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GENERALIZED LOGISTIC MODEL AND ITS ORTHANT TAIL DEPENDENCE

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The Multivariate Extreme Value distributions have shown their usefulness in environmental studies, financial and insurance mathematics. The Logistic or Gumbel–Hougaard distribution is one of the oldest multivariate extreme value models and it has been extended to asymmetric models. In this paper we introduce generalized logistic multivariate distributions. Our tools are mixtures of copulas and stable mixing variables, extending approaches in Tawn [14], Joe and Hu [6] and Fougères et al. [3]. The parametric family of multivariate extreme value distributions considered presents a flexible dependence structure and we compute for it the multivariate tail dependence coefficients considered in Li [7].

Keywords: multivariate extreme value distribution, tail dependence, logistic model, mixture

Classification: 60G70

1. INTRODUCTION

In multivariate extreme value theory, the main interest has been in building multivariate extreme value distributions. Joe [5] and McNeil et al. [9], among others, provide a source of multivariate dependence models turned into its copulas functions. Extreme value copulas arise naturally in the extreme value theory but they are also itself a suitable choice to model dependence structures (Capéraà et al. [1]).

Recently several authors proposed construction schemes of d-variate copulas and some of them can be seen as transformations of given copulas, enlarging the number of parameters and allowing tail asymmetries. The probabilistic representations of the transformations, not always available, allows the understanding and interpretation of the dependence structure and can suggest a sampling strategy for the new copulas.

In this paper we start from a q-variate random vector $\mathbf{S}$ which margins S_j are standard positive α-stable variables and q d-variate random vectors $\mathbf{X}_j$, $j = 1, \ldots, q$, that conditionally on each S_j have dependence structure regulated by given copulas C_j. We then study the componentwise maxima model from the $\mathbf{X}_j$'s. From this model we derive a new family of copulas and analyze its orthant tail dependence by computing the multivariate tail dependence coefficients considered in Li [7].

In section 2 we introduce the model and the resulting copula as the main result and discuss several special cases. It is shown how several copulas of the literature arise as special cases of the present construction. In section 3 we investigate how the proposed construction affects the multivariate tail dependence. It allows tail dependence and different dependence coefficients between each variable pair. Finally we apply the results to the particular case of C_j being the copula arising from the distribution of the variables in a M4 process (Smith and Weissman [13]).

2. THE MODEL

Let $\mathcal{L}(Z|W)$ denotes the conditional distribution of a random variable or vector Z given another random variable or vector W. For the vectors $\mathbf{X}_j = (X_{j,1}, \ldots, X_{j,d})$, $j = 1, \ldots, q$, and $\mathbf{S} = (S_1, \ldots, S_q)$, defined on the same probability space, we shall assume that:

(a) $\mathcal{L}((\mathbf{X}_1, \ldots, \mathbf{X}_q)|\mathbf{S}) = \prod_{j=1}^{q} \mathcal{L}(\mathbf{X}_j|\mathbf{S})$,

(b) $\mathcal{L}(\mathbf{X}_j|\mathbf{S}) = \mathcal{L}(\mathbf{X}_j|S_j)$,

(c) $P\left(\bigcap_{i=1}^{d} X_{ji} \le x_i|S_j\right) = C_j\left(e^{-\left(\frac{x_1}{\beta_{j1}}\right)^{-1/\alpha_j} S_j}, \ldots, e^{-\left(\frac{x_d}{\beta_{jd}}\right)^{-1/\alpha_j} S_j}\right)$, $x_j > 0$,

$j = 1, \ldots, q$, where C_j's are max-stable copulas and $\{\beta_{ji}, j = 1, \ldots, q, i = 1, \ldots, d\}$ are non-negative constants such that $\sum_{j=1}^{q} \beta_{ji} = 1$, $i = 1, \ldots, d$,

(d) $E\left(e^{-tS_j}\right) = e^{-t^{\alpha_j}}$, $t \ge 0$, $j = 1, \ldots, q$, where α_j's are constants in $(0, 1]$

and

(e) $\mathcal{L}(\mathbf{S}) = \prod_{j=1}^{q} \mathcal{L}(S_j)$.

Thus every X_{ji} is a scale mixture with mixing variable $\beta_{ji} S_j^{\alpha_j}$ and $\mathbf{X}_j$, $j = 1, \ldots, q$, are conditionally independent given $\mathbf{S}$.

Scale mixtures have been studied and used in a variety of applications (Marshall and Olkin [8], Joe and Hu [6] and Fougères et al. [3], Li [7]).

We shall consider here a componentwise maxima model from the $\mathbf{X}_j$'s and we present in the next result its distribution.

Theorem 2.1. If the random vectors $\mathbf{X}_j$, $j = 1, \ldots, q$, and $\mathbf{S}$ satisfy the conditions (a)–(e) then $\mathbf{Y} = (Y_1, \ldots, Y_d)$ defined by $Y_i = \bigvee_{j=1}^{q} X_{ji}$, $i = 1, \ldots, d$, has multivariate extreme value distribution with unit Fréchet margins and copula

$$C_{\mathbf{Y}}(u_1, \ldots, u_d) \quad (1)$$
$$= \exp\left\{-\sum_{j=1}^{q} \left(-\ln C_j\left(e^{-(-\beta_{j1} \ln u_1)^{1/\alpha_j}}, \ldots, e^{-(-\beta_{jd} \ln u_d)^{1/\alpha_j}}\right)\right)^{\alpha_j}\right\}.$$

P r o o f. To obtain $C_{\mathbf{Y}}$ we just apply the conditional independence of the $\mathbf{X}_j$'s followed by the max-stability of C_j's and the α_j-stability of each S_j, as follows:

$$P\left(\bigcap_{i=1}^{d}\{Y_i\le x_i\}\right)=\int P\left(\bigcap_{j=1}^{q}\bigcap_{i=1}^{d}\{X_{ji}eqx_i\}\,|S=s\right)\mathrm{d}\mathbf{S}\left(s_1,\ldots,s_q\right)$$

$$=\int\prod_{j=1}^{q}C_j\left(e^{-\left(\frac{x_1}{\beta_{j1}}\right)^{-1/\alpha_j}s_j},\ldots,e^{-\left(\frac{x_d}{\beta_{jd}}\right)^{-1/\alpha_j}s_j}\right)\mathrm{d}\mathbf{S}\left(s_1,\ldots,s_q\right)$$

$$=\prod_{j=1}^{q}\exp\left\{-\left(-\ln C_j\left(e^{-\left(\frac{x_1}{\beta_{j1}}\right)^{-1/\alpha_j}},\ldots,e^{-\left(\frac{x_d}{\beta_{jd}}\right)^{-1/\alpha_j}}\right)\right)^{\alpha_j}\right\}.$$

For each j and i, X_{ji} is a positive α_j-stable size mixture of a Fréchet distribution with location β_{ji}, scale $\beta_{ji}\alpha_j$ and shape parameter α_j and has itself Fréchet distribution with same location and the same right end point, but scale β_{ji} and shape parameter 1. Since $\sum_{j=1}^{q}\beta_{ji}=1$, $i=1,\ldots,d$, each Y_i has unit Fréchet distribution. The max-stability of $C_{\mathbf{Y}}$ follows from its expression and the max-stability of the C_j's. □

We now discuss some particular cases of (1) that has been explored.

(I) If $q=1$ then $\beta_{1i}=1$, $i=1,\ldots d$, and

$$C_{\mathbf{Y}}\left(u_1,\ldots,u_d\right)=\exp\left\{-\left(-\ln C_1\left(e^{-(-\ln u_1)^{1/\alpha_1}},\ldots,e^{-(-\ln u_d)^{1/\alpha_1}}\right)\right)^{\alpha_1}\right\}$$

is a generalisation of the Archimedean copula (Joe [5]), which for the particular case of the product copula $C_1=\Pi$ leads to the Gumbel–Hougaard or logistic copula.

The above copula is a particular case of the copula C_φ considered in Morillas [10] with $\varphi(x)=\exp\left(-(-\ln x)^{1/\alpha_1}\right)$.

The dependence properties of the special case of

$$C_1(u_1,\ldots,u_d)=\prod_{1\le s<t\le d}C_{\{s,t\}}(u_s^{p_s},u_t^{p_t})\prod_{i=1}^{d}u_i^{p_i\nu_i},$$

where $C_{\{s,t\}}$, $1\le s<t\le d$, are bivariate copulas and $(d-1)p_i+p_i\nu_i=1$, $i=1,\ldots,d$, were analysed in Joe and Hu [6].

(II) If $C_j=\Pi$, $j=1,\ldots,q$, then

$$\prod_{j=1}^{q}\exp\left\{-\left(-\ln C_j\left(e^{-\left(\frac{x_1}{\beta_{j1}}\right)^{-1/\alpha_j}},\ldots,e^{-\left(\frac{x_d}{\beta_{jd}}\right)^{-1/\alpha_j}}\right)\right)^{\alpha_j}\right\}$$

$$=\exp\left\{-\sum_{j=1}^{q}\left(\sum_{i=1}^{d}\left(\frac{x_i}{\beta_{ji}}\right)^{-1/\alpha_j}\right)^{\alpha_j}\right\},$$

which leads to an asymmetric logistic copula

$$C_{\mathbf{Y}}\left(u_1,\ldots,u_d\right)=\exp\left\{-\sum_{j=1}^{q}\left(\sum_{i=1}^{d}\left(-\beta_{ji}\ln u_i\right)^{-1/\alpha_j}\right)^{\alpha_j}\right\}. \tag{2}$$

In (2), if we take $\alpha_j = \alpha$, $j = 1, \ldots, q \leq +\infty$, we find an analogous mixture of extreme value distributions to those considered in Fougeres et al. [3] by departing just from a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ satisfying $\mathcal{L}(\mathbf{X}_j|\mathbf{S}) = \prod_{i=1}^{d} \mathcal{L}(X_i|\mathbf{S})$ and $P(X_i \leq x|\mathbf{S}) = \exp\left\{-\left(\sum_{j=1}^{q} c_{ji}S_j\right)\left(1+\gamma_i \frac{x-\mu_i}{\sigma_i}\right)^{-1/\gamma_i}\right\}$, $i = 1, \ldots, d$. In other words, in this different approach, conditionally on $\mathbf{S}$, the vector $\mathbf{X}$ has independent margins and each margin is a power mixture of an extreme value distribution with mixing variable $\sum_{j=1}^{q} c_{ji}S_j$, where the c_{ji} are non-negative constants.

(III) Assume now, in (2), that each j corresponds to an element A of the set $\mathcal{S}$, the class of all nonempty subsets of $D = \{1, \ldots, d\}$. If $\beta_{Ai} = 0$ for each $i \notin A$ then the copula (2) becomes

$$C_{\mathbf{Y}}(u_1, \ldots, u_d) = \exp\left\{-\sum_{A\subset\mathcal{S}}\left(\sum_{i\in A}^{d}(-\beta_{Ai}\ln u_i)^{-1/\alpha_A}\right)^{\alpha_A}\right\}, \tag{3}$$

with $\sum_{A\subset\mathcal{S}} \beta_{Ai} = 1$, $i = 1, \ldots, d$. This is the asymmetric logistic model considered in Tawn [14], by following a different probabilistic approach. More generally, by applying the same interpretation of the constants β_{ji} in (1), we obtain

$$\begin{aligned} &C_{\mathbf{Y}}(u_1, \ldots, u_d) \qquad (4)\\ = \;& \exp\left\{\sum_{A\subset\mathcal{S}}\left(-\ln C_A\left(e^{-\left(-\beta_{Ai_1(A)}\ln u_1\right)^{1/\alpha_A}}, \ldots, e^{-\left(-\beta_{Ai_s(A)}\ln u_d\right)^{1/\alpha_A}}\right)\right)^{\alpha_A}\right\}, \end{aligned}$$

where C_A's are copulas with different dimensions and we denote by $(i_1(A), \ldots, i_s(A))$ the sub-vector of $(1, \ldots, d)$ corresponding to indices in A. In particular, if we begin with one copula $C_j = C$, $j = 1, \ldots, q$, then C_A, $A \subset \mathcal{S}$, are all the sub-copulas of C.

(IV) Finally, let us suppose that $\beta_{ji} = \beta_j$, $i = 1, \ldots, d$, in (1). Then

$$C_{\mathbf{Y}}(u_1, \ldots, u_d) = \prod_{j=1}^{q} \exp\left\{-\left(\ln C_j\left(e^{-(-\ln u_1)^{1/\alpha_j}}, \ldots, e^{-(-\ln u_d)^{1/\alpha_j}}\right)\right)^{\alpha_j}\beta_j\right\}, \tag{5}$$

with $\sum_{j=1}^{q} \beta_j = 1$, that is, $C_{\mathbf{Y}}$ is a geometric mean of mixtures of powers of multivariate extreme value distributions. The particular case of the weighted geometric mean $C_{\mathbf{Y}}(u_1, u_2) = (u_1 \wedge u_2)^{\beta_1}(u_1u_2)^{1-\beta_1}$ is due to Cuadras and Augé [2].

3. ORTHANT TAIL DEPENDENCE

For a random vector $\mathbf{Y} = (Y_1, \ldots, Y_d)$ with continuos margins $F_1, \ldots, F_d$ and copula C, the bivariate (upper) tail dependence coefficients are defined by

$$\lambda_{\{s,t\}}^{(\mathbf{Y})} \equiv \lambda_{\{s,t\}}^{(C)} = \lim_{u\uparrow 1} P\Big(F_s(Y_s) > u | F_t(Y_t) > u\Big), \quad 1 \leq s < t \leq d. \tag{6}$$

The tail dependence is a copula based measure and it holds

$$\lambda_{\{s,t\}}^{(C)} = 2 - \lim_{u\uparrow 1} \frac{\ln C_{\{s,t\}}(u,u)}{\ln u}, \tag{7}$$

where $C_{\{s,t\}}$ is the copula of the sub-vector (Y_s, Y_t) (Joe [5], Nelsen [11]).

To characterise the relative strength of extremal dependence with respect to a particular subset of random variables of $\mathbf{Y}$ one can use conditional orthant tail probabilities of $\mathbf{Y}$ given that the components with indices in the subset J are extreme. The tail dependence of bivariate copulas can be extended as done in Schmid and Schmidt [12] and Li [7].

For $\emptyset \neq J \subset D = \{1, \ldots, d\}$, let

$$\lambda_J^{(\mathbf{Y})} \equiv \lambda_J^{(C)} = \lim_{u \uparrow 1} P\left(\bigcap_{j \notin J} \{F_j(Y_j) > u\} \mid \bigcap_{j \in J} \{F_j(Y_j) > u\}\right). \tag{8}$$

If for some $\emptyset \neq J \subset \{1, \ldots, d\}$ the coefficient $\lambda_J^{(C)}$ exists and is positive then we say that $\mathbf{Y}$ is (upper) orthant tail dependent.

We have $\lambda_J^{(C)} = \frac{\lambda_{\{s\}}^{(C)}}{\lambda_{\{s\}}^{(C_J)}}$, if $\lambda_{\{s\}}^{(C_J)} \neq 0$ and the relation (7) between the tail dependence coefficient and the bivariate copula can also be generalized by

$$\lambda_J^{(C)} = \lim_{u \uparrow 1} \frac{\sum_{\emptyset \neq A \subset D} (-1)^{|A|-1} \ln C_A(\mathbf{u}_A)}{\sum_{\emptyset \neq A \subset J} (-1)^{|A|-1} \ln C_A(\mathbf{u}_A)}, \tag{9}$$

where C_A denotes the sub-copula of C corresponding to margins with indices in A and and $\mathbf{u}_A$ the $|A|$-dimensional vector $(u, \ldots, u)$. By applying this relation and the max-stability of the copulas C_j, we get the following result.

Theorem 3.1. For a copula C defined by (1), it holds

(a)

$$\lambda_J^{(C)} = \frac{\displaystyle\sum_{j=1}^{q} \sum_{\emptyset \neq A \subset D} (-1)^{|A|-1} \left(-\ln C_{j,A}\left(e^{-\beta_{j1}^{1/\alpha_j}}, \ldots, e^{-\beta_{jd}^{1/\alpha_j}}\right)_A\right)^{\alpha_j}}{\displaystyle\sum_{j=1}^{q} \sum_{\emptyset \neq A \subset J} (-1)^{|A|-1} \left(-\ln C_{j,A}\left(e^{-\beta_{j1}^{1/\alpha_j}}, \ldots, e^{-\beta_{jd}^{1/\alpha_j}}\right)_A\right)^{\alpha_j}}, \tag{10}$$

where $C_{j,A}$ denotes the sub-copula of C_j corresponding to the margins with indices in A.

(b) If $C_j = \Pi$, for each $j = 1, \ldots, q$, then

$$\lambda_J^{(C)} = \frac{\displaystyle\sum_{j=1}^{q} \sum_{\emptyset \neq A \subset D} (-1)^{|A|-1} \left(\sum_{i \in A} \beta_{ji}^{1/\alpha_j}\right)^{\alpha_j}}{\displaystyle\sum_{j=1}^{q} \sum_{\emptyset \neq A \subset J} (-1)^{|A|-1} \left(\sum_{i \in A} \beta_{ji}^{1/\alpha_j}\right)^{\alpha_j}}. \tag{11}$$

The tail dependence result in (10) depends on the mixing variables through the parameters α_j, even for the case of $q = 1$, that is the global dependence added by the mixing variables doesn't vanish in extremes of maxima. This contrast with the result in Li [7], where the scale mixture of MEV distributions $(RX_1, \ldots, RX_d)$ is considered with the mixing variable R satisfying $\frac{E(e^{-ctR})}{E(e^{-tR})} \to c^{-\alpha}$, as t tends to ∞, and $c \geq 1$, $\alpha > 0$. In this case the upper tail dependence coefficients are exactly the same as the coefficients of the MEV distribution without mixing.

We remark that, for $\beta_{ji} = \beta_j$, $i = 1, \ldots, d$, the numerator in (10) is, for each $A \subset D$,

$$\lambda_{\{s\}}^{(C_A)} = \sum_{j=1}^{q} \beta_j \lambda_{\{s\}}^{(C_{j,A})}$$

that is, the tail dependence coefficient $\lambda_{\{s\}}^{(C_A)}$ is a linear convex combination of the corresponding tail dependence coefficients for the sub-copulas $C_{j,A}$ of C_j, $j = 1, \ldots, q$.

The result in (11) leads to

$$\lambda_{\{s,t\}}^{(C)} = 2 - \sum_{j=1}^{q} \left(\beta_{js}^{1/\alpha_j} + \beta_{jt}^{1/\alpha_j} \right)^{\alpha_j},$$

extending the the known result

$$\lambda_{\{s,t\}}^{(C)} = 2 - 2^{\alpha}, \tag{12}$$

corresponding to $q = 1$ (Joe [5], Nelsen [11]). The result in (10) enables to extend the equation (12) for other copulae C_1 than the product copula as

$$\lambda_{\{s,t\}}^{(C)} = 2 - (2 - \lambda_{\{s,t\}}^{(C_1)})^{\alpha}. \tag{13}$$

The above results include a large number of possibilities for the tail dependence. We now illustrate the results with an example.

Example 3.2. We will suppose that $C_j = C$, $j = 1, \ldots, d$, with

$$C(u_1, \ldots, u_d) = \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} \left(\bigwedge_{i=1}^{d} u_i^{a_{lki}} \right), \quad u_j \in [0,1], \quad j = 1, \ldots, d,$$

where $\{a_{lkj}, l \geq 1, \ -\infty < k < \infty, 1 \leq j \leq d\}$, are nonnegative constants satisfying

$$\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{lkj} = 1 \quad \text{for } j = 1, \ldots, d.$$

That copula arises from the common distribution of the variables of an M4 process (Smith and Weissman [13]).

Then the copula in (1) becomes

$$C_{\mathbf{Y}}(u_1,\ldots,u_d)=\exp\left\{-\sum_{j=1}^{q}\left(\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\bigvee_{i=1}^{d}\left(-\beta_{ji}a_{lki}^{\alpha_j}\ln u_i\right)^{1/\alpha_j}\right)^{\alpha_j}\right\}. \qquad (14)$$

By applying the result in Proposition 2.1. (a), we obtain for the numerator in (10)

$$\lambda_{\{s\}}^{(C_{\mathbf{Y}})}=\sum_{j=1}^{q}\sum_{\emptyset\neq A\subset D}(-1)^{|A|-1}\left(\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\bigvee_{i\in A}\left(a_{lki}\beta_{ji}^{1/\alpha_j}\right)\right)^{\alpha_j}.$$

For the bivariate tail dependence it holds

$$\lambda_{\{s,t\}}^{(C_{\mathbf{Y}})}=2-\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{j=1}^{q}\left(a_{lks}\beta_{js}^{1/\alpha_j}\vee a_{lkt}\beta_{jt}^{1/\alpha_j}\right)^{\alpha_j},$$

which, for the case $q=1$ leads to the result $\lambda_{\{s,t\}}^{(C_{\mathbf{Y}})}=2-\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}(a_{lks}\vee a_{lkt})$ in Heffernan et al. [4].

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