Oboudi and Al-Amoudi [3] and Aouf and Mostafa [5]);

(ii) For $q=2$, $s=1$, $\alpha_1=a$ ($a>0$), $\alpha_2=1$, $\beta_1=c$ ($c>0$) and $l=0$, we have $D_{\lambda,0}^{n,2,1}f(z)=I_{a,c,\lambda}^{n}f(z)$ (see Prajapat and Raina [26]);

(iii) For $q=2$, $s=1$ and $\alpha=\alpha_2=\beta_1=1$, we have $D_{\lambda,l}^{n,2,1}f(z)=I^n(\lambda,l)f(z)$ (see Catas [11]);

$D_{\lambda,0}^{n,2,1}f(z)=D_{\lambda}^{n}f(z)$ (see Al-Oboudi [2]) and $D_{1,0}^{n,2,1}f(z)=D^nf(z)$ (see Salagean [32]);

(iv) $D_{0,0}^{1,q,s}f(z)=H_{q,s}(\alpha_1)$ (see Dziok and Srivastava [13]).

Using the operator $D_{\lambda,l}^{n,q,s}f(z)$, we define the following classes. Let $\mathrm{UCV}_{\lambda,l}^{n,q,s}(\alpha_1,\beta_1;\beta,\gamma)$ be the class of functions $f\in A$ satisfying

$$\mathrm{Re}\Big\{1+\frac{z(D_{\lambda,l}^{n,q,s}f(z))''}{(D_{\lambda,l}^{n,q,s}f(z))'}-\gamma\Big\}>\beta\Big|\frac{z(D_{\lambda,l}^{n,q,s}f(z))''}{(D_{\lambda,l}^{n,q,s}f(z))'}\Big|, \tag{1.15}$$

where $\beta \geqslant 0$, $\gamma \in [-1, 1)$, $\beta + \gamma \geqslant 0$, $q \leqslant s + 1$, $q, s \in \mathbb{N}_0$, $\lambda \geqslant 0$, $l \geqslant 0$ and $n \in \mathbb{N}_0$. Observe that $D^{n,q,s}_{\lambda,l} f(z) \in \mathrm{UCV}(\beta, \gamma)$.

Let $\mathrm{SP}^{n,q,s}_{\lambda,l}(\alpha_1, \beta_1; \beta, \gamma)$ be the corresponding class consisting of functions $f(z) \in A$ satisfying

$$\mathrm{Re}\left\{ \frac{z(D^{n,q,s}_{\lambda,l} f(z))'}{D^{n,q,s}_{\lambda,l} f(z)} - \gamma \right\} > \beta \left| \frac{z(D^{n,q,s}_{\lambda,l} f(z))'}{D^{n,q,s}_{\lambda,l} f(z)} - 1 \right|, \tag{1.16}$$

where $\beta \geqslant 0$, $\gamma \in [-1, 1)$, $\beta + \gamma \geqslant 0$, $q \leqslant s + 1$, $q, s \in \mathbb{N}_0$, $\lambda \geqslant 0$, $l \geqslant 0$ and $n \in \mathbb{N}_0$. Observe that $D^{n,q,s}_{\lambda,l} f(z) \in \mathrm{SP}(\beta, \gamma)$.

From (1.15) and (1.16) we have

$$f(z) \in \mathrm{UCV}^{n,q,s}_{\lambda,l}(\alpha_1, \beta_1; \beta, \gamma) \Leftrightarrow z f'(z) \in \mathrm{SP}^{n,q,s}_{\lambda,l}(\alpha_1, \beta_1; \beta, \gamma), \tag{1.17}$$

and

$$\mathrm{UCV}^{n,q,s}_{\lambda,l}(\alpha_1, \beta_1; \beta, \gamma) \subseteq \mathrm{SP}^{n,q,s}_{\lambda,l}(\alpha_1, \beta_1; \beta, \gamma).$$

Geometric interpretation. By virtue of (1.15) and (1.16), $f(z) \in \mathrm{UCV}^{n,q,s}_{\lambda,l}$ $(\alpha_1, \beta_1; \beta, \gamma)$ and $f(z) \in \mathrm{SP}^{n,q,s}_{\lambda,l}$ $(\alpha_1, \beta_1; \beta, \gamma)$ if and only if

$$p(z) = 1 + \frac{z(D^{n,q,s}_{\lambda,l} f(z))''}{(D^{n,q,s}_{\lambda,l} f(z))'} \quad \text{and} \quad p(z) = \frac{z(D^{n,q,s}_{\lambda,l} f(z))'}{D^{n,q,s}_{\lambda,l} f(z)},$$

respectively, take all values in the conic domain $R_{\beta,\gamma}$ given in (1.2) which is included in the right half plane. We may rewrite the conditions (1.15) and (1.16) in the form

$$p \prec P_{\beta,\gamma} \tag{1.18}$$

where the functions $P_{\beta,\gamma}$ are given by (1.3).

By virtue of (1.15), (1.16) and by the properties of the domain $P_{\beta,\gamma}$ we have, respectively,

$$\mathrm{Re}\left\{ 1 + \frac{z(D^{n,q,s}_{\lambda,l} f(z))''}{(D^{n,q,s}_{\lambda,l} f(z))'} \right\} > \frac{\beta + \gamma}{1 + \beta} \quad (z \in U), \tag{1.19}$$

and

$$\mathrm{Re}\left\{ \frac{z(D^{n,q,s}_{\lambda,l} f(z))'}{D^{n,q,s}_{\lambda,l} f(z)} \right\} > \frac{\beta + \gamma}{1 + \beta} \quad (z \in U), \tag{1.20}$$

which means that

$$f(z) \in \mathrm{UCV}^{n,q,s}_{\lambda,l}(\alpha_1, \beta_1; \beta, \gamma) \Rightarrow D^{n,q,s}_{\lambda,l} f(z) \in \mathrm{CV}\left(\frac{\beta + \gamma}{1 + \beta}\right) \subseteq \mathrm{CV}, \tag{1.21}$$

and

$$f(z) \in \mathrm{SP}_{\lambda,l}^{n,q,s}(\alpha_1, \beta_1; \beta, \gamma) \Rightarrow D_{\lambda,l}^{n,q,s} f(z) \in \mathrm{ST}\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq \mathrm{ST}\,.$$

We note that:

(i) $\mathrm{SP}_{\lambda,0}^{n,q,s}(\alpha_1, \beta_1; \beta, \gamma) = \mathrm{SP}^{q,s}(n, \lambda; \beta, \gamma)$ (see Srivastava et al. [38]);

(ii) $\mathrm{SP}_{\lambda,0}^{n,2,1}(2, 1; 2-\alpha; \beta, \gamma) = \mathrm{SP}_{\alpha,\lambda}^{n}(\beta, \gamma)$ (Al-Oboudi and Al-Amoudi [3] and Aouf and Mostafa [4]);

(iii) $\mathrm{SP}_{0,0}^{1,q,s}(\alpha_1, \beta_1; \beta, \gamma) = \mathrm{SP}^{q,s}(\alpha_1; \beta, \gamma)$ (see Aouf and Murugusundarmoorthy [6]);

(iv) $\mathrm{SP}_{0,0}^{1,2,1}(a, 1; c; \beta, \gamma) = \mathrm{SP}(a, c; \beta, \gamma)$ $(a > 0, c > 0)$ (see Murugusungaramoorthy and Magesh [24] and Frasin [14]);

(v) $\mathrm{SP}_{0,0}^{1,2,1}(\delta + 1, 1; 1; \beta, \gamma) = \mathrm{SP}(\delta; \beta, \gamma)$ $(\delta > -1)$ (see Rosy et al. [31]);

(vi) $\mathrm{SP}_{0,0}^{1,2,1}(2, 1; 2 - \mu; \beta, \gamma) = \mathrm{SP}(\mu; \beta, \gamma)$ $(0 \leqslant \mu \leqslant 1)$ (see Srivastava and Mishra [37]);

(vii) $\mathrm{SP}_{1,0}^{n,2,1}(1, 1; 1; \beta, \gamma) = \mathrm{SP}(n; \beta, \gamma)$ (see Rosy and Murugusungaramoorthy [30]).

Also, we note that:

(i)

$$\begin{aligned} &\mathrm{SP}_{0,0}^{1,2,1}([2, 1; m+1]; \beta, \gamma) = \mathrm{SP}(m; \beta, \gamma) \\ &= \Big\{ f(z) \in A\colon\ \mathrm{Re}\Big\{ \frac{z(I_m f(z))'}{I_m f(z)} - \gamma \Big\} > \beta \Big| \frac{z(I_m f(z))'}{I_m f(z)} - 1 \Big|, \\ &\quad -1 \leqslant \gamma < 1,\ \beta \geqslant 0,\ m > -1, z \in U \Big\}, \end{aligned}$$

where I_m is the Noor operator considered by Noor and Noor [25];

(ii)

$$\begin{aligned} &\mathrm{SP}_{0,0}^{1,2,1}(\mu, 1; \delta+1; \beta, \gamma) = \mathrm{SP}(\mu, \delta; \beta, \gamma) \\ &= \Big\{ f(z) \in A\colon\ \mathrm{Re}\Big\{ \frac{z(I_{\delta,\mu} f(z))'}{I_{\delta,\mu} f(z)} - \gamma \Big\} > \beta \Big| \frac{z(I_{\delta,\mu} f(z))'}{I_{\delta,\mu} f(z)} - 1 \Big|, \\ &\quad -1 \leqslant \gamma < 1,\ \beta \geqslant 0,\ \delta > -1, \mu > 0, z \in U \Big\}, \end{aligned}$$

where $I_{\delta,\mu}$ is the Choi-Saigo-Srivastava operator [12];

(iii)

$$\begin{aligned} &\mathrm{SP}_{0,0}^{1,2,1}(\nu+1, 1; \nu+2; \beta, \gamma) = \mathrm{SP}(\nu; \beta, \gamma) \\ &= \Big\{ f(z) \in A\colon\ \mathrm{Re}\Big\{ \frac{z(J_\nu f(z))'}{J_\nu f(z)} - \gamma \Big\} > \beta \Big| \frac{z(J_\nu f(z))'}{J_\nu f(z)} - 1 \Big|, \\ &\quad -1 \leqslant \gamma < 1,\ \beta \geqslant 0,\ \nu > -1,\ z \in U \Big\}, \end{aligned}$$

where $J_\nu (\nu > -1)$ is the generalized Bernardi-Libera-Livingston operator ([9], [19] and [20]);

(iv)

$$\begin{aligned}&\mathrm{SP}^{n,2,1}_{\lambda,l}(\mu,1;\delta+1;\beta,\gamma)=\mathrm{SP}^{n}_{\lambda,l}(\mu,\delta+1;\beta,\gamma)\\&\quad=\Big\{f(z)\in A\colon\ \mathrm{Re}\Big\{\frac{z(I^{n,\mu,\delta}_{\lambda,l}f(z))'}{I^{n,\mu,\delta}_{\lambda,l}f(z)}-\gamma\Big\}>\beta\Big|\frac{z(I^{n,\mu,\delta}_{\lambda,l}f(z))'}{I^{n,\mu,\delta}_{\lambda,l}f(z)}-1\Big|,\\&\qquad-1\leqslant\gamma<1,\ \beta\geqslant 0,\ \lambda\geqslant 0,\ l\geqslant 0,\ \mu>0,\ \delta>-1,\ n\in\mathbb{N}_0,\ z\in U\Big\},\end{aligned}$$

where

$$I^{n,\mu,\delta}_{\lambda,l}f(z)=z+\sum_{k=2}^{\infty}\Big[\frac{l+1+\lambda(k-1)}{l+1}\frac{(\mu)_{k-1}}{(\delta+1)_{k-1}}\Big]^n a_k z^k;$$

(v)

$$\begin{aligned}&\mathrm{SP}^{n,2,1}_{\lambda,l}(\delta+1,a,c;\beta,\gamma)=\mathrm{SP}^{n}_{\lambda,l}(\delta+1,a,c;\beta,\gamma)\\&\quad=\Big\{f(z)\in A\colon\ \mathrm{Re}\Big\{\frac{z(I^{n,\delta}_{\lambda,l}(a,c)f(z))'}{I^{n,\delta}_{\lambda,l}(a,c)f(z)}-\gamma\Big\}>\beta\Big|\frac{z(I^{n,\delta}_{\lambda,l}(a,c)f(z))'}{I^{n,\delta}_{\lambda,l}(a,c)f(z)}-1\Big|,\\&\qquad-1\leqslant\gamma<1,\ \beta\geqslant 0,\ \lambda\geqslant 0,\ l\geqslant 0,\ \delta>0,\ a>0,\ c>0,\ n\in\mathbb{N}_0,\ z\in U\Big\},\end{aligned}$$

where

$$I^{n,\delta}_{\lambda,l}(a,c)f(z)=z+\sum_{k=2}^{\infty}\Big[\frac{l+1+\lambda(k-1)}{l+1}\frac{(a)_{k-1}(\delta+1)_{k-1}}{(c)_{k-1}}\Big]^n a_k z^k;$$

(vi)

$$\begin{aligned}&\mathrm{SP}^{n,2,1}_{\lambda,l}([2,1;m+1];\beta,\gamma)=\mathrm{SP}^{n}_{\lambda,l}(2,m+1;\beta,\gamma)\\&\quad=\Big\{f(z)\in A\colon\ \mathrm{Re}\Big\{\frac{z(I^{n,m}_{\lambda,l}f(z))'}{I^{n,m}_{\lambda,l}f(z)}-\gamma\Big\}>\beta\Big|\frac{z(I^{n,m}_{\lambda,l}f(z))'}{I^{n,m}_{\lambda,l}f(z)}-1\Big|,\\&\qquad-1\leqslant\gamma<1,\ \beta\geqslant 0,\ \lambda\geqslant 0,\ l\geqslant 0,\ m>-1,\ n\in N_0,z\in U\Big\},\end{aligned}$$

where

$$I^{n,m}_{\lambda,l}f(z)=z+\sum_{k=2}^{\infty}\Big[\frac{l+1+\lambda(k-1)}{l+1}\frac{(2)_{k-1}}{(m+1)_{k-1}}\Big]^n a_k z^k;$$

(vii)

$$\begin{aligned}&\mathrm{SP}^{n,2,1}_{\lambda,l}(\nu+1,1;\nu+2;\beta,\gamma)=\mathrm{SP}^{n}_{\lambda,l}(\nu;\beta,\gamma)\\&\quad=\Big\{f(z)\in A\colon\ \mathrm{Re}\Big\{\frac{z(J^{n,\nu}_{\lambda,l}f(z))'}{J^{n,\nu}_{\lambda,l}f(z)}-\gamma\Big\}>\beta\Big|\frac{z(J^{n,\nu}_{\lambda,l}f(z))'}{J^{n,\nu}_{\lambda,l}f(z)}-1\Big|,\\&\qquad-1\leqslant\gamma<1,\ \beta\geqslant 0,\ \lambda\geqslant 0,\ l\geqslant 0,\ \nu>-1,\ n\in\mathbb{N}_0,\ z\in U\Big\},\end{aligned}$$

where

$$J_{\lambda,l}^{n,\nu} f(z) = z + \sum_{k=2}^{\infty} \left[\frac{l+1+\lambda(k-1)}{l+1}\frac{\nu+1}{\nu+k}\right]^n a_k z^k.$$

In this paper we establish subordination results for the associated classes $\hat{\mathrm{SP}}_{\lambda,l}^{n,q,s}(\alpha_1,\beta_1;\beta,\gamma)$ and $\tilde{U}\,\mathrm{CV}_{\lambda,l}^{n,q,s}(\alpha_1,\beta_1;\beta,\gamma)$ in U whose coefficients satisfy (2.9) and (2.10), respectively (defined below). Some consequences of the subordination results are mentioned in the concluding section.

Before we state and prove our main results we need the following definitions and lemmas.

Definition 1. Let $g(z)$ be analytic and univalent in U. If $f(z)$ is analytic in U, $f(0) = g(0)$ and $f(U) \subset g(U)$, then we say that the function $f(z)$ is *subordinate* to $g(z)$ in U and we write $f(z) \prec g(z)$, see [8], [22] and [23].

Definition 2. A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called a *subordinating factor sequence* if, whenever $f(z)$of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by

$$\sum_{k=2}^{\infty} a_k b_k z^k \prec f(z) \quad (z \in U, a_1 = 1). \tag{1.22}$$

Lemma 1 [39]**.** *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\mathrm{Re}\left[1 + 2\sum_{k=1}^{\infty} b_k z^k\right] > 0 \quad (z \in U). \tag{1.23}$$

Lemma 2 [Rogosinski's Theorem, 27]**.** *Let $h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ be subordinate to $H(z) = 1 + \sum_{k=1}^{\infty} C_k z^k$ in U. If $H(z)$ is univalent in U and $H(U)$ is convex, then*

$$|c_k| \leqslant |C_1|, \quad k \geqslant 1.$$

2. Coefficient bounds

Unless otherwise mentioned we shall assume throughout the paper that $-1 \leqslant \gamma < 1$, $\beta \geqslant 0$, $\beta + \gamma \geqslant 0$, $q \leqslant s+1$; $q, s \in \mathbb{N}_0$, $\lambda \geqslant 0$, $l \geqslant 0$, $n \in \mathbb{N}_0$, $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ are positive and real.

In this section we give bounds for the coefficients of series expansions of functions belonging to the classes $\mathrm{SP}_{\lambda,l}^{n,q,s}(\alpha_1, \beta_1; \beta, \gamma)$ and $\mathrm{UCV}_{\lambda,l}^{n,q,s}(\alpha_1, \beta_1; \beta, \gamma)$, and sufficient conditions for a function to belong to these classes.

Taking into account the fundamental relation

$$P_{\beta,\gamma}(z) = \frac{z(D_{\lambda,l}^{n,q,s} f_{\beta,\gamma}(z))'}{D_{\lambda,l}^{n,q,s} f_{\beta,\gamma}(z)} \tag{2.1}$$

between the extremal functions in the class $\varrho(P_{\beta,\gamma})$ and the extremal functions $f_{\beta,\gamma}$ of the class $\mathrm{SP}_{\lambda,l}^{n,q,s}(\alpha_1, \beta_1; \beta, \gamma)$ and in view of (1.12) and (2.1), we have for $P_{\beta,\gamma}(z) = 1 + P_1 z + \ldots$ which is defined in (1.3) and for

$$f_{\beta,\gamma}(z) = z + A_2 z^2 + \ldots \tag{2.2}$$

a coefficient relation

$$(k-1) A_k \Phi_{k,n}(\alpha_1, \lambda, l) = \sum_{j=1}^{k-1} P_{k-j} A_j \Phi_{j,n}(\alpha_1, \lambda, l), \qquad A_1 = 1.$$

In particular, by a straightforward computation we obtain

$$A_2 = \frac{1}{\Phi_{2,n}(\alpha_1, \lambda, l)} P_1; \tag{2.3}$$

observe also that the coefficients A_k are nonnegative, since $\Phi_{k,n}(\alpha_1, \lambda, l) > 0$ and P_k are nonnegative.

As simple consequences of the above and the result given in [17], we give the sharp bound on the second coefficient for functions of the class $\mathrm{SP}_{\lambda,l}^{n,q,s}(\alpha_1, \beta_1; \beta, \gamma)$.

Theorem 1. *If a function $f(z)$ of the form (1.1) is in $\mathrm{SP}_{\lambda,l}^{n,q,s}(\alpha_1, \beta_1; \beta, \gamma)$, then*

$$|a_2| \leqslant A_2.$$

Theorem 2. *If a function $f(z)$ of the form (1.1) is in $\mathrm{SP}_{\lambda,l}^{n,q,s}(\alpha_1,\beta_1;\beta,\gamma)$, then*

$$|a_k| \leqslant \frac{1}{\Phi_{k,n}(\alpha_1,\lambda,l)}\frac{(P_1)_{k-1}}{(1)_{k-1}}, \qquad k \geqslant 2, \tag{2.4}$$

where

$$P_1 = P_1(\beta,\gamma) = \begin{cases} \dfrac{8(1-\gamma)(\cos^{-1}\beta)^2}{\pi^2(1-\beta^2)}, & 0 \leqslant \beta < 1, \\ \dfrac{8}{\pi^2}(1-\gamma), & \beta = 1, \\ \dfrac{\pi^2(1-\gamma)}{4\sqrt{t}(\beta^2-1)K^2(t)(1+t)}, & \beta > 1. \end{cases} \tag{2.5}$$

P r o o f. Let $f(z)$ defined by (1.1) be in the class $\mathrm{SP}_{\lambda,l}^{n,q,s}(\alpha_1,\beta_1;\beta,\gamma)$. By (1.18), we obtain

$$\frac{z(D_{\lambda,l}^{n,q,s}f(z))'}{D_{\lambda,l}^{n,q,s}f(z)} \prec P_{\beta,\gamma}(z).$$

Define

$$q(z) = \frac{z(D_{\lambda,l}^{n,q,s}f(z))'}{D_{\lambda,l}^{n,q,s}f(z)} = 1 + \sum_{k=1}^{\infty} c_k z^k.$$

The function $P_{\beta,\gamma}$ is univalent in U and $P_{\beta,\gamma}(U)$, the conic domain, is a convex domain. Then by using Lemma 2 we have

$$|c_k| \leqslant P_1, \quad k \geqslant 1, \tag{2.6}$$

where $P_1 = P_1(\beta,\gamma)$ is given by (2.5). Now writing

$$z(D_{\lambda,l}^{n,q,s}f(z))' = q(z)(D_{\lambda,l}^{n,q,s}f(z))$$

and comparing the coefficients at z^k on both sides, we get

$$(k-1)a_k\Phi_{k,n}(\alpha_1,\lambda,l) = \sum_{j=1}^{k-1} c_{k-j}a_j\Phi_{j,n}(\alpha_1,\lambda,l), \qquad a_1 = 1. \tag{2.7}$$

From (2.7) we get

$$|a_2| = \frac{1}{\Phi_{2,n}(\alpha_1,\lambda,l)}|c_1| \leqslant \frac{P_1}{\Phi_{2,n}(\alpha_1,\lambda,l)}.$$

So the result is true for $k = 2$. Let $k \geqslant 2$ and assume that the inequality (2.4) is true for all $j \leqslant k-1$. By using (2.6), (2.7) and applying the induction hypothesis to $|a_j|$ we get

$$
\begin{aligned}
|a_k| &\leqslant \frac{1}{(k-1)\Phi_{k,n}(\alpha_1,\lambda,l)}\left[|c_1| + \sum_{j=2}^{k-1} |c_{k-j}|\,|a_j|\,\Phi_{j,n}(\alpha_1,\lambda,l)\right] \\
&\leqslant \frac{P_1}{(k-1)\Phi_{k,n}(\alpha_1,\lambda,l)}\left[1 + \sum_{j=2}^{k-1} \frac{(P_1)_{j-1}}{(1)_{j-1}}\right].
\end{aligned}
$$

By applying mathematical induction one more time, we find that

$$
1 + \sum_{j=2}^{k-1} \frac{(P_1)_{j-1}}{(1)_{j-1}} = \frac{(1+P_1)(2+P_2)\ldots((k-2)+P_1)}{(k-2)!}.
$$

Thus we get the inequality (2.4).

Applying (1.17), we observe that the extremal function of $\mathrm{UCV}_{\lambda,l}^{n,q,s}(\alpha_1,\beta_1;\beta,\gamma)$, denoted by $F_{\beta,\gamma}(z)$, is given by

$$
F_{\beta,\gamma}(z) = \int_0^z \frac{f_{\beta,\gamma}(\xi)}{\xi}\,\mathrm{d}\xi, \tag{2.8}
$$

where $f_{\beta,\gamma}(z)$ is given by (2.2). By (2.3), for

$$
F_{\beta,\gamma}(z) = z + B_2 z^2 + \ldots
$$

we get

$$
B_2 = \frac{1}{2\Phi_{2,n}(\alpha_1,\lambda,l)} P_1.
$$

Applying relation (1.17), we can prove the following two corollaries.

Corollary 1. *If a function $f(z)$ of the form (1.1) is in $\mathrm{UCV}_{\lambda,l}^{n,q,s}(\alpha_1,\beta_1;\beta,\gamma)$, then*

$$
|a_2| \leqslant B_2.
$$

Corollary 2. *If a function $f(z)$ of the form* (1.1) *is in* $\mathrm{UCV}^{n,q,s}_{\lambda,l}(\alpha_1,\beta_1;\beta,\gamma)$, *then*

$$|a_k| \leqslant \frac{1}{\Phi_{k,n}(\alpha_1,\lambda,l)}\frac{(P_1)_{k-1}}{(1)_k},$$

where $P_1 = P_1(\beta,\gamma)$ is given by (2.5).

R e m a r k 1. The results of Theorem 2 and Corollary 1 are sharp for $k = 2$ or $\beta = 0$.

R e m a r k 2. Putting $q = 2$, $s = 1$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 2-\alpha$ ($\alpha \neq 2,3,4,\ldots$) and $l = 0$ in Theorem 2, we obtain the result obtained by Al-Oboudi and Al-Amoudi [3, Theorem 4].

Now we obtain a sufficient condition for f to be in the class $\mathrm{SP}^{n,q,s}_{\lambda,l}(\alpha_1,\beta_1;\beta,\gamma)$.

Theorem 3. *A function $f(z)$ of the form* (1.1) *is in* $\mathrm{SP}^{n,q,s}_{\lambda,l}(\alpha_1,\beta_1;\beta,\gamma)$ *if*

$$\sum_{k=2}^{\infty}[k(1+\beta)-(\beta+\gamma)]\,\Phi_{k,n}(\alpha_1,\lambda,l)\,|a_k| \leqslant 1-\gamma, \tag{2.9}$$

where $\Phi_{k,n}(\alpha_1,\lambda,l)$ is defined by (1.13).

P r o o f. It suffices to show that

$$\beta\left|\frac{z(D^{n,q,s}_{\lambda,l}f(z))'}{D^{n,q,s}_{\lambda,l}f(z)}-1\right| - \mathrm{Re}\left\{\frac{z(D^{n,q,s}_{\lambda,l}f(z))'}{D^{n,q,s}_{\lambda,l}f(z)}-1\right\} < 1-\gamma.$$

We have

$$\begin{aligned}
&\beta\left|\frac{z(D^{n,q,s}_{\lambda,l}f(z))'}{D^{n,q,s}_{\lambda,l}f(z)}-1\right| - \mathrm{Re}\left\{\frac{z(D^{n,q,s}_{\lambda,l}f(z))'}{D^{n,q,s}_{\lambda,l}f(z)}-1\right\}\\
&\quad\leqslant (1+\beta)\left|\frac{z(D^{n,q,s}_{\lambda,l}f(z))'}{D^{n,q,s}_{\lambda,l}f(z)}-1\right|\\
&\quad\leqslant \frac{(1+\beta)\sum\limits_{k=2}^{\infty}(k-1)\Phi_{k,n}(\alpha_1,\lambda,l)|a_k||z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}\Phi_{k,n}(\alpha_1,\lambda,l)|a_k||z|^{k-1}}\\
&\quad< \frac{(1+\beta)\sum\limits_{k=2}^{\infty}(k-1)\Phi_{k,n}(\alpha_1,\lambda,l)\,|a_k|}{1-\sum\limits_{k=2}^{\infty}\Phi_{k,n}(\alpha_1,\lambda,l)\,|a_k|}.
\end{aligned}$$

The last expression is bounded above by $(1-\gamma)$ if (2.9) is satisfied.

By virtue of (1.17) and Theorem 3 we have

Corollary 3. *A function $f(z)$ of the form* (1.1) *is in* $\mathrm{UCV}^{n,q,s}_{\lambda,l}(\alpha_1,\beta_1;\beta,\gamma)$ *if*

$$\sum_{k=2}^{\infty} k\,[k(1+\beta)-(\beta+\gamma)]\,\Phi_{k,n}(\alpha_1,\lambda,l)\,|a_k| \leqslant 1-\gamma. \tag{2.10}$$

Remark 3. Putting $q=2$, $s=1$, $\alpha_1=2$, $\alpha_2=1$, $\beta_1=2-\alpha$ ($\alpha\neq 2,3,4,\ldots$) and $l=0$ in Theorem 3 and Corollary 3 we obtain the results obtained by Al-Oboudi and Al-Amoudi [3, Theorem 3.3 and Corollary 3.3, respectively].

3. Subordination results

Employing the techniques used earlier by Attiya [7], Srivastava and Attiya [35], and Singh [34], we state and prove the following theorem.

Theorem 4. *Let $f(z)\in A$ defined by* (1.1) *be in the class* $\widehat{\mathrm{SP}}^{n,q,s}_{\lambda,l}(\alpha_1,\beta_1;\beta,\gamma)$. *Then*

$$\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{2\,\{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)\}}(f*\varphi)(z)\prec\varphi(z) \qquad (z\in U;\varphi\in \mathrm{CV}), \tag{3.1}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)}{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)} \qquad (z\in U). \tag{3.2}$$

The constant $\frac{[2(1+\beta)-(\beta+\gamma)]\Phi_{2,n}(\alpha_1,\lambda,l)}{2\{[2(1+\beta)-(\beta+\gamma)]\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)\}}$ *is the best estimate.*

Proof. Let $f(z)\in\widehat{\mathrm{SP}}^{n,q,s}_{\lambda,l}(\alpha_1,\beta_1;\beta,\gamma)$, and let $\varphi(z)=z+\sum\limits_{k=2}^{\infty}c_kz^k$ be any function in the class CV. Then

$$\begin{aligned}\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{2\,\{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)\}}&(f*\varphi)(z)\\ =\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{2\,\{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)\}}&\Bigl(z+\sum_{k=2}^{\infty}a_kc_kz^k\Bigr).\end{aligned} \tag{3.3}$$

Thus, by Definition 2, the assertion of the theorem will hold if the sequence

$$\Bigl\{\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{2\,\{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)\}}a_k\Bigr\}_{k=1}^{\infty} \tag{3.4}$$

is a subordination factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to

$$\text{(3.5)}\quad \mathrm{Re}\left\{1+\sum_{k=1}^{\infty}\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)}a_k z^k\right\}>0 \quad (z\in U).$$

Now

$$\begin{aligned}
&\mathrm{Re}\left\{1+\sum_{k=1}^{\infty}\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)}a_k z^k\right\}\\
&\quad=\mathrm{Re}\left\{1+\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)}z\right.\\
&\qquad+\frac{1}{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)}\\
&\qquad\qquad\left.\times\sum_{k=2}^{\infty}[2(1+\beta)-(\beta+\gamma)]\,\Phi_{k,n}(\alpha_1,\lambda,l)a_k z^k\right\}\\
&\quad\geqslant 1-\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)}r\\
&\qquad-\frac{1}{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)}\\
&\qquad\qquad\times\sum_{k=2}^{\infty}[2(1+\beta)-(\beta+\gamma)]\,\Phi_{k,n}(\alpha_1,\lambda,l)\,|a_k|\,r^k\\
&\quad> 1-\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)}r\\
&\qquad-\frac{(1-\gamma)}{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)}r\\
&\quad=1-r>0 \quad (|z|=r).
\end{aligned}$$

Thus (3.5) holds true in U. The proof of (3.1), (3.2) follows by taking $\varphi(z) = z/(1-z)$ in (3.1).

Now we consider the function $f_0(z) \in \widehat{\mathrm{SP}}^{n,q,s}_{\lambda,l}(\alpha_1,\beta_1,\beta,\gamma)$ given by

$$\text{(3.6)}\qquad f_0(z)=z-\frac{(1-\gamma)}{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}z^2 \quad (-1\leqslant\gamma<1;\ \beta\geqslant 0)$$

which is a member of the class $\widehat{\mathrm{SP}}^{n,q,s}_{\lambda,l}(\alpha_1,\beta_1;\beta,\gamma)$. By using (3.1), we have

$$\text{(3.7)}\qquad \frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{2\left\{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)\right\}}f_0(z)\prec\frac{z}{1-z}.$$

It can be easily verified that

(3.8)
$$\min_{|z|\leqslant 1} \operatorname{Re}\Big[\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{2\,\{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)\}} f_0(z)\Big] = -\frac{1}{2} \quad (z\in U),$$

hence the constant $\frac{[2(1+\beta)-(\beta+\gamma)]\Phi_{2,n}(\alpha_1,\lambda,l)}{2\{[2(1+\beta)-(\beta+\gamma)]\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)\}}$ is the best possible.

This completes the proof of Theorem 4.

Remark 4. (i) Putting $q=2$, $s=1$, $\alpha_1=2$, $\alpha_2=1$, $\beta_1=2-\alpha$ ($\alpha\neq 2,3,4,\ldots$) and $l=0$ in Theorem 4, we obtain the result obtained by Aouf and Mostafa [5, Theorem 2.4].

(ii) Putting $q=2$, $s=1$, $\alpha_1=a$ ($a>0$), $\alpha_2=1$, $\beta_1=c$ ($c>0$) and $l=0$ in Theorem 4, we obtain the result obtained by Prajapat and Raina [26 Theorem 1].

(iii) Putting $q=2$, $s=1$, $\alpha_1=a$ ($a>0$), $\alpha_2=1$, $\beta_1=c$ ($c>0$) and $l=\lambda=0$ in Theorem 4, we obtain the result obtained by Frasin [14, Theorem 2.1].

Similarly, by using (1.17) and Theorem 4 we can prove the following corollary.

Corollary 4. *Let $f(z)\in A$ defined by* (1.1) *be in the class* $\widehat{\mathrm{UCV}}^{n,q,s}_{\lambda,l}(\alpha_1,\beta_1;\beta,\gamma)$. *Then*

(3.9)
$$\frac{[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)}{\{2\,[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)\}}(f*\varphi)(z)\prec\varphi(z) \qquad (z\in U;\varphi\in \mathrm{CV}),$$

and

(3.10)
$$\operatorname{Re}\{f(z)\} > -\frac{2\,[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)}{2\,[2(1+\beta)-(\beta+\gamma)]\,\Phi_{2,n}(\alpha_1,\lambda,l)} \quad (z\in U).$$

The constant $\frac{[2(1+\beta)-(\beta+\gamma)]\Phi_{2,n}(\alpha_1,\lambda,l)}{\{2[2(1+\beta)-(\beta+\gamma)]\Phi_{2,n}(\alpha_1,\lambda,l)+(1-\gamma)\}}$ *is the best estimate.*

Also, we establish subordination results for the associated subclasses $\widehat{\mathrm{SP}}(m;\beta,\gamma)$, $\widehat{\mathrm{SP}}(\mu,\delta;\beta,\gamma)$, $\widehat{\mathrm{SP}}(\nu;\beta,\gamma)$, $\widehat{\mathrm{SP}}^n_{\lambda,l}(\mu,\delta+1;\beta,\gamma)$, $\widehat{\mathrm{SP}}^n_{\lambda,l}(\delta+1,a,c;\beta,\gamma)$, $\widehat{\mathrm{SP}}^n_{\lambda,l}(2,m+1;\beta,\gamma)$ and $\widehat{\mathrm{SP}}^n_{\lambda,l}(\nu;\beta,\gamma)$ whose coefficients satisfy (2.9) or (2.10), in the special cases as mentioned above.

Putting $\lambda=l=0$, $n=1$, $q=2$, $s=1$, $\alpha_1=2$, $\alpha_2=1$ and $\beta_1=m+1$ ($m>-1$) in Theorem 4, we obtain the following corollary.

Corollary 5. *Let* $f(z) \in A$ *defined by* (1.1) *be in the class* $\widehat{\mathrm{SP}}(m;\beta,\gamma)$. *Then*

$$\frac{[2(1+\beta)-(\beta+\gamma)]\, I_m f(z)}{2\left\{[2(1+\beta)-(\beta+\gamma)]\, I_m f(z)+(1-\gamma)\right\}}(f*\varphi)(z) \prec \varphi(z) \ (z\in U; \varphi \in \mathrm{CV}), \tag{3.11}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{[2(1+\beta)-(\beta+\gamma)]\, I_m f(z)+(1-\gamma)}{[2(1+\beta)-(\beta+\gamma)]\, I_m f(z)} \ (z\in U).$$

The constant $\frac{[2(1+\beta)-(\beta+\gamma)]I_m f(z)}{2\{[2(1+\beta)-(\beta+\gamma)]I_m f(z)+(1-\gamma)\}}$ *is the best estimate.*

Putting $\lambda = l = 0$, $n = 1$, $q = 2$, $s = 1$, $\alpha_1 = \mu$ $(\mu > 0)$, $\alpha_2 = 1$ and $\beta_1 = \delta + 1$ $(\delta > -1)$ in Theorem 4, we obtain the following corollary.

Corollary 6. *Let* $f(z) \in A$ *defined by* (1.1) *be in the class* $\widehat{\mathrm{SP}}$ $(\mu,\delta;\beta,\gamma)$. *Then*

$$\frac{[2(1+\beta)-(\beta+\gamma)]\, I_{\delta,\mu} f(z)}{2\left\{[2(1+\beta)-(\beta+\gamma)]\, I_{\delta,\mu} f(z)+(1-\gamma)\right\}} (f*\varphi)(z) \prec \varphi(z) \quad (z\in U; \varphi \in \mathrm{CV}), \tag{3.12}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{[2(1+\beta)-(\beta+\gamma)]\, I_{\delta,\mu} f(z)+(1-\gamma)}{[2(1+\beta)-(\beta+\gamma)]\, I_{\delta,\mu} f(z)} \ (z\in U).$$

The constant $\frac{[2(1+\beta)-(\beta+\gamma)]I_{\delta,\mu} f(z)}{2\{[2(1+\beta)-(\beta+\gamma)]I_{\delta,\mu} f(z)+(1-\gamma)\}}$ *is the best estimate.*

Putting $\lambda = l = 0$, $n = 1$, $q = 2$, $s = 1$, $\alpha_1 = \nu + 1$, $\alpha_2 = 1$ and $\beta_1 = \nu + 2$ $(\nu > -1)$ in Theorem 4, we obtain the following corollary.

Corollary 7. *Let* $f(z) \in A$ *defined by* (1.1) *be in the class* $\widehat{\mathrm{SP}}(\nu;\beta,\gamma)$. *Then*

$$\frac{[2(1+\beta)-(\beta+\gamma)]\, J_\nu f(z)}{2\left\{[2(1+\beta)-(\beta+\gamma)]\, J_\nu f(z)+(1-\gamma)\right\}} (f*\varphi)(z) \prec \varphi(z) \ (z\in U; \varphi \in \mathrm{CV}), \tag{3.13}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{[2(1+\beta)-(\beta+\gamma)]\, J_\nu f(z)+(1-\gamma)}{[2(1+\beta)-(\beta+\gamma)]\, J_\nu f(z)} \ (z\in U).$$

The constant $\frac{[2(1+\beta)-(\beta+\gamma)]J_\nu f(z)}{2\{[2(1+\beta)-(\beta+\gamma)]J_\nu f(z)+(1-\gamma)\}}$ *is the best estimate.*

Putting $q = 2$, $s = 1$, $\alpha_1 = \mu$ $(\mu > 0)$, $\alpha_2 = 1$ and $\beta_1 = \delta + 1$ $(\delta > -1)$ in Theorem 4, we obtain the following corollary.

Corollary 8. *Let* $f(z) \in A$ *defined by* (1.1) *be in the class* $\widehat{\mathrm{SP}}^{n}_{\lambda,l}$ $(\mu, \delta+1; \beta, \gamma)$. *Then*

$$\frac{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{\mu}{\delta+1}]^n}{2\{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{\mu}{\delta+1}]^n+(1-\gamma)\}}(f*\varphi)(z) \prec \varphi(z) \quad (z \in U; \varphi \in \mathrm{CV}), \tag{3.14}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{\mu}{\delta+1}]^n+(1-\gamma)}{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{\mu}{\delta+1}]^n} \quad (z \in U).$$

The constant $\frac{[2(1+\beta)-(\beta+\gamma)]\left[\frac{l+1+\lambda}{l+1}\frac{\mu}{\delta+1}\right]^n}{2\left\{[2(1+\beta)-(\beta+\gamma)]\left[\frac{l+1+\lambda}{l+1}\frac{\mu}{\delta+1}\right]^n+(1-\gamma)\right\}}$ *is the best estimate.*

Putting $q=2$, $s=1$, $\alpha_1=\delta+1$ $(\delta>0)$, $\alpha_2=a$ $(a>0)$ and $\beta_1=c$ $(c>0)$ in Theorem 4, we obtain the following corollary.

Corollary 9. *Let* $f(z) \in A$ *defined by* (1.1) *be in the class* $\widehat{\mathrm{SP}}^{n}_{\lambda,l}$ $(\delta+1, a, c; \beta, \gamma)$. *Then*

$$\frac{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{a(\delta+1)}{c}]^n}{2\{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{a(\delta+1)}{c}]^n+(1-\gamma)\}}(f*\varphi)(z) \prec \varphi(z) \quad (z \in U; \varphi \in \mathrm{CV}), \tag{3.15}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{a(\delta+1)}{c}]^n+(1-\gamma)}{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{a(\delta+1)}{c}]^n} \quad (z \in U).$$

The constant $\frac{[2(1+\beta)-(\beta+\gamma)]\left[\frac{l+1+\lambda}{l+1}\frac{a(\delta+1)}{c}\right]^n}{2\left\{[2(1+\beta)-(\beta+\gamma)]\left[\frac{l+1+\lambda}{l+1}\frac{a(\delta+1)}{c}\right]^n+(1-\gamma)\right\}}$ *is the best estimate.*

Putting $q=2$, $s=1$, $\alpha_1=2$, $\alpha_2=1$ and $\beta_1=m+1$ $(m>-1)$ in Theorem 4, we obtain the following corollary.

Corollary 10. *Let* $f(z) \in A$ *defined by* (1.1) *be in the class* $\widehat{\mathrm{SP}}^{n}_{\lambda,l}$ $(2, m+1; \beta, \gamma)$. *Then*

$$\frac{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{2}{m+1}]^n}{2\{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{2}{m+1}]^n+(1-\gamma)\}}(f*\varphi)(z) \prec \varphi(z) \quad (z \in U; \varphi \in \mathrm{CV}), \tag{3.16}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{2}{m+1}]^n+(1-\gamma)}{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{2}{m+1}]^n} \quad (z \in U).$$

The constant $\frac{[2(1+\beta)-(\beta+\gamma)]\left[\frac{l+1+\lambda}{l+1}\frac{2}{m+1}\right]^n}{2\left\{[2(1+\beta)-(\beta+\gamma)]\left[\frac{l+1+\lambda}{l+1}\frac{2}{m+1}\right]^n+(1-\gamma)\right\}}$ *is the best estimate.*

Putting $q=2$, $s=1$, $\alpha_1=\nu+1$, $\alpha_2=1$ and $\beta_1=\nu+2$ $(\nu>-1)$ in Theorem 4, we obtain the following corollary.

Corollary 11. *Let* $f(z) \in A$ *defined by* (1.1) *be in the class* $\widehat{\mathrm{SP}}^n_{\lambda,l}\ (\nu;\beta,\gamma)$. *Then*

$$\frac{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{\nu+1}{\nu+2}]^n}{2\{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{\nu+1}{\nu+2}]^n+(1-\gamma)\}}(f*\varphi)(z) \prec \varphi(z) \quad (z\in U;\varphi\in \mathrm{CV}), \tag{3.17}$$

and

$$\mathrm{Re}\{f(z)\} > -\frac{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{\nu+1}{\nu+2}]^n+(1-\gamma)}{[2(1+\beta)-(\beta+\gamma)][\frac{l+1+\lambda}{l+1}\frac{\nu+1}{\nu+2}]^n} \quad (z \in U).$$

The constant $\frac{[2(1+\beta)-(\beta+\gamma)]\left[\frac{l+1+\lambda}{l+1}\frac{\nu+1}{\nu+2}\right]^n}{2\left\{[2(1+\beta)-(\beta+\gamma)]\left[\frac{l+1+\lambda}{l+1}\frac{\nu+1}{\nu+2}\right]^n+(1-\gamma)\right\}}$ *is the best estimate.*

Remark 5. (i) Putting $\beta=0$, $n=1$, $\lambda=l=0$, $q=2$, $s=1$ and $\alpha_1=\alpha_2=\beta_1=1$ in Theorem 4, we obtain the result obtained by Prajapat and Raina [26, Corollary 3].

(ii) Putting $\gamma=\beta=0$, $n=1$, $\lambda=l=0$, $q=2$, $s=1$ and $\alpha_1=\alpha_2=\beta_1=1$ in Theorem 4, we obtain the result obtained by Singh [33, Corollary 2.2].

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