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SPECTRUM OF THE WEIGHTED LAPLACE OPERATOR IN UNBOUNDED DOMAINS

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Abstract. We investigate the spectral properties of the differential operator $-r^s\Delta$, $s \geqslant 0$ with the Dirichlet boundary condition in unbounded domains whose boundaries satisfy some geometrical condition. Considering this operator as a self-adjoint operator in the space with the norm $\|u\|^2_{L_{2,s}(\Omega)} = \int_\Omega r^{-s}|u|^2\,\mathrm{d}x$, we study the structure of the spectrum with respect to the parameter s. Further we give an estimate of the rate of condensation of discrete spectra when it changes to continuous.

Keywords: Laplace operator, multiplicative perturbation, Dirichlet problem, Friedrichs extension, purely discrete spectra, purely continuous spectra

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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be an unbounded domain whose closure does not contain the origin, with a boundary Γ. Let us consider the differential expression

$$lu = -r^s\Delta u, \quad r = |x|, \quad s \geqslant 0. \tag{1}$$

We shall treat the differential operator (1) in the Hilbert space $L_{2,s}(\Omega)$ with the norm $\|u\|^2_{L_{2,s}(\Omega)} = \int_\Omega r^{-s}|u|^2\,\mathrm{d}x$. Let L be the self-adjoint Friedrichs extension in $L_{2,s}(\Omega)$ of the minimal operator generated by the differential expression (1). Then, L is a non-negative self-adjoint operator in $L_{2,s}(\Omega)$ that is an operator of the first boundary value problem for the differential expression (1).

We will study spectral properties of the operator L (location of spectrum on the real axis, density of spectrum on some sets, structure of the spectrum) with respect

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to the parameter s. For the operator (1) we know conditions on s and the domain Ω, guaranteeing the discreteness of the spectrum of the operator L ([1]). In particular, in [1] it was proved that the spectrum of the operator L is discrete at $s > 2$. On the other side, as follows from the results of [2], in the case $\Omega = \mathbb{R}^n$ with $0 \leqslant s \leqslant 2$ the spectrum of the operator with coefficients equal to (1) in the neighborhood of infinity is continuous and for $0 \leqslant s < 2$ fills the complete positive semi-axis.

Let $S_\eta = \Omega \cap \{r = \eta\}$, $\eta > 0$, and Σ_η be the set of points x belonging to the unit sphere Σ and satisfying $\eta x \in S_\eta$. In the sequel we will consider domains Ω such that

$$\Sigma_{\eta_1} \subset \Sigma_{\eta_2}, \quad \eta_1 < \eta_2, \tag{2}$$

(it is the star-shapeness condition for the set $\mathbb{R}^n \setminus \Omega$ with respect to the origin). Denote by $\hat{\lambda}(\eta)$ the modulus of the first eigenvalue of the Laplace-Beltrami operator in Σ_η with zero Dirichlet data on $\partial\Sigma_\eta$. By our supposition $\hat{\lambda}(\eta)$ is a decreasing nonnegative function on $[\inf_{x\in\Omega} r, +\infty)$. Denote $\Lambda = \lim\limits_{\eta\to\infty} \hat{\lambda}(\eta)$. We will also suppose without loss of generality that $\ln r > 1$ in Ω.

Our first statement localizes the spectrum set $\sigma(L)$ of the operator L on the real axis.

Theorem 1. *The spectrum of L has the following properties:*
i) *if $0 \leqslant s < 2$, then $\sigma(L) = [0, +\infty)$;*
ii) *if $s = 2$, then $\sigma(L) = [\frac{1}{4}(n-2)^2 + \Lambda, +\infty)$;*
iii) *if $s > 2$, then $\sigma(L) \subset (\frac{1}{4}(n-2)^2 + \Lambda, +\infty)$.*

The next statement declares that there exists a critical value of s for which the spectrum of L becomes discrete.

Theorem 2. *The spectrum of L has the following properties:*
i) *if $0 \leqslant s \leqslant 2$ and $\Gamma \in C^2$, then the spectrum of the operator L is continuous;*
ii) *if $s > 2$, then the spectrum of the operator L is discrete.*

It is natural to expect that the discrete spectrum condenses on the semi-axis $[\frac{1}{4}(n-2)^2 + \Lambda, +\infty)$ at $s \to 2+0$. In the next statement we establish an estimate of the rate of this condensation.

Theorem 3. *For any $\lambda \in [\frac{1}{4}(n-2)^2 + \Lambda, +\infty)$ there exist a constant $C > 0$ and a number $s_0 > 2$ such that for any $s \in (2, s_0]$ the following relation holds:*

$$\sigma(L) \cap (\lambda - \delta(s), \lambda + \delta(s)) \neq \emptyset, \tag{3}$$

where
$$\delta(s)=\hat\lambda(\ln\ln(1/(s-2)))-\Lambda+C\,\frac{\ln\ln\ln(1/(s-2))}{\ln\ln(1/(s-2))}.$$
The constant C depends on λ.

2. Energy space and domain

Let us define the space of functions
$$H^1_s(\Omega)=\{u\colon\ u\in L_{2,s}(\Omega)\cap H^1(\Omega_R),\ R>0,\ u_{x_j}\in L_2(\Omega),\ j=1,\dots,n\},$$
where $\Omega_R=\Omega\cap\{r<R\}$, with the norm $\|u\|^2_{H^1_s(\Omega)}=\int_\Omega(|\nabla u|^2+r^{-s}|u|^2)\,\mathrm{d}x$. By $\overset{\circ}{H}{}^1_s(\Omega)$ denote the subspace of $H^1_s(\Omega)$ which is the closure of the set of functions $u\in H^1_s(\Omega)$ vanishing in a neighborhood of Γ. Consider the quadratic form $A[u]=\int_\Omega|\nabla u|^2\,\mathrm{d}x$ on the set of functions $\overset{\circ}{C}{}^\infty(\Omega)\subset L_{2,s}(\Omega)$.

Lemma 1. *The form $A[u]$ is closeable.*

P r o o f. Let $\{u_j\}\subset\overset{\circ}{C}{}^\infty(\Omega)$, $j=1,2,\dots$ be a sequence of functions such that $A[u_j-u_l]\to0$, $j,l\to\infty$ and $\|u_j\|_{L_{2,s}(\Omega)}\to0$, $j\to\infty$. Now, by $\|u_j\|^2_{H^1_s(\Omega)}=A[u_j]+\|u_j\|^2_{L_{2,s}(\Omega)}$ we have that the sequence u_j is fundamental in the space $H^1_s(\Omega)$. By $\hat u\in H^1_s(\Omega)$ denote the limit function: $\lim\limits_{j\to\infty}\|\hat u-u_j\|_{H^1_s(\Omega)}=0$. Then $\lim\limits_{j\to\infty}\|u_j-\hat u\|_{L_{2,s}(\Omega)}=0$, i.e. $\|\hat u\|_{L_{2,s}(\Omega)}\leqslant\lim\limits_{j\to\infty}\|u_j\|_{L_{2,s}(\Omega)}+\lim\limits_{j\to\infty}\|\hat u-u_j\|_{L_{2,s}(\Omega)}=0$. Hence, $\hat u=0$ and $\lim\limits_{j\to\infty}\|u_j\|_{L_{2,s}(\Omega)}=0$. So, the possibility to close the form $A[u]$ is proved.

By Lemma 1 the energy space H_A of the operator L is the closure of the set of functions $\overset{\circ}{C}{}^\infty(\Omega)$ in the norm $\|u\|^2_{H^1_s(\Omega)}=A[u]+\|u_j\|^2_{L_{2,s}(\Omega)}$.

Lemma 2. *The energy space of the operator L is*
$$H_A=\Big\{u\colon\ u\in\overset{\circ}{H}{}^1_s(\Omega),\int_\Omega r^{-2}\ln^{-2q}r|u|^2\,\mathrm{d}x<\infty\Big\},\tag{4}$$
where $q=0$ for $n\geqslant3$ and $q=1$ for $n=2$.

P r o o f. It is sufficient to prove that for any function $u\in\overset{\circ}{H}{}^1_s(\Omega)$ such that $\int_\Omega r^{-2}\ln^{-2q}r|u|^2\,\mathrm{d}x<\infty$ and for any $\varepsilon>0$ there exists a function $\tilde u\in\overset{\circ}{C}{}^\infty(\Omega)$, such that $A[u-\tilde u]<\varepsilon$.

First let us prove that for any function $u \in H_A$ the integral $\int_\Omega r^{-2} \ln^{-2q} r|u|^2 \,\mathrm{d}x$ converges. We use the inequalities

$$\int_{\Omega_R} r^{-2} \ln^{-2} r|u|^2 \,\mathrm{d}x \leqslant 4 \int_{\Omega_R} |u_r|^2 \,\mathrm{d}x, \quad n = 2, \tag{5}$$

$$\int_{\Omega_R} r^{-2}|u|^2 \,\mathrm{d}x \leqslant \frac{2}{(n-2)R} \int_{S_R} |u|^2 \,\mathrm{d}s + \frac{4}{(n-2)^2} \int_{\Omega_R} |u_r|^2 \,\mathrm{d}x, \quad n \geqslant 3, \tag{6}$$

which are valid for all $R > 1$ for functions $u \in H^1(\Omega_R)$ such that $u|_\Gamma = 0$.

For any function $u \in H_A$ there exists a sequence of functions $\{u_j\} \subset \overset{\circ}{C}{}^\infty(\Omega)$, $j = 1, 2, \dots$ such that $A[u - u_j] \to 0$, $\|u - u_j\|_{L_{2,s}(\Omega)} \to 0$, $j \to \infty$. Apply (5) (6) to u_j with sufficiently large R. Since the term on the right hand side containing $(u_j)_r$ is bounded for all j and R, we obtain $\int_\Omega r^{-2} \ln^{-2q} r|u_j|^2 \,\mathrm{d}x \leqslant C_1$. Since $r^{-1} \ln^{-q} r u_j$ must converge in $L_2(\Omega)$ weakly to $r^{-1} \ln^{-q} r u$, we obtain

$$\int_\Omega r^{-2} \ln^{-2q} r|u|^2 \,\mathrm{d}x \leqslant C_1. \tag{7}$$

Conversely, let us suppose that $u \in \overset{\circ}{H}{}^1_s(\Omega)$ and $\int_\Omega r^{-2} \ln^{-2q} r|u|^2 \,\mathrm{d}x < \infty$. Let $n \geqslant 3$. Consider functions $\xi_m(x) = \eta(\ln(r/m) + 1)$, $m = 1, 2, \dots$, where $\eta(t) \in \overset{\circ}{C}{}^\infty([0, +\infty))$ is a nonnegative function satisfying the condition $0 \leqslant \eta \leqslant 1$ and such that $\eta = 1$ for $0 < t < 1$, $\eta = 0$ for $t > 2$. Hence $\xi_m = 1$, $x \in \Omega_m$, $\xi_m = 0$, $x \in \Omega \setminus \Omega_{me}$. We have an estimate

$$|\nabla \xi_m| = \left| \frac{\mathrm{d}}{\mathrm{dr}} \xi_m \right| = |\eta'(\ln(r/m) + 1)| r^{-1} \leqslant C_2 r^{-1}, \quad x \in \Omega. \tag{8}$$

The function $u\xi_m$ belongs to the space H_A. Let us prove that $\|u - u\xi_m\|_{H^1_s(\Omega)} \to 0$, $m \to \infty$. We get

$$\|u - u\xi_m\|^2_{H^1_s(\Omega)} \leqslant 2(I_{1,m} + I_{2,m}), \tag{9}$$

where

$$I_{1,m} = \int_{\Omega \setminus \Omega_m} (|\nabla u|^2 + r^{-s}|u|^2)(1 - \xi_m)^2 \,\mathrm{d}x, \quad I_{2,m} = \int_{\Omega_{me} \setminus \Omega_m} |u|^2 |\nabla \xi_m|^2 \,\mathrm{d}x.$$

Since $u \in \overset{\circ}{H}{}^1_s(\Omega)$, we obtain $I_{1,m} \to 0$, $m \to \infty$. Furthermore, it follows from (7), (8) that

$$I_{2,m} \leqslant C_2^2 \int_{\Omega_{me} \setminus \Omega_m} r^{-2}|u|^2 \,\mathrm{d}x \to 0, \quad m \to \infty.$$

Thus, $\|u - u\xi_m\|_{H_s^1(\Omega)} \to 0$, $m \to \infty$. Now, by virtue of $u\xi_m \in \overset{\circ}{H}{}_s^1(\Omega_{me})$ there exist functions $\tilde{u}_m \in \overset{\circ}{C}{}^\infty(\Omega_{me})$, such that $A[\tilde{u}_m - u\xi_m] \leqslant \|\tilde{u}_m - u\xi_m\|^2_{H_s^1(\Omega_{me})} \to 0$, $m \to \infty$. Consider the zero continuation of the function $\tilde{u}_m$ to the set $\Omega \setminus \Omega_{me}$ and denote the continued function also by $\tilde{u}_m$. Therefore $\tilde{u}_m \in \overset{\circ}{C}{}^\infty(\Omega)$ and $A[u - \tilde{u}_m] \leqslant 2(A[u - u\xi_m] + A[u\xi_m - \tilde{u}_m]) \to 0$, $m \to \infty$. The existence of a function $\tilde{u} \in \overset{\circ}{C}{}^\infty(\Omega)$ such that $A[u - \tilde{u}] < \varepsilon$ in the case $n \geqslant 3$ is proved.

Let us consider the case $n = 2$. Put $\xi_m(x) = \eta(\ln(\ln r/\ln m))$ where the function η is the same as for $n \geqslant 3$. Then $\xi_m = 1$ for $x \in \Omega_{m^e}$ and $\xi_m = 0$ for $x \in \Omega \setminus \Omega_{m^{e^2}}$. For the function ξ_m we obtain

$$|\nabla \xi_m| = \left|\frac{\mathrm{d}}{\mathrm{dr}}\xi_m\right| = |\eta'(\ln(\ln r/\ln m))|(r\ln r)^{-1} \leqslant C_2(r\ln r)^{-1}, \quad x \in \Omega. \tag{10}$$

The estimate (9) with

$$I_{1,m} = \int_{\Omega\setminus\Omega_{m^e}} (|\nabla u|^2 + r^{-s}|u|^2)(1-\xi_m)^2\,\mathrm{d}x, \quad I_{2,m} = \int_{\Omega_{m^{e^2}}\setminus\Omega_{m^e}} |u|^2|\nabla\xi_m|^2\,\mathrm{d}x$$

holds. As in the case $n \geqslant 3$, we obtain that $I_{1,m} \to 0$, $m \to \infty$. It follows from the estimate (7) with $q = 1$ and (10) that

$$I_{2,m} \leqslant C_2^2 \int_{\Omega_{m^{e^2}}\setminus\Omega_{m^e}} r^{-2}\ln^{-2} r|u|^2\,\mathrm{d}x \to 0, \quad m \to \infty.$$

Thus, $A[u - u\xi_m] \to 0$, $m \to \infty$. Now, we get the existence of a sequence $\tilde{u}_m \in \overset{\circ}{C}{}^\infty(\Omega)$, $\operatorname{supp}\tilde{u}_m \subset \Omega_{m^{e^2}}$ such that $A[\tilde{u}_m - u\xi_m] \leqslant \|\tilde{u}_m - u\xi_m\|^2_{H_s^1(\Omega_{m^{e^2}})} \to 0$, $m \to \infty$. Hence the existence of a function $\tilde{u} \in \overset{\circ}{C}{}^\infty(\Omega)$ such that $A[u - \tilde{u}] < \varepsilon$ for $n = 2$ is proved. This completes the proof of Lemma 2.

Lemma 3. *The domain of the operator L is*

$$D(L) = \left\{u\colon\ u \in \overset{\circ}{H}{}_s^1(\Omega) \cap H^2_{\mathrm{loc}}(\Omega), lu \in L_{2,s}(\Omega), \int_\Omega r^{-2}\ln^{-2q} r|u|^2\,\mathrm{d}x < \infty\right\}.$$

In the case $\Gamma \in C^2$ the domain of the operator L is

$$D(L) = \left\{u\colon\ u \in \overset{\circ}{H}{}_s^1(\Omega) \cap H^2(\Omega_R), R > 0, lu \in L_{2,s}(\Omega), \int_\Omega r^{-2}\ln^{-2q} r|u|^2\,\mathrm{d}x < \infty\right\}.$$

P r o o f. Applying interior estimates for the derivatives of solutions of elliptic equations ([3], p. 204, Lemma 7.1) to $u \in \overset{\circ}{H}{}_s^1(\Omega)$, $lu \in L_{2,s}(\Omega)$ and any domain

$\Omega' \Subset \Omega$, we get $u \in H^2(\Omega')$. If, furthermore, $\Gamma \in C^2$, applying the boundary estimates for derivatives of solutions of elliptic equations ([3], p. 224, Theorem 9.2), for all $R > 0$ we obtain $u \in H^2(\Omega_R)$. This completes the proof of Lemma 3.

3. Localization of spectrum

It follows from the inequalities (5), (6) that for functions $u \in H_A$ we have the lower estimate

$$\begin{aligned} A[u] &= \int_\Omega |u_r|^2 \,\mathrm{d}x + \int_\Omega r^{-2} |\nabla_\Theta u|^2 \,\mathrm{d}x \\ &\geqslant \frac{(n-2-q)^2}{4} \int_\Omega r^{-2} \ln^{-2q} r |u|^2 \,\mathrm{d}x + \int_\Omega r^{-2} |\nabla_\Theta u|^2 \,\mathrm{d}x, \quad u(x) = u(r,\Theta). \end{aligned}$$

Since $\hat{\lambda}(r)$ is the modulus of the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator in Σ_r, we get

$$\int_{\Sigma_r} |\nabla_\Theta u|^2 \,\mathrm{d}\Theta \geqslant \hat{\lambda}(r) \int_{\Sigma_r} |u|^2 \,\mathrm{d}\Theta \geqslant \Lambda \int_{\Sigma_r} |u|^2 \,\mathrm{d}\Theta, \quad r > 0. \tag{11}$$

Therefore,

$$\begin{aligned} A[u] &\geqslant \frac{(n-2-q)^2}{4} \int_\Omega r^{-2} \ln^{-2q} r |u|^2 \,\mathrm{d}x \\ &\quad + \Lambda \int_\Omega r^{-2} |u|^2 \,\mathrm{d}x \geqslant \Big(\frac{(n-2)^2}{4} + \Lambda \Big) \int_\Omega r^{-2} |u|^2 \,\mathrm{d}x, \end{aligned} \tag{12}$$

and for $s \geqslant 2$ we have an estimate $A[u] \geqslant (\frac{1}{4}(n-2)^2 + \Lambda) \|u\|^2_{L_{2,s}(\Omega)}$ and, consequently, $\sigma(L) \subset [\frac{1}{4}(n-2)^2 + \Lambda, +\infty)$, $s \geqslant 2$.

Let us prove that for $s > 2$ the number $\frac{1}{4}(n-2)^2 + \Lambda$ does not belong to the spectrum of the operator L. Assume the converse, let $s > 2$ and $\frac{1}{4}(n-2)^2 + \Lambda \in \sigma(L)$. We show that there exists a non-zero function $\hat{u} \in H_A$ such that

$$\int_\Omega |\nabla \hat{u}|^2 \,\mathrm{d}x = \Big(\frac{(n-2)^2}{4} + \Lambda \Big) \int_\Omega r^{-s} |\hat{u}|^2 \,\mathrm{d}x. \tag{13}$$

If $\frac{1}{4}(n-2)^2 + \Lambda$ is an eigenvalue of the operator L, the relation (13) holds for the corresponding eigenfunction $\hat{u}$. If $\frac{1}{4}(n-2)^2 + \Lambda$ is the continuous spectrum point, let us use I. M. Glazman lemma for quadratic forms ([5], Supplement 1, Lemma 3.1′), which is a modification of the corresponding operator statement ([4], Chapter 1, Section 1, Theorem 9^{bis}). By this lemma

$$N(\lambda - 0) = \sup_{\{F \subset H_A,\, A[u] < \lambda \|u\|^2_H,\, u \in F \setminus \{0\}\}} \dim F, \tag{14}$$

where H is the main space ($H = L_{2,s}(\Omega)$ in our case), F is a linear subspace of H_A, $N(\lambda) = \dim(E_\lambda H)$ where E_λ denotes the spectral projector of the spectral family corresponding to the self-adjoint operator L. As follows from this lemma, if λ is a continuous spectrum point, for any $\delta > 0$ the relation $N(\lambda+\delta) - N(\lambda-\delta) = \infty$ holds. Thus for any $\delta > 0$ there exists a function $u \in H_A$, $u \neq 0$, such that $A[u] \leqslant (\frac{1}{4}(n-2)^2 + \Lambda + \delta)\|u\|^2_{L_{2,s}(\Omega)}$.

Let us choose a sequence $\delta_j > 0$, $\delta_j \to 0$, $j \to \infty$. Then there exists a non-zero sequence $u_j \in H_A$ such that

$$\int_\Omega |\nabla u_j|^2 \,\mathrm{d}x \leqslant \Big(\frac{(n-2)^2}{4} + \Lambda + \delta_j\Big) \int_\Omega r^{-s}|u_j|^2 \,\mathrm{d}x. \tag{15}$$

Let $\|u_j\|_{L_{2,s}(\Omega)} = 1$. Then $\int_\Omega |\nabla u_j|^2 \,\mathrm{d}x \leqslant \frac{1}{4}(n-2)^2 + \Lambda + \delta_j$ and, clearly, the inequalities (5), (6) imply

$$\int_\Omega r^{-2} \ln^{-2q} r|u_j|^2 \,\mathrm{d}x \leqslant C_3. \tag{16}$$

From the sequence $\{u_j\}$ let us choose a subsequence which is weakly convergent in the space $L_{2,s}(\Omega)$ and show that it is pre-compact in $L_{2,s}(\Omega)$. In the same way as in Rellich's theorem about the compact imbedding of $\overset{\circ}{H}{}^1(\Omega')$ into $L_2(\Omega')$ in a bounded domain Ω', we can prove that for any $R > 0$ the space $H^1_s(\Omega_R)$ imbeds compactly into $L_{2,s}(\Omega_R)$. Hence, there exists a function $\hat{u} \in L_{2,s}(\Omega)$ such that for any $R > 0$ we have $\lim\limits_{j\to\infty} \|\hat{u} - u_j\|_{L_{2,s}(\Omega_R)} = 0$. It means that for any sequence $\{R_j\}$, $R_j \to \infty$, $j \to \infty$, it is possible to choose a subsequence $\{u_j\}$ (denoted also by $\{u_j\}$) such that $\|\hat{u} - u_j\|_{L_{2,s}(\Omega_{R_j})} < j^{-1}$. Therefore by (16) we have

$$\begin{aligned}
\|\hat{u} - u_j\|^2_{L_{2,s}(\Omega)} &= \|\hat{u} - u_j\|^2_{L_{2,s}(\Omega_{R_j})} + \|\hat{u} - u_j\|^2_{L_{2,s}(\Omega\setminus\Omega_{R_j})} \\
&< j^{-2} + 2\big(\|\hat{u}\|^2_{L_{2,s}(\Omega\setminus\Omega_{R_j})} + \|u_j\|^2_{L_{2,s}(\Omega\setminus\Omega_{R_j})}\big) \\
&\leqslant j^{-2} + C_4 R_j^{2-s} \ln^{2q} R_j \to 0, \quad j \to \infty.
\end{aligned}$$

So, the convergence of the sequence $\{u_j\}$ to $\hat{u}$ in the space $L_{2,s}(\Omega)$ is proved. This, in particular, yields that $\|\hat{u}\|_{L_{2,s}(\Omega)} = 1$. It implies that

$$\int_\Omega |\nabla \hat{u}|^2 \,\mathrm{d}x \leqslant \liminf\nolimits_{j\to\infty} \int_\Omega |\nabla u_j|^2 \,\mathrm{d}x = \Big(\frac{(n-2)^2}{4} + \Lambda\Big) \int_\Omega r^{-s}|\hat{u}|^2 \,\mathrm{d}x. \tag{17}$$

The relation (13) is proved.

For the proof of the relation $\frac{1}{4}(n-2)^2+\Lambda \notin \sigma(L)$ let us first consider the case $\frac{1}{4}(n-2)^2+\Lambda > 0$. In this case

$$A[\hat{u}] = \Big(\frac{(n-2)^2}{4}+\Lambda\Big)\int_\Omega r^{-s}|\hat{u}|^2\,\mathrm{d}x < \Big(\frac{(n-2)^2}{4}+\Lambda\Big)\int_\Omega r^{-2}|\hat{u}|^2\,\mathrm{d}x, \tag{18}$$

which contradicts (12).

In the case $\frac{1}{4}(n-2)^2+\Lambda = 0$ we have by (13) the equality $\int_\Omega |\nabla\hat{u}|^2\,\mathrm{d}x = 0$. Thus, $\nabla\hat{u} = 0$ and $\hat{u} = \text{const}$. But then $\hat{u} = 0$, which contradicts $\|\hat{u}\|_{L_{2,s}(\Omega)} = 1$. So, $\frac{1}{4}(n-2)^2+\Lambda \notin \sigma(L)$ and the relation $\sigma(L) \subset (\frac{1}{4}(n-2)^2+\Lambda, +\infty)$ holds true. Proof of the point iii) of Theorem 1 is now complete.

Finally, for $s > 2$ we have that any sequence bounded in the space H_A is precompact in $L_{2,s}(\Omega)$. By F. Rellich criterion ([5], Supplement 1, Par. 3), the spectrum of the operator L is discrete at $s > 2$. This completes the proof of the point ii) of Theorem 2.

4. Density of spectrum on the semi-axis

First consider the case $0 \leqslant s < 2$. Let us use the relation (14). By this relation the number of points of the spectrum for the operator L in the interval $(\lambda-\delta, \lambda+\delta)$ with account of multiplicity is equal to the maximal dimension of the linear manifolds $F \subset H_A$ for which the following inequality is valid:

$$\big|A[u] - \lambda\|u\|_H^2\big| < \delta\|u\|_H^2, \quad u \neq 0. \tag{19}$$

In our case the relation (19) can be written as

$$\Big|\int_\Omega (|\nabla u|^2 - \lambda r^{-s}|u|^2)\,\mathrm{d}x\Big| < \delta\int_\Omega r^{-s}|u|^2\,\mathrm{d}x. \tag{20}$$

Denote by $v_\varrho(\Theta) \in \overset{\circ}{H}{}^1(\Sigma_\varrho)$, $\varrho > 0$, the first eigenfunction of the Laplace-Beltrami operator in the domain Σ_ϱ. Hence $\int_{\Sigma_\varrho} |\nabla_\Theta v_\varrho|^2\,\mathrm{d}\Theta = \hat{\lambda}(\varrho)\int_{\Sigma_\varrho} v_\varrho^2\,\mathrm{d}\Theta$. Let us continue the function v_ϱ by zero to the set Σ. Therefore $v_\varrho \in H^1(\Sigma)$ and $\int_\Sigma |\nabla_\Theta v_\varrho|^2\,\mathrm{d}\Theta = \hat{\lambda}(\varrho)\int_\Sigma v_\varrho^2\,\mathrm{d}\Theta$. We choose a nonzero real-valued function $\varphi(t) \in \overset{\circ}{C}{}^\infty(0,+\infty)$ such that $\operatorname{supp}\varphi = [1,2]$. Consider functions

$$u_\varepsilon(r,\Theta) = \sqrt{\varepsilon}r^{1-n/2}H^{(1)}_{\frac{n-2}{2-3}}\Big(\frac{2\sqrt{\lambda}}{2-s}\,r^{1-s/2}\Big)\varphi(\varepsilon r^{1-s/2})v_{\varepsilon^{-2/(2-s)}}(\Theta), \tag{21}$$
$$\varepsilon > 0,\ \lambda > 0,$$

where $H_p^{(1)}(z)$ is a Hankel function. In this case we have $\operatorname{supp} u_\varepsilon \subset \Omega$, $u_\varepsilon \in \overset{\circ}{H}{}_s^1(\Omega)$ and the inequality (20) has the form

$$\left| \int_{R^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-s} |u_\varepsilon|^2) \, \mathrm{d}x \right| < \delta \int_{R^n} r^{-s} |u_\varepsilon|^2 \, \mathrm{d}x. \tag{22}$$

Consider the behavior of the left hand and right hand parts of the inequality (22) for $\varepsilon \to 0$. Let us note that $\inf\limits_{x \in \operatorname{supp} u_\varepsilon} r > \varepsilon^{-2/(2-s)} \to \infty$, $\varepsilon \to 0$. Now we use the relations for derivatives of Hankel functions and the asymptotic expansions of Hankel functions of large argument ([6], Chapter 9):

$$H_p^{(1)\prime}(z) = p z^{-1} H_p^{(1)}(z) - H_{p+1}^{(1)}(z), \tag{23}$$

$$H_p^{(1)}(z) = \sqrt{2(\pi z)^{-1}} \exp\bigl(\mathrm{i}(z - \pi(p + \tfrac{1}{2})/2)\bigr)(1 + O(|z|^{-1})), \quad |z| \to \infty. \tag{24}$$

By (24) we have

$$\int_{R^n} r^{-s} |u_\varepsilon|^2 \, \mathrm{d}x = \frac{\varepsilon(2-s)}{\pi\sqrt{\lambda}} \int_0^\infty (r^{-s/2} + f_1(r)) \varphi^2(\varepsilon r^{1-s/2}) \, \mathrm{d}r \int_\Sigma v^2_{\varepsilon^{-2/(2-s)}} \, \mathrm{d}\Theta,$$

where $|f_1(r)| \leqslant C_5 r^{-1}$. Therefore,

$$\begin{aligned} \int_{R^n} r^{-s} |u_\varepsilon|^2 \, \mathrm{d}x &= \left(\frac{2\varepsilon}{\pi\sqrt{\lambda}} \int_0^\infty \varphi^2(\varepsilon z) \, \mathrm{d}z + J_1(\varepsilon) \right) \int_\Sigma v^2_{\varepsilon^{-2/(2-s)}} \, \mathrm{d}\Theta \\ &= \left(\frac{2}{\pi\sqrt{\lambda}} \int_0^\infty \varphi^2(t) \, \mathrm{d}t + J_1(\varepsilon) \right) \int_\Sigma v^2_{\varepsilon^{-2/(2-s)}} \, \mathrm{d}\Theta, \end{aligned}$$

where

$$|J_1| \leqslant \frac{(2-s)\varepsilon}{\pi\sqrt{\lambda}} \int_0^\infty |f_1(r)| \, \varphi^2(\varepsilon r^{1-s/2}) \, \mathrm{d}r \leqslant \frac{2C_5\varepsilon}{\pi\sqrt{\lambda}} \int_0^\infty t^{-1} \varphi^2(t) \, \mathrm{d}t.$$

Thus,

$$\int_{\mathbb{R}^n} r^{-s} |u_\varepsilon|^2 \, \mathrm{d}x = (C_6 + O(\varepsilon)) \int_\Sigma v^2_{\varepsilon^{-2/(2-s)}} \, \mathrm{d}\Theta, \quad \varepsilon \to 0, \ C_6 > 0. \tag{25}$$

Consider now the left hand side of the relation (22). It follows from (23)–(24) that

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-s} |u_\varepsilon|^2) \, \mathrm{d}x \right| \\ &\quad = \frac{\varepsilon(2-s)}{\pi\sqrt{\lambda}} \left| \int_0^\infty ((f_2(r) + \hat{\lambda}(\varepsilon^{-2/(2-s)}) f_3(r)) \varphi^2(\varepsilon r^{1-s/2}) \right. \\ &\qquad + \varepsilon((2-s)(2-2n+s)(4r)^{-1} + f_4(r)) \varphi(\varepsilon r^{1-s/2}) \varphi'(\varepsilon r^{1-s/2}) \\ &\qquad \left. + \varepsilon^2((2-s)^2 r^{-s/2}/4 + f_5(r)) {\varphi'}^2(\varepsilon r^{1-s/2})) \, \mathrm{d}r \right| \int_\Sigma v^2_{\varepsilon^{-2/(2-s)}} \, \mathrm{d}\Theta, \end{aligned} \tag{26}$$

where f_j, $j = 2, 3, 4, 5$, are functions satisfying the inequalities $|f_j| \leqslant C_7 r^{-1}$, $j = 2, 5$, $|f_j| \leqslant C_7 r^{s/2-2}$, $j = 3, 4$. Using equality (26), we get the estimate

$$\begin{aligned}
(27) \quad \left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-s} |u_\varepsilon|^2) \, \mathrm{d}x \right| &\leqslant C_8 \varepsilon \int_0^\infty r^{-1} (\varphi^2(\varepsilon r^{1-s/2}) + {\varphi'}^2(\varepsilon r^{1-s/2})) \, \mathrm{d}r \\
&= C_9 \varepsilon \int_0^\infty t^{-1} \big(\varphi^2(t) + {\varphi'}^2(t)\big) \, \mathrm{d}t = C_{10} \varepsilon.
\end{aligned}$$

It follows from (25) and (27) that for any $\delta > 0$ there exists $\varepsilon > 0$ such that the function u_ε satisfies inequality (20). This implies $\sigma(L) \cap (\lambda - \delta, \lambda + \delta) \neq \emptyset$. Thus, $\sigma(L) = [0, \infty)$. Point i) of Theorem 1 is proved.

Let us investigate now the case $s = 2$. Consider the functions

$$\begin{gathered}
(28) \quad u_\varepsilon(r, \Theta) = \sqrt{\varepsilon} r^{1-n/2} \mathrm{e}^{\mathrm{i}\sqrt{\lambda - (n-2)^2/4 - \Lambda} \ln r} \varphi(\varepsilon \ln r) v_{e^{1/\varepsilon}}(\Theta), \\
\lambda > \tfrac{1}{4}(n-2)^2 + \Lambda, \ \varepsilon > 0,
\end{gathered}$$

where the function φ is the same as for $0 \leqslant s < 2$.

In this case we have $\operatorname{supp} u_\varepsilon \subset \Omega$, $u_\varepsilon \in \overset{\circ}{H}{}^1_2(\Omega)$, and the inequality (20) can be written as

$$(29) \quad \left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-2} |u_\varepsilon|^2) \, \mathrm{d}x \right| < \delta \int_{\mathbb{R}^n} r^{-2} |u_\varepsilon|^2 \, \mathrm{d}x.$$

Let us study the behavior of the left hand and right hand sides of the inequality (29) when $\varepsilon \to 0$. We have

$$\begin{aligned}
(30) \quad & \int_{\mathbb{R}^n} r^{-2} |u_\varepsilon|^2 \, \mathrm{d}x \\
&= \varepsilon \int_0^\infty r^{-1} \varphi^2(\varepsilon \ln r) \, \mathrm{d}r \int_\Sigma v_{e^{1/\varepsilon}}^2 \, \mathrm{d}\Theta \\
&= \int_0^\infty \varphi^2(t) \, \mathrm{d}t \int_\Sigma v_{e^{1/\varepsilon}}^2 \, \mathrm{d}\Theta = C_{11} \int_\Sigma v_{e^{1/\varepsilon}}^2 \, \mathrm{d}\Theta,
\end{aligned}$$

$$\begin{aligned}
(31) \quad & \left| \int_{\mathbb{R}^n} (|\nabla u_\varepsilon|^2 - \lambda r^{-2} |u_\varepsilon|^2) \, \mathrm{d}x \right| \\
&= \Big| \varepsilon^2 \int_0^\infty r^{-1} ((2-n) \varphi(\varepsilon \ln r) \varphi'(\varepsilon \ln r) + \varepsilon {\varphi'}^2(\varepsilon \ln r)) \, \mathrm{d}r \int_\Sigma v_{e^{1/\varepsilon}}^2 \, \mathrm{d}\Theta \\
&\quad + \varepsilon \int_0^\infty r^{-1} \varphi^2(\varepsilon \ln r) \, \mathrm{d}r \int_\Sigma (|\nabla_\Theta v_{e^{1/\varepsilon}}|^2 - \Lambda v_{e^{1/\varepsilon}}^2) \, \mathrm{d}\Theta \Big| \\
&\leqslant \int_0^\infty (\varepsilon(n-2) |\varphi(t)| |\varphi'(t)| \\
&\quad + \varepsilon^2 {\varphi'}^2(t) + (\hat{\lambda}(e^{1/\varepsilon}) - \Lambda) \varphi^2(t)) \, \mathrm{d}t \int_\Sigma v_{e^{1/\varepsilon}}^2 \, \mathrm{d}\Theta.
\end{aligned}$$

From the relations (30), (31) we get that for any $\delta > 0$ there exists $\varepsilon > 0$ such that the function u_ε satisfies the inequality (29). This implies that $\sigma(L) \cap (\lambda - \delta, \lambda + \delta) \neq \emptyset$. Thus, $\sigma(L) = [\frac{1}{4}(n-2)^2 + \Lambda, \infty)$. Point ii) of Theorem 1 is proved.

5. On the rate of condensation of the discrete spectrum

Let $s > 2$ and let the spectrum of the operator L be discrete. For any $\lambda \in (\frac{1}{4}(n-2)^2 + \Lambda, \infty)$ consider functions

$$u_s(r,\Theta) = r^{1-n/2} \mathrm{e}^{\mathrm{i}\sqrt{\lambda-(n-2)^2/4-\Lambda}\ln r} \eta_s(r) v_{\ln\ln(1/(s-2))}(\Theta), \tag{32}$$

where $\eta_s(r) = r/\ln\ln(1/(s-2)) - 1$ for $\ln\ln(1/(s-2)) < r < 2\ln\ln(1/(s-2))$, $\eta_s(r) = 1$ for $2\ln\ln(1/(s-2)) < r < \ln(1/(s-2))$, $\eta_s(r) = 2 - r/\ln(1/(s-2))$ for $\ln(1/(s-2)) < r < 2\ln(1/(s-2))$ and $\eta_s(r) = 0$ in the other cases, the function $v_\varrho(\Theta)$ being the same as in the proof of point ii) of Theorem 1. Let us continue the function v_ϱ by zero to Σ. As follows from (19), to prove the relation (3) it is sufficient for some $s_0 > 2$ and some constant $C > 0$ for all $2 < s < s_0$ establish inequality

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} (|\nabla u_s|^2 - \lambda r^{-s}|u_s|^2)\,\mathrm{d}x \right| \\ &\qquad < \left(\hat{\lambda}\Big(\ln\ln\frac{1}{s-2}\Big) - \Lambda + C\frac{\ln\ln\ln(1/(s-2))}{\ln\ln(1/(s-2))} \right) \int_{\mathbb{R}^n} r^{-s}|u_s|^2\,\mathrm{d}x. \end{aligned} \tag{33}$$

By (32) we have

$$\begin{aligned} &\int_{\mathbb{R}^n} r^{-s}|u_s|^2\,\mathrm{d}x \\ &\quad = \int_0^\infty r^{1-s}\eta_s^2(r)\,\mathrm{d}r \int_\Sigma v^2_{\ln\ln(1/(s-2))}\,\mathrm{d}\Theta \\ &\quad = \left(\frac{\ln^{2-s}\ln(1/(s-2)) - \ln^{2-s}(1/(s-2))}{s-2} + O(1) \right) \int_\Sigma v^2_{\ln\ln(1/(s-2))}\,\mathrm{d}\Theta \\ &\quad = \left(\ln\ln\frac{1}{s-2} + O\Big(\ln\ln\ln\frac{1}{s-2}\Big) \right) \int_\Sigma v^2_{\ln\ln(1/(s-2))}\,\mathrm{d}\Theta. \end{aligned} \tag{34}$$

Consider the behavior of the left hand side of the inequality (33) for $s \to 2+0$:

$$
\begin{aligned}
(35)\quad & \Big| \int_{\mathbb{R}^n} (|\nabla u_s|^2 - \lambda r^{-s} |u_s|^2)\,\mathrm{d}x \Big| \\
&= \Big| \int_0^\infty \big(\eta_s'^2 + (2-n) r^{-1} \eta_s' \eta_s + (\lambda (r^{-1} - r^{1-s}) - \Lambda r^{-1}) \eta_s^2\big)\,\mathrm{d}r \\
&\quad \times \int_\Sigma v^2_{\ln\ln(1/(s-2))}\,\mathrm{d}\Theta + \int_0^\infty r^{-1} \eta_s^2\,\mathrm{d}r \int_\Sigma |\nabla_\Theta v_{\ln\ln(1/(s-2))}|^2\,\mathrm{d}\Theta \Big| \\
&= \Big| \int_0^\infty \Big(\eta_s'^2 + (2-n) r^{-1} \eta_s' \eta_s \\
&\quad + \Big(\lambda(r^{-1} - r^{1-s}) + \Big(\hat\lambda\Big(\ln\ln\frac{1}{s-2}\Big) - \Lambda\Big) r^{-1}\Big)\eta_s^2\Big)\,\mathrm{d}r \Big| \int_\Sigma v^2_{\ln\ln(1/(s-2))}\,\mathrm{d}\Theta \\
&\leqslant \Big(3(n-1) + \lambda \int_0^\infty (r^{-1} - r^{1-s}) \eta_s^2\,\mathrm{d}r + \Big(\hat\lambda\Big(\ln\ln\frac{1}{s-2}\Big) - \Lambda\Big) \int_0^\infty r^{-1}\eta_s^2\,\mathrm{d}r\Big) \\
&\quad \times \int_\Sigma v^2_{\ln\ln(1/(s-2))}\,\mathrm{d}\Theta \\
&< \Big(\hat\lambda\Big(\ln\ln\frac{1}{s-2}\Big) - \Lambda + C_{12}\,\frac{\ln\ln\ln(1/(s-2))}{\ln\ln(1/(s-2))}\Big) \\
&\quad \times \ln\ln(1/(s-2)) \int_\Sigma v^2_{\ln\ln(1/(s-2))}\,\mathrm{d}\Theta, \quad C_{12} > 0.
\end{aligned}
$$

Hence, the inequality (33) follows from (34), (35). Proof of Theorem 3 is complete.

6. Continuity of spectrum

Let us prove continuity of the spectrum of the operator L for $0 \leqslant s \leqslant 2$. Let $\lambda > 0$ and $u \in D(L)$ be non-zero functions, satisfying the equation $\Delta u + \lambda r^{-s} u = 0$ and vanishing on Γ (we consider the function u to be real-valued). By Lemma 3 for $\Gamma \in C^2$ we have the inclusion $u \in H^2(\Omega_R)$, $R > 0$. We multiply the equation by $2ru_r$ and integrate over the domain Ω_R. Thus we have the equality

$$
\begin{aligned}
(36)\qquad & R \int_{S_R} \Big(2u_r^2 - |\nabla u|^2 + \lambda R^{-s} u^2 + \frac{n-2}{R} u u_r\Big)\,\mathrm{d}s_x \\
&\quad + \int_{\Gamma_R} (\nu, x)\Big(\frac{\partial u}{\partial \nu}\Big)^2\,\mathrm{d}s_x - \lambda(2-s)\int_{\Omega_R} r^{-s} u^2\,\mathrm{d}x = 0,
\end{aligned}
$$

where ν is the outward unit normal vector to Γ. From (36) we get the inequalities:

$$
\begin{aligned}
(37)\quad & \frac{R}{2}\int_{S_R}(nu_r^2+(2\lambda+(n-2))R^{-s}u^2)\,\mathrm{d}s_x\\
&\geqslant R\int_{S_R}(u_r^2+\lambda R^{-s}u^2+\frac{n-2}{2}(u_r^2+R^{-2}u^2))\,\mathrm{d}s_x\\
&\geqslant R\int_{S_R}\Big(u_r^2+\lambda R^{-s}u^2+\frac{n-2}{R}u_r u\Big)\,\mathrm{d}s_x\geqslant-\int_{\Gamma_R}(\nu,x)\Big(\frac{\partial u}{\partial\nu}\Big)^2\,\mathrm{d}s_x.
\end{aligned}
$$

Let us note that the star-shapeness condition for the set $\mathbb{R}^n\setminus\Omega$ for a smooth surface Γ means that $(\nu,x)\leqslant 0$, $x\in\Gamma$. The surface Γ is not a cone, so there exists a point $x_0\in\Gamma$ such that $(\nu,x_0)<0$. So, $u|_\Gamma=0$, and then by the uniqueness theorem for the solution of the Cauchy problem for elliptic equations ([7]) there exists a neighborhood $U(x_0)$ such that $\int_{\Gamma\cap U(x_0)}(\nu,x)(\partial u/\partial\nu)^2\,\mathrm{d}s_x<0$. Therefore, $\int_{S_R}(u_r^2+R^{-s}u^2)\,\mathrm{d}s_x\geqslant C_{13}R^{-1}$, $C_{13}>0$, $R\geqslant R_0$ and $\|u\|^2_{H^1_s(\Omega)}\geqslant\int_{\Omega\cap\{r\geqslant R_0\}}(u_r^2+r^{-s}u^2)\,\mathrm{d}x=\int_{R_0}^\infty\mathrm{d}r\int_{S_R}(u_r^2+r^{-s}u^2)\,\mathrm{d}s_x\geqslant C_{13}\int_{R_0}^\infty r^{-1}\,\mathrm{d}r=+\infty$, i.e. u is not an eigenfunction of the operator L. This completes the proof of point i) of Theorem 2.

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