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SPECTRUM OF THE WEIGHTED LAPLACE OPERATOR IN UNBOUNDED DOMAINS
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Abstract. We investigate the spectral properties of the differential operator $-r^s\Delta$, $s \geq 0$ with the Dirichlet boundary condition in unbounded domains whose boundaries satisfy some geometrical condition. Considering this operator as a self-adjoint operator in the space with the norm $\|u\|_{L_{2,s}(\Omega)}^2 = \int_{\Omega} r^{-s}|u|^2 \, dx$, we study the structure of the spectrum with respect to the parameter $s$. Further we give an estimate of the rate of condensation of discrete spectra when it changes to continuous.

Keywords: Laplace operator, multiplicative perturbation, Dirichlet problem, Friedrichs extension, purely discrete spectra, purely continuous spectra

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an unbounded domain whose closure does not contain the origin, with a boundary $\Gamma$. Let us consider the differential expression

$$(1) \quad lu = -r^s\Delta u, \quad r = |x|, \quad s \geq 0.$$ 

We shall treat the differential operator (1) in the Hilbert space $L_{2,s}(\Omega)$ with the norm $\|u\|_{L_{2,s}(\Omega)}^2 = \int_{\Omega} r^{-s}|u|^2 \, dx$. Let $L$ be the self-adjoint Friedrichs extension in $L_{2,s}(\Omega)$ of the minimal operator generated by the differential expression (1). Then, $L$ is a non-negative self-adjoint operator in $L_{2,s}(\Omega)$ that is an operator of the first boundary value problem for the differential expression (1).

We will study spectral properties of the operator $L$ (location of spectrum on the real axis, density of spectrum on some sets, structure of the spectrum) with respect
to the parameter $s$. For the operator (1) we know conditions on $s$ and the domain $\Omega$, guaranteeing the discreteness of the spectrum of the operator $L$ ([1]). In particular, in [1] it was proved that the spectrum of the operator $L$ is discrete at $s > 2$. On the other side, as follows from the results of [2], in the case $\Omega = \mathbb{R}^n$ with $0 \leq s \leq 2$ the spectrum of the operator with coefficients equal to (1) in the neighborhood of infinity is continuous and for $0 \leq s < 2$ fills the complete positive semi-axis.

Let $S_\eta = \Omega \cap \{r = \eta\}$, $\eta > 0$, and $\Sigma_\eta$ be the set of points $x$ belonging to the unit sphere $\Sigma$ and satisfying $\eta x \in S_\eta$. In the sequel we will consider domains $\Omega$ such that

\begin{equation}
\Sigma_\eta_1 \subset \Sigma_\eta_2, \quad \eta_1 < \eta_2,
\end{equation}

(it is the star-shapeness condition for the set $\mathbb{R}^n \setminus \Omega$ with respect to the origin). Denote by $\hat{\lambda}(\eta)$ the modulus of the first eigenvalue of the Laplace-Beltrami operator in $\Sigma_\eta$ with zero Dirichlet data on $\partial \Sigma_\eta$. By our supposition $\hat{\lambda}(\eta)$ is a decreasing nonnegative function on $[\inf_{x \in \Omega} r, +\infty)$. Denote $\Lambda = \lim_{\eta \to \infty} \hat{\lambda}(\eta)$. We will also suppose without loss of generality that $\ln r > 1$ in $\Omega$.

Our first statement localizes the spectrum set $\sigma(L)$ of the operator $L$ on the real axis.

**Theorem 1.** The spectrum of $L$ has the following properties:

i) if $0 \leq s < 2$, then $\sigma(L) = [0, +\infty)$;

ii) if $s = 2$, then $\sigma(L) = [\frac{1}{4}(n - 2)^2 + \Lambda, +\infty)$;

iii) if $s > 2$, then $\sigma(L) \subset (\frac{1}{4}(n - 2)^2 + \Lambda, +\infty)$.

The next statement declares that there exists a critical value of $s$ for which the spectrum of $L$ becomes discrete.

**Theorem 2.** The spectrum of $L$ has the following properties:

i) if $0 \leq s \leq 2$ and $\Gamma \in C^2$, then the spectrum of the operator $L$ is continuous;

ii) if $s > 2$, then the spectrum of the operator $L$ is discrete.

It is natural to expect that the discrete spectrum condenses on the semi-axis $[\frac{1}{4}(n - 2)^2 + \Lambda, +\infty)$ at $s \to 2 + 0$. In the next statement we establish an estimate of the rate of this condensation.

**Theorem 3.** For any $\lambda \in \left[\frac{1}{4}(n - 2)^2 + \Lambda, +\infty\right)$ there exist a constant $C > 0$ and a number $s_0 > 2$ such that for any $s \in (2, s_0]$ the following relation holds:

\begin{equation}
\sigma(L) \cap (\lambda - \delta(s), \lambda + \delta(s)) \neq \emptyset,
\end{equation}

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where
\[ \delta(s) = \lambda(\ln \ln(1/(s - 2))) - \Lambda + C \frac{\ln \ln(1/(s - 2))}{\ln \ln(1/(s - 2))}. \]

The constant \( C \) depends on \( \lambda \).

2. ENERGY SPACE AND DOMAIN

Let us define the space of functions

\[ H^1_s(\Omega) = \{ u: u \in L_{2,s}(\Omega) \cap H^1(\Omega_R), R > 0, u_{x_j} \in L_2(\Omega), j = 1, \ldots, n \}, \]

where \( \Omega_R = \Omega \cap \{ r < R \} \), with the norm \( \| u \|^2_{H^1_s(\Omega)} = \int_\Omega (|\nabla u|^2 + r^{-s}|u|^2) \, dx \). By \( \tilde{H}^1_s(\Omega) \) denote the subspace of \( H^1_s(\Omega) \) which is the closure of the set of functions \( u \in H^1_s(\Omega) \) vanishing in a neighborhood of \( \Gamma \). Consider the quadratic form \( A[u] = \int_\Omega |\nabla u|^2 \, dx \) on the set of functions \( \tilde{C}^\infty(\Omega) \subset L_{2,s}(\Omega) \).

**Lemma 1.** The form \( A[u] \) is closeable.

**Proof.** Let \( \{ u_j \} \subset \tilde{C}^\infty(\Omega), j = 1, 2, \ldots \) be a sequence of functions such that \( A[u_j - u_l] \to 0, j, l \to \infty \) and \( \| u_j \|_{L_{2,s}(\Omega)} \to 0, j \to \infty \). Now, by \( \| u_j \|^2_{H^1_s(\Omega)} = A[u_j] + \| u_j \|^2_{L_{2,s}(\Omega)} \) we have that the sequence \( u_j \) is fundamental in the space \( H^1_s(\Omega) \).

By \( \hat{u} \in H^1_s(\Omega) \) denote the limit function: \( \lim_{j \to \infty} \| \hat{u} - u_j \|_{H^1_s(\Omega)} = 0 \). Then \( \lim_{j \to \infty} \| u_j - \hat{u} \|_{L_{2,s}(\Omega)} = 0 \), i.e. \( \| \hat{u} \|_{L_{2,s}(\Omega)} \leq \lim_{j \to \infty} \| u_j \|_{L_{2,s}(\Omega)} + \lim_{j \to \infty} \| \hat{u} - u_j \|_{L_{2,s}(\Omega)} = 0 \). Hence, \( \hat{u} = 0 \) and \( \lim_{j \to \infty} \| u_j \|_{L_{2,s}(\Omega)} = 0 \). So, the possibility to close the form \( A[u] \) is proved.

By Lemma 1 the energy space \( H_A \) of the operator \( L \) is the closure of the set of functions \( \tilde{C}^\infty(\Omega) \) in the norm \( \| u \|^2_{H^1_s(\Omega)} = A[u] + \| u_j \|^2_{L_{2,s}(\Omega)} \).

**Lemma 2.** The energy space of the operator \( L \) is

\[ H_A = \left\{ u: u \in \tilde{H}^1_s(\Omega), \int_\Omega r^{-2} \ln^{-2q} r|u|^2 \, dx < \infty \right\}, \]

where \( q = 0 \) for \( n \geq 3 \) and \( q = 1 \) for \( n = 2 \).

**Proof.** It is sufficient to prove that for any function \( u \in \tilde{H}^1_s(\Omega) \) such that \( \int_\Omega r^{-2} \ln^{-2q} r|u|^2 \, dx < \infty \) and for any \( \varepsilon > 0 \) there exists a function \( \hat{u} \in \tilde{C}^\infty(\Omega) \), such that \( A[u - \hat{u}] < \varepsilon \).
First let us prove that for any function \( u \in H_A \) the integral \( \int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 \, dx \) converges. We use the inequalities

\[
\begin{align*}
(5) \quad & \int_{\Omega_R} r^{-2} \ln^{-2} r |u|^2 \, dx \leq 4 \int_{\Omega_R} |u_r|^2 \, dx, \quad n = 2, \\
(6) \quad & \int_{\Omega_R} r^{-2} |u|^2 \, dx \leq \frac{2}{(n-2)R} \int_{\mathcal{S}_R} |u|^2 \, ds + \frac{4}{(n-2)^2} \int_{\Omega_R} |u_r|^2 \, dx, \quad n \geq 3,
\end{align*}
\]

which are valid for all \( R > 1 \) for functions \( u \in H^1(\Omega_R) \) such that \( u|_\Gamma = 0 \).

For any function \( u \in H_A \) there exists a sequence of functions \( \{u_j\} \subset \mathcal{C}_\infty(\Omega) \), \( j = 1, 2, \ldots \) such that \( A[u - u_j] \to 0, \|u - u_j\|_{L^2(\Omega)} \to 0, j \to \infty \). Apply (5) (6) to \( u_j \) with sufficiently large \( R \). Since the term on the right hand side containing \( (u_j)_r \) is bounded for all \( j \) and \( R \), we obtain \( \int_{\Omega} r^{-2} \ln^{-2q} r |u_j|^2 \, dx \leq C_1 \). Since \( r^{-1} \ln^{-q} ru_j \) must converge in \( L^2(\Omega) \) weakly to \( r^{-1} \ln^{-q} ru \), we obtain

\[
(7) \quad \int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 \, dx \leq C_1.
\]

Conversely, let us suppose that \( u \in H^{1}(\Omega) \) and \( \int_{\Omega} r^{-2} \ln^{-2q} r |u|^2 \, dx < \infty \). Let \( n \geq 3 \). Consider functions \( \xi_m(x) = \eta(\ln(r/m) + 1), m = 1, 2, \ldots \), where \( \eta(t) \in \mathcal{C}_\infty([0, +\infty)) \) is a nonnegative function satisfying the condition \( 0 \leq \eta \leq 1 \) and such that \( \eta = 1 \) for \( 0 < t < 1 \), \( \eta = 0 \) for \( t > 2 \). Hence \( \xi_m = 1, x \in \Omega_m, \xi_m = 0, x \in \Omega \setminus \Omega_m \). We have an estimate

\[
(8) \quad |\nabla \xi_m| = \left| \frac{d}{dr} \xi_m \right| = |\eta'(\ln(r/m) + 1)|r^{-1} \leq C_2 r^{-1}, \quad x \in \Omega.
\]

The function \( u\xi_m \) belongs to the space \( H_A \). Let us prove that \( \|u - u\xi_m\|_{H^1(\Omega)} \to 0, m \to \infty \). We get

\[
(9) \quad \|u - u\xi_m\|_{H^1(\Omega)}^2 \leq 2(I_{1,m} + I_{2,m}),
\]

where

\[
I_{1,m} = \int_{\Omega \setminus \Omega_m} (|\nabla u|^2 + r^{-s} |u|^2)(1 - \xi_m)^2 \, dx, \quad I_{2,m} = \int_{\Omega_m \setminus \Omega_m} |u|^2 |\nabla \xi_m|^2 \, dx.
\]

Since \( u \in H^1(\Omega) \), we obtain \( I_{1,m} \to 0, m \to \infty \). Furthermore, it follows from (7), (8) that

\[
I_{2,m} \leq C_2^2 \int_{\Omega_m \setminus \Omega_m} r^{-2} |u|^2 \, dx \to 0, \quad m \to \infty.
\]

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Thus, \(|u - u\xi_m|\|_{H^1_s(\Omega)} \to 0, m \to \infty\). Now, by virtue of \(u\xi_m \in \mathring{H}^1_s(\Omega_{me})\) there exist functions \(\tilde{u}_m \in \mathring{C}^\infty(\Omega_{me})\), such that \(A[\tilde{u}_m - u\xi_m] \leq \|\tilde{u}_m - u\xi_m\|_{H^1_s(\Omega_{me})}^2 \to 0, m \to \infty\). Consider the zero continuation of the function \(\tilde{u}_m\) to the set \(\Omega \setminus \Omega_{me}\) and denote the continued function also by \(\tilde{u}_m\). Therefore \(\tilde{u}_m \in \mathring{C}^\infty(\Omega)\) and \(A[u - \tilde{u}_m] \leq 2(A[u - u\xi_m] + A[u\xi_m - \tilde{u}_m]) \to 0, m \to \infty\). The existence of a function \(\tilde{u} \in \mathring{C}^\infty(\Omega)\) such that \(A[u - \tilde{u}] < \varepsilon\) in the case \(n \geq 3\) is proved.

Let us consider the case \(n = 2\). Put \(\xi_m(x) = \eta(\ln(\ln r/\ln m))\) where the function \(\eta\) is the same as for \(n \geq 3\). Then \(\xi_m = 1\) for \(x \in \Omega_{me}\) and \(\xi_m = 0\) for \(x \in \Omega \setminus \Omega_{me}\).

For the function \(\xi_m\) we obtain

\[
|\nabla \xi_m| = \left| \frac{d}{dx} \xi_m \right| = |\eta'(\ln(\ln r/\ln m))| (r \ln r)^{-1} \leq C_2 (r \ln r)^{-1}, \quad x \in \Omega.
\]

The estimate (9) with

\[
I_{1,m} = \int_{\Omega \setminus \Omega_{me}} (|\nabla u|^2 + r^{-s}|u|^2)(1 - \xi_m)^2 \, dx, \quad I_{2,m} = \int_{\Omega_{me} \setminus \Omega_m} |u|^2 |\nabla \xi_m|^2 \, dx
\]

holds. As in the case \(n \geq 3\), we obtain that \(I_{1,m} \to 0, m \to \infty\). It follows from the estimate (7) with \(q = 1\) and (10) that

\[
I_{2,m} \leq C_2^2 \int_{\Omega_{me} \setminus \Omega_m} r^{-2} \ln^{-2} r |u|^2 \, dx \to 0, \quad m \to \infty.
\]

Thus, \(A[u - u\xi_m] \to 0, m \to \infty\). Now, we get the existence of a sequence \(\tilde{u}_m \in \mathring{C}^\infty(\Omega)\), \(\text{supp} \tilde{u}_m \subset \Omega_{me}\) such that \(A[\tilde{u}_m - u\xi_m] \leq \|\tilde{u}_m - u\xi_m\|_{H^1_s(\Omega_{me})}^2 \to 0, m \to \infty\). Hence the existence of a function \(\tilde{u} \in \mathring{C}^\infty(\Omega)\) such that \(A[u - \tilde{u}] < \varepsilon\) for \(n = 2\) is proved. This completes the proof of Lemma 2.

**Lemma 3.** The domain of the operator \(L\) is

\[
D(L) = \left\{ u : u \in \mathring{H}^1_s(\Omega) \cap H^2_{\text{loc}}(\Omega), lu \in L_{2,s}(\Omega), \int_{\Omega} r^{-2} \ln^{-2g} r |u|^2 \, dx < \infty \right\}.
\]

In the case \(\Gamma \in C^2\) the domain of the operator \(L\) is

\[
D(L) = \left\{ u : u \in \mathring{H}^1_s(\Omega) \cap H^2(\Omega_R), R > 0, lu \in L_{2,s}(\Omega), \int_{\Omega} r^{-2} \ln^{-2g} r |u|^2 \, dx < \infty \right\}.
\]

**Proof.** Applying interior estimates for the derivatives of solutions of elliptic equations ([3], p. 204, Lemma 7.1) to \(u \in \mathring{H}^1_s(\Omega)\), \(lu \in L_{2,s}(\Omega)\) and any domain
\( \Omega' \subset \Omega \), we get \( u \in H^2(\Omega') \). If, furthermore, \( \Gamma \subset C^2 \), applying the boundary estimates for derivatives of solutions of elliptic equations ([3], p. 224, Theorem 9.2), for all \( R > 0 \) we obtain \( u \in H^2(\Omega_R) \). This completes the proof of Lemma 3.

3. Localization of spectrum

It follows from the inequalities (5), (6) that for functions \( u \in H_A \) we have the lower estimate

\[
A[u] = \int_{\Omega} |u|^2 \, dx + \int_{\Omega} r^{-2}\nabla u|^2 \, dx \\
\geq \frac{(n-2-q)^2}{4} \int_{\Omega} r^{-2} \ln^{-2q} r|u|^2 \, dx + \int_{\Omega} r^{-2}\nabla u|^2 \, dx,
\]

\( u(x) = u(r, \Theta) \).

Since \( \hat{\lambda}(r) \) is the modulus of the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator in \( \Sigma_r \), we get

\[
\int_{\Sigma_r} |\nabla \Theta u|^2 \, d\Theta \geq \hat{\lambda}(r) \int_{\Sigma_r} |u|^2 \, d\Theta \geq \Lambda \int_{\Sigma_r} |u|^2 \, d\Theta, \quad r > 0.
\]

Therefore,

\[
A[u] \geq \frac{(n-2-q)^2}{4} \int_{\Omega} r^{-2} \ln^{-2q} r|u|^2 \, dx \\
+ \Lambda \int_{\Omega} r^{-2}|u|^2 \, dx \geq \left( \frac{(n-2)^2}{4} + \Lambda \right) \int_{\Omega} r^{-2}|u|^2 \, dx,
\]

and for \( s \geq 2 \) we have an estimate \( A[u] \geq \left( \frac{1}{4}(n-2)^2 + \Lambda \right) \|u\|^2_{L^2(\Omega)} \) and, consequently, \( \sigma(L) \subset \left[ \frac{1}{4}(n-2)^2 + \Lambda, +\infty \right) \), \( s \geq 2 \).

Let us prove that for \( s > 2 \) the number \( \frac{1}{4}(n-2)^2 + \Lambda \) does not belong to the spectrum of the operator \( L \). Assume the converse, let \( s > 2 \) and \( \frac{1}{4}(n-2)^2 + \Lambda \in \sigma(L) \). We show that there exists a non-zero function \( \hat{u} \in H_A \) such that

\[
\int_{\Omega} |\nabla \hat{u}|^2 \, dx = \left( \frac{(n-2)^2}{4} + \Lambda \right) \int_{\Omega} r^{-s}|\hat{u}|^2 \, dx.
\]

If \( \frac{1}{4}(n-2)^2 + \Lambda \) is an eigenvalue of the operator \( L \), the relation (13) holds for the corresponding eigenfunction \( \hat{u} \). If \( \frac{1}{4}(n-2)^2 + \Lambda \) is the continuous spectrum point, let us use I.M. Glazman lemma for quadratic forms ([5], Supplement 1, Lemma 3.1'), which is a modification of the corresponding operator statement ([4], Chapter 1, Section 1, Theorem 9bis). By this lemma

\[
N(\lambda - 0) = \sup_{\{F \subset H_A \mid A[u] < \lambda\|u\|^2_{L^2(\Omega)}, u \in F \setminus \{0\} \}} \dim F.
\]

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where $H$ is the main space ($H = L_{2, s}(\Omega)$ in our case), $F$ is a linear subspace of $H_A$, $N(\lambda) = \text{dim}(E_\lambda H)$ where $E_\lambda$ denotes the spectral projector of the spectral family corresponding to the self-adjoint operator $L$. As follows from this lemma, if $\lambda$ is a continuous spectrum point, for any $\delta > 0$ the relation $N(\lambda + \delta) - N(\lambda - \delta) = \infty$ holds. Thus for any $\delta > 0$ there exists a function $u \in H_A$, $u \neq 0$, such that 

\[
\langle u \rangle \leq (\frac{1}{4}(n - 2)^2 + \Lambda + \delta)\|u\|_{L_{2, s}(\Omega)}^2.
\]

Let us choose a sequence $\delta_j > 0$, $\delta_j \to 0$, $j \to \infty$. Then there exists a non-zero sequence $u_j \in H_A$ such that

\[
\langle u_j \rangle \leq 1. \quad \text{Then} \quad \int_{\Omega} |\nabla u_j|^2 \, dx \leq \left(\frac{(n - 2)^2}{4} + \Lambda + \delta_j\right) \int_{\Omega} r^{-s}|u_j|^2 \, dx.
\]

Let $\|u_j\|_{L_{2, s}(\Omega)} = 1$. Then $\int_{\Omega} |\nabla u_j|^2 \, dx \leq \frac{1}{4}(n - 2)^2 + \Lambda + \delta_j$ and, clearly, the inequalities (5), (6) imply

\[
\int_{\Omega} r^{-2} \ln^{-2q} r |u_j|^2 \, dx \leq C_3.
\]

From the sequence $\{u_j\}$ let us choose a subsequence which is weakly convergent in the space $L_{2, s}(\Omega)$ and show that it is pre-compact in $L_{2, s}(\Omega)$. In the same way as in Rellich’s theorem about the compact imbedding of $H^1(\Omega')$ into $L_2(\Omega')$ in a bounded domain $\Omega'$, we can prove that for any $R > 0$ the space $H^1_0(\Omega_R)$ imbeds compactly into $L_{2, s}(\Omega_R)$. Hence, there exists a function $\hat{u} \in L_{2, s}(\Omega)$ such that for any $R > 0$ we have $\lim_{j \to \infty} \|\hat{u} - u_j\|_{L_{2, s}(\Omega_R)} = 0$. It means that for any sequence $\{R_j\}$, $R_j \to \infty$, $j \to \infty$, it is possible to choose a subsequence $\{u_j\}$ (denoted also by $\{u_j\}$) such that $\|\hat{u} - u_j\|_{L_{2, s}(\Omega_{R_j})} < j^{-1}$. Therefore by (16) we have

\[
\|\hat{u} - u_j\|_{L_{2, s}(\Omega)}^2 = \|\hat{u} - u_j\|_{L_{2, s}(\Omega_{R_j})}^2 + \|\hat{u} - u_j\|_{L_{2, s}(\Omega \setminus \Omega_{R_j})}^2 < j^{-2} + 2\left(\|\hat{u}\|_{L_{2, s}(\Omega \setminus \Omega_{R_j})}^2 + \|u_j\|_{L_{2, s}(\Omega \setminus \Omega_{R_j})}^2\right) \\
\leq j^{-2} + C_4 R_j^{-2q} \ln^{-2q} R_j \to 0, \quad j \to \infty.
\]

So, the convergence of the sequence $\{u_j\}$ to $\hat{u}$ in the space $L_{2, s}(\Omega)$ is proved. This, in particular, yields that $\|\hat{u}\|_{L_{2, s}(\Omega)} = 1$. It implies that

\[
\int_{\Omega} |\nabla \hat{u}|^2 \, dx \leq \liminf_{j \to \infty} \int_{\Omega} |\nabla u_j|^2 \, dx = \left(\frac{(n - 2)^2}{4} + \Lambda\right) \int_{\Omega} r^{-s}|\hat{u}|^2 \, dx.
\]

The relation (13) is proved.
For the proof of the relation $\frac{1}{4}(n - 2)^2 + \Lambda \notin \sigma(L)$ let us first consider the case $\frac{1}{4}(n - 2)^2 + \Lambda > 0$. In this case

$$(18) \quad A[\hat{u}] = \left(\frac{(n - 2)^2}{4} + \Lambda\right) \int_{\Omega} r^{-s}|\hat{u}|^2 \, dx < \left(\frac{(n - 2)^2}{4} + \Lambda\right) \int_{\Omega} r^{-2}|\hat{u}|^2 \, dx,$$

which contradicts (12).

In the case $\frac{1}{4}(n - 2)^2 + \Lambda = 0$ we have by (13) the equality $\int_{\Omega} |\nabla \hat{u}|^2 \, dx = 0$. Thus, $\nabla \hat{u} = 0$ and $\hat{u} = \text{const}$. But then $\hat{u} = 0$, which contradicts $\|\hat{u}\|_{L_{2,s}(\Omega)} = 1$. So, $\frac{1}{4}(n - 2)^2 + \Lambda \notin \sigma(L)$ and the relation $\sigma(L) \subset (\frac{1}{4}(n - 2)^2 + \Lambda, +\infty)$ holds true. Proof of the point iii) of Theorem 1 is now complete.

Finally, for $s > 2$ we have that any sequence bounded in the space $H_A$ is pre-compact in $L_{2,s}(\Omega)$. By F. Rellich criterion ([5], Supplement 1, Par. 3), the spectrum of the operator $L$ is discrete at $s > 2$. This completes the proof of the point ii) of Theorem 2.

4. Density of spectrum on the semi-axis

First consider the case $0 \leq s < 2$. Let us use the relation (14). By this relation the number of points of the spectrum for the operator $L$ in the interval $(\lambda - \delta, \lambda + \delta)$ with account of multiplicity is equal to the maximal dimension of the linear manifolds $F \subset H_A$ for which the following inequality is valid:

$$\quad (19) \quad |A[u] - \lambda\|u\|_H^2| < \delta\|u\|_H^2, \quad u \neq 0.$$

In our case the relation (19) can be written as

$$\quad (20) \quad \left| \int_{\Omega} (|\nabla u|^2 - \lambda r^{-s}|u|^2) \, dx \right| < \delta \int_{\Omega} r^{-s}|u|^2 \, dx.$$

Denote by $v_\varrho(\Theta) \in \overset{\circ}{H}^1(\Sigma_\varrho)$, $\varrho > 0$, the first eigenfunction of the Laplace-Beltrami operator in the domain $\Sigma_\varrho$. Hence $\int_{\Sigma_\varrho} |\nabla_\Theta v_\varrho|^2 \, d\Theta = \hat{\lambda}(\varrho) \int_{\Sigma_\varrho} v_\varrho^2 \, d\Theta$. Let us continue the function $v_\varrho$ by zero to the set $\Sigma$. Therefore $v_\varrho \in H^1(\Sigma)$ and $\int_{\Sigma} |\nabla_\Theta v_\varrho|^2 \, d\Theta = \hat{\lambda}(\varrho) \int_{\Sigma} v_\varrho^2 \, d\Theta$. We choose a nonzero real-valued function $\varphi(t) \in C^\infty(0, +\infty)$ such that $\text{supp}\, \varphi = [1, 2]$. Consider functions

$$\quad (21) \quad u_\varepsilon(r, \Theta) = \sqrt{\varepsilon} r^{1-n/2} H_{\frac{n}{2}-\frac{3}{2}}^{(1)} \left(\frac{2\sqrt{\lambda}}{2 - s} r^{1-s/2}\right) \varphi(\varepsilon r^{1-s/2}) v_{\varepsilon-2/(2-s)}(\Theta),$$

$\varepsilon > 0$, $\lambda > 0$, $n \geq 3$. 

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Consider now the left hand side of the relation (22). It follows from (23)–(24) that
\[(25) \quad \left| \int_{R^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-s}|u_\varepsilon|^2) \, dx \right| < \delta \int_{R^n} r^{-s}|u_\varepsilon|^2 \, dx.\]

Consider the behavior of the left hand and right hand parts of the inequality (22) for \(\varepsilon \to 0\). Let us note that \(\inf_{x \in \text{supp } u_\varepsilon} r > \varepsilon^{-2/(2-s)} \to \infty, \varepsilon \to 0\). Now we use the relations for derivatives of Hankel functions and the asymptotic expansions of Hankel functions of large argument ([6], Chapter 9):

\[(23) \quad H_p^{(1)}(z) = pz^{-1}H_p^{(1)}(z) - H_{p+1}^{(1)}(z),\]
\[(24) \quad H_p^{(1)}(z) = \sqrt{2/(\pi z)} \exp(i(z - \pi(p + 1/2)))(1 + O(|z|^{-1})), \quad |z| \to \infty.\]

By (24) we have
\[\int_{R^n} r^{-s}|u_\varepsilon|^2 \, dx = \frac{\varepsilon(2 - s)}{\pi\sqrt{\lambda}} \int_0^\infty (r^{-s/2} + f_1(r))\varphi^2(\varepsilon r^{1-s/2}) \, dr \int_\Sigma v_{\varepsilon-2/(2-s)}^2 \, d\Theta,\]
where \(|f_1(r)| \leq C_5 r^{-1}\). Therefore,
\[\int_{R^n} r^{-s}|u_\varepsilon|^2 \, dx = \left(\frac{2\varepsilon}{\pi\sqrt{\lambda}} \int_0^\infty \varphi^2(\varepsilon z) \, dz \right) \int_\Sigma v_{\varepsilon-2/(2-s)}^2 \, d\Theta \leq \left(\frac{2}{\pi\sqrt{\lambda}} \int_0^\infty \varphi^2(\varepsilon t) \, dt \right) \int_\Sigma v_{\varepsilon-2/(2-s)}^2 \, d\Theta,
\]
where
\[|J_1| \leq \frac{(2 - s)\varepsilon}{\pi\sqrt{\lambda}} \int_0^\infty |f_1(r)| \varphi^2(\varepsilon r^{1-s/2}) \, dr \leq \frac{2C_5\varepsilon}{\pi\sqrt{\lambda}} \int_0^\infty t^{-1}\varphi^2(t) \, dt.\]

Thus,
\[(25) \quad \int_{R^n} r^{-s}|u_\varepsilon|^2 \, dx = (C_6 + O(\varepsilon)) \int_\Sigma v_{\varepsilon-2/(2-s)}^2 \, d\Theta, \quad \varepsilon \to 0, \quad C_6 > 0.\]

Consider now the left hand side of the relation (22). It follows from (23)–(24) that
\[(26) \quad \left| \int_{R^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-s}|u_\varepsilon|^2) \, dx \right|
\[= \frac{\varepsilon(2 - s)}{\pi\sqrt{\lambda}} \int_0^\infty ((f_2(r) + \lambda(\varepsilon^{-2/(2-s)}f_3(r))\varphi^2(\varepsilon r^{1-s/2})
\[+ \varepsilon((2 - s)(2 - 2n + s)(4r)^{-1} + f_4(r))\varphi(\varepsilon r^{1-s/2})\varphi'(\varepsilon r^{1-s/2})
\[+ \varepsilon^2((2 - s)^2r^{-s/2}/4 + f_5(r))\varphi'^2(\varepsilon r^{1-s/2})) \, dr \right| \int_\Sigma v_{\varepsilon-2/(2-s)}^2 \, d\Theta,\]

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where \( f_j, j = 2, 3, 4, 5, \) are functions satisfying the inequalities \(|f_j| \leq C_7 r^{-1}, j = 2, 5, |f_j| \leq C_7 r^{s/2-2}, j = 3, 4.\) Using equality (26), we get the estimate
\[
\left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-s}|u_\varepsilon|^2) \, dx \right| \leq C_8 \varepsilon \int_{0}^{\infty} r^{-1}(\varepsilon^2 r^{-1/2} + \varepsilon^2 r^{-1/2}) \, dr = C_9 \varepsilon \int_{0}^{\infty} t^{-1}(\varphi^2(t) + \varphi'^2(t)) \, dt = C_{10} \varepsilon.
\]

It follows from (25) and (27) that for any \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that the function \( u_\varepsilon \) satisfies inequality (20). This implies \( \sigma(L) \cap (\lambda - \delta, \lambda + \delta) \neq \emptyset. \) Thus, \( \sigma(L) = [0, \infty). \) Point i) of Theorem 1 is proved.

Let us investigate now the case \( s = 2. \) Consider the functions
\[
u_\varepsilon(r, \Theta) = \sqrt{\varepsilon} r^{1-n/2} \varepsilon^{\lambda-(n-2)^2/4} \Lambda r \varphi(\varepsilon \ln r) v_{e^{1/\varepsilon}}(\Theta),
\]
where \( \varphi \) is the same as for \( 0 \leq s < 2. \)

In this case we have \( \text{supp} \, u_\varepsilon \subset \Omega, \, u_\varepsilon \in \dot{H}^1_2(\Omega), \) and the inequality (20) can be written as
\[
\left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-2}|u_\varepsilon|^2) \, dx \right| < \delta \int_{\mathbb{R}^n} r^{-2}|u_\varepsilon|^2 \, dx.
\]

Let us study the behavior of the left hand and right hand sides of the inequality (29) when \( \varepsilon \to 0. \) We have
\[
\left| \int_{\mathbb{R}^n} r^{-2}|u_\varepsilon|^2 \, dx \right| = \varepsilon \int_{0}^{\infty} r^{-1}\varphi^2(\varepsilon \ln r) \, dr \int_{\Sigma} v_{e^{1/\varepsilon}}^2 \, d\Theta = \int_{0}^{\infty} \varphi^2(t) \, dt \int_{\Sigma} v_{e^{1/\varepsilon}}^2 \, d\Theta = C_{11} \int_{\Sigma} v_{e^{1/\varepsilon}}^2 \, d\Theta,
\]

\[
\left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-2}|u_\varepsilon|^2) \, dx \right| = \left| \varepsilon^2 \int_{0}^{\infty} r^{-1}((2-n)\varphi(\varepsilon \ln r)\varphi'(\varepsilon \ln r) + \varepsilon\varphi'^2(\varepsilon \ln r)) \, dr \int_{\Sigma} v_{e^{1/\varepsilon}}^2 \, d\Theta \right|
\]
\[
+ \varepsilon \int_{0}^{\infty} r^{-1}\varphi^2(\varepsilon \ln r) \, dr \int_{\Sigma} (|\nabla \Theta v_{e^{1/\varepsilon}}|^2 - \Lambda v_{e^{1/\varepsilon}}^2) \, d\Theta \leq \int_{0}^{\infty} (\varepsilon(n-2)|\varphi(t)||\varphi'(t)|
\]
\[
+ \varepsilon^2 \varphi'^2(t) + (\Lambda(e^{1/\varepsilon}) - \Lambda)\varphi^2(t)) \, dt \int_{\Sigma} v_{e^{1/\varepsilon}}^2 \, d\Theta.
\]
From the relations (30), (31) we get that for any $\delta > 0$ there exists $\varepsilon > 0$ such that the function $u_\varepsilon$ satisfies the inequality (29). This implies that $\sigma(L) \cap (\lambda - \delta, \lambda + \delta) \neq \emptyset$. Thus, $\sigma(L) = [\frac{1}{4}(n - 2)^2 + \Lambda, \infty)$. Point ii) of Theorem 1 is proved.

5. ON THE RATE OF CONDENSATION OF THE DISCRETE SPECTRUM

Let $s > 2$ and let the spectrum of the operator $L$ be discrete. For any $\lambda \in (\frac{1}{4}(n - 2)^2 + \Lambda, \infty)$ consider functions

$$u_s(r, \Theta) = r^{1-n/2}e^{\sqrt{\lambda - (n-2)^2/4} - \Lambda \ln r} \eta_s(r) v_{\ln \ln(1/(s-2))}(\Theta),$$

where $\eta_s(r) = r/\ln \ln(1/(s-2)) - 1$ for $\ln \ln(1/(s-2)) < r < 2 \ln \ln(1/(s-2))$, $\eta_s(r) = 1$ for $2 \ln \ln(1/(s-2)) < r < \ln(1/(s-2))$, $\eta_s(r) = 2 - r/\ln(1/(s-2))$ for $\ln(1/(s-2)) < r < 2 \ln(1/(s-2))$ and $\eta_s(r) = 0$ in the other cases, the function $v_\varrho(\Theta)$ being the same as in the proof of point ii) of Theorem 1. Let us continue the function $v_\varrho$ by zero to $\Sigma$. As follows from (19), to prove the relation (3) it is sufficient for some $s_0 > 2$ and some constant $C > 0$ for all $2 < s < s_0$ establish inequality

$$\left| \int_{\mathbb{R}^n} (|\nabla u_s|^2 - \lambda r^{-s}|u_s|^2) \, dx \right| < \left( \hat{\lambda} \left( \ln \ln \frac{1}{s-2} \right) - \Lambda + C \frac{\ln \ln(1/(s-2))}{\ln \ln(1/(s-2))} \right) \int_{\mathbb{R}^n} r^{-s}|u_s|^2 \, dx.$$

By (32) we have

$$\int_{\mathbb{R}^n} r^{-s}|u_s|^2 \, dx$$

$$= \int_0^\infty r^{1-s}\eta_s^2(r) \, dr \int_{\Sigma} v_{\ln \ln(1/(s-2))}^2 \, d\Theta$$

$$= \left( \frac{\ln^{2-s} \ln(1/(s-2)) - \ln^{2-s}(1/(s-2))}{s-2} + O(1) \right) \int_{\Sigma} v_{\ln \ln(1/(s-2))}^2 \, d\Theta$$

$$= \left( \ln \ln \frac{1}{s-2} + O\left( \ln \ln \ln \frac{1}{s-2} \right) \right) \int_{\Sigma} v_{\ln \ln(1/(s-2))}^2 \, d\Theta.$$
Consider the behavior of the left hand side of the inequality (33) for $s \to 2 + 0$:

\[
(35) \left| \int_{\mathbb{R}^n} (|\nabla u_s|^2 - \lambda r^{-s}|u_s|^2) \, dx \right|
\]

\[
= \left| \int_0^\infty (\eta_s^2 + (2 - n)r^{-1}\eta_s\eta_s + (\lambda(r^{-1} - r^{-s}) - \Lambda r^{-1})\eta_s^2) \, dr \right.
\]

\[
\times \int_\Sigma v_{\ln(1/(s-2))}^2 \, d\Theta + \int_0^\infty r^{-1}\eta_s^2 \, dr \int_\Sigma |\nabla \Theta v_{\ln(1/(s-2))}|^2 \, d\Theta \right|
\]

\[
= \left| \int_0^\infty (\eta_s^2 + (2 - n)r^{-1}\eta_s\eta_s
\right.
\]

\[
+ (\lambda(r^{-1} - r^{-s}) + (\hat{\lambda}(\ln(1/(s-2)) - \Lambda))r^{-1})\eta_s^2 \, dr \int_\Sigma v_{\ln(1/(s-2))}^2 \, d\Theta
\]

\[
\leq \left( 3(n - 1) + \lambda \int_0^\infty (r^{-1} - r^{-s})\eta_s^2 \, dr + (\hat{\lambda}(\ln(1/(s-2)) - \Lambda) \int_0^\infty r^{-1}\eta_s^2 \, dr \right)
\]

\[
\times \int_\Sigma v_{\ln(1/(s-2))}^2 \, d\Theta
\]

\[
< \left( \hat{\lambda}(\ln(1/(s-2)) - \Lambda + C_{12} \frac{\ln(1/(s-2))}{\ln(1/(s-2))} \right)
\]

\[
\times \ln(1/(s-2)) \int_\Sigma v_{\ln(1/(s-2))}^2 \, d\Theta, \quad C_{12} > 0.
\]

Hence, the inequality (33) follows from (34), (35). Proof of Theorem 3 is complete.

6. Continuity of spectrum

Let us prove continuity of the spectrum of the operator $L$ for $0 \leq s \leq 2$. Let $\lambda > 0$ and $u \in D(L)$ be non-zero functions, satisfying the equation $\Delta u + \lambda r^{-s} u = 0$ and vanishing on $\Gamma$ (we consider the function $u$ to be real-valued). By Lemma 3 for $\Gamma \in C^2$ we have the inclusion $u \in H^2(\Omega_R)$, $R > 0$. We multiply the equation by $2ru_r$ and integrate over the domain $\Omega_R$. Thus we have the equality

\[
(36) \quad R \int_{S_R} \left( 2u_r^2 - |\nabla u|^2 + \lambda R^{-s}u^2 + \frac{n-2}{R} uu_r \right) \, ds_x
\]

\[
+ \int_{\Gamma_R} (\nu, x) \left( \frac{\partial u}{\partial \nu} \right)^2 \, ds_x - \lambda(2-s) \int_{\Omega_R} r^{-s}u^2 \, dx = 0,
\]
where $\nu$ is the outward unit normal vector to $\Gamma$. From (36) we get the inequalities:

\[
R \frac{1}{2} \int_{S_R} (nu_r^2 + (2\lambda + (n - 2))R^{-s}u^2) \, ds_x \\
\geq R \int_{S_R} (u_r^2 + \lambda R^{-s}u^2 + \frac{n-2}{2}(u_r^2 + R^{-2}u^2)) \, ds_x \\
\geq R \int_{S_R} \left( u_r^2 + \lambda R^{-s}u^2 + \frac{n-2}{R} u_r u \right) \, ds_x \geq -\int_{\Gamma_R} (\nu, x) \left( \frac{\partial u}{\partial \nu} \right)^2 \, ds_x.
\]

Let us note that the star-shapeness condition for the set $\mathbb{R}^n \setminus \Omega$ for a smooth surface $\Gamma$ means that $(\nu, x) \leq 0$, $x \in \Gamma$. The surface $\Gamma$ is not a cone, so there exists a point $x_0 \in \Gamma$ such that $(\nu, x_0) < 0$. So, $u|_\Gamma = 0$, and then by the uniqueness theorem for the solution of the Cauchy problem for elliptic equations ([7]) there exists a neighborhood $U(x_0)$ such that $\int_{\Gamma \cap U(x_0)} (\nu, x)(\partial u/\partial \nu)^2 \, ds_x < 0$. Therefore, $\int_{S_R} (u_r^2 + R^{-s}u^2) \, ds_x \geq C_{13} R^{-1}$, $C_{13} > 0$, $R \geq R_0$ and $\|u\|_{H^1(\Omega)}^2 \geq \int_{\Omega \setminus \{r \geq R_0\}} (u_r^2 + r^{-s}u^2) \, dx = \int_{R_0}^{\infty} dr \int_{S_R} (u_r^2 + r^{-s}u^2) \, ds_x \geq C_{13} \int_{R_0}^{\infty} r^{-1} \, dr = +\infty$, i.e. $u$ is not an eigenfunction of the operator $L$. This completes the proof of point i) of Theorem 2.

References


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