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The Type A Uncertainty*

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Abstract

If in the model of measurement except useful parameters, which are to be determined, other auxiliary parameters occur as well, which were estimated from another experiment, then the type A and B uncertainties of measurement results must be taken into account. The type A uncertainty is caused by the new experiment and the type B uncertainty characterizes an accuracy of the parameters which must be used in estimation of useful parameters. The problem is to estimate of the type A uncertainty in the case that the type B uncertainty is known.

Key words: two stage linear model, the type A and B uncertainties, insensitivity region, linearization region

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1 Introduction

Mainly in metrology two types of unertainties have been distinguished, i.e. the uncertainty A and B. From the mathematical viewpoint they can be characterized as follows. In the two stage regression model (1) the covariance matrix of the estimator of the parameter $\boldsymbol{\beta}$ can be decomposed into two parts. The one depends on the matrix $\boldsymbol{\Sigma}$ only (it is the uncertainty A) and in the oter parts the matrix $\mathbf{W}$ occurs (this part is the uncertainty B).

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There are several ways how to characterize the type A and B uncertainties in the measured results (see e.g. [1]), The simplest situation occurs when the covariance matrix $\mathbf{W}$ of the auxiliary parameter estimator and also the covariance matrix $\mathbf{\Sigma}$ of the observation vector of the new experiment are known. A little more complicated situation occurs when the covariance matrix of the observation vector depends on some unknown parameter. Then it is necessary to estimate it from data of the new experiment.

If the regression model of the new experiment is nonlinear, then the problem of linearization occurs.

It is to be remarked that the mentioned problems lead to the two stage regression models with/without constraints. Relatively large class of statistical problems arise there, see e.g. [2], [3], [5], [7], [8], [9]. Small attention is devoted to estimation of covariance matrix parameters, which serve as a starting point for a determination of the type A uncertainty.

The aim of the paper is to analyze some simple problems arising in the evaluation of an accuracy of measurement results, mainly of the type A uncertainty in the case that some unknown parameter occurs in covariance matrix $\mathbf{\Sigma}$ of the new experiment.

2 Symbols and auxiliary statements

The notation

$$\begin{pmatrix} \widehat{\mathbf{\Theta}} \\ \mathbf{Y} \end{pmatrix} \sim \left[\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{D}, & \mathbf{X} \end{pmatrix} \begin{pmatrix} \mathbf{\Theta} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \mathbf{W}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{\Sigma} \end{pmatrix} \right] \tag{1}$$

means that the estimator $\widehat{\mathbf{\Theta}}$ of the auxiliary vector parameter $\mathbf{\Theta}$ is unbiased and its covariance matrix $\mathrm{Var}(\widehat{\mathbf{\Theta}})$ is equal to $\mathbf{W}$ (which is assumed to be positive definite). The mean value $E(\mathbf{Y})$ of the observation vector $\mathbf{Y}$ of the new experiment is equal to $\mathbf{D\Theta}+\mathbf{X}\boldsymbol{\beta}$, where $n\times l$ matrix $\mathbf{D}$ and $n \times k$ matrix $\mathbf{X}$ are known and $\boldsymbol{\beta}$ is the useful parameter of the new experiment. The matrix $\mathbf{X}$ is assumed to be of the rank $r(\mathbf{X}) = k < n$. The symbol $\mathbf{\Sigma}$ means the covariance matrix of the observation vector $\mathbf{Y}$ and it is assumed that it is positive definite. The symbol $\mathbf{I}_n$ means the $n \times n$ identity matrix, $\mathbf{A}^-$ is the generalized inverse of a matrix $\mathbf{A}$, i.e. $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$ (in more detail see [10]) and $\mathbf{M}_X = \mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ is the projection matrix (in the Euclidean norm) on the orthogonal subspace of the column subspace $\mathcal{M}(\mathbf{X}) = \{\mathbf{Xu}\colon \mathbf{u} \in R^k\}$ of the $n \times k$ matrix $\mathbf{X}$.

Theorem 2.1 *The best linear unbiased estimator (BLUE) of the vector* $\begin{pmatrix} \mathbf{\Theta} \\ \boldsymbol{\beta} \end{pmatrix}$ *in the model (1) is*

$$\begin{pmatrix} \widehat{\widehat{\mathbf{\Theta}}} \\ \widehat{\boldsymbol{\beta}} \end{pmatrix} = \left[\begin{pmatrix} \mathbf{I}, & \mathbf{D}' \\ \mathbf{0}, & \mathbf{X}' \end{pmatrix} \begin{pmatrix} \mathbf{W}^{-1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{\Sigma}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{D}, & \mathbf{X} \end{pmatrix} \right]^{-1} \begin{pmatrix} \mathbf{I}, & \mathbf{D}', \\ \mathbf{0} & \mathbf{X}' \end{pmatrix}$$
$$\times \begin{pmatrix} \mathbf{W}^{-1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{\Sigma}^{-1} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{\Theta}} \\ \mathbf{Y} \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{\boldsymbol{\Theta}} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}(\mathbf{Y}-\mathbf{D}\widehat{\boldsymbol{\Theta}}-\mathbf{X}\widehat{\boldsymbol{\beta}}) \\ \left[\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\right]^{-1}\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}(\mathbf{Y}-\mathbf{D}\widehat{\boldsymbol{\Theta}}) \end{pmatrix},$$

$$\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Theta}}}) = \mathbf{W} - \mathbf{W}\mathbf{D}'[\mathbf{M}_X(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')\mathbf{M}_X]^{+}\mathbf{D}\mathbf{W},$$

$$\mathrm{Var}(\widehat{\boldsymbol{\beta}}) = \left[\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\right]^{-1}.$$

Proof The model

$$\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ -\mathbf{D}, & \mathbf{I} \end{pmatrix}\begin{pmatrix} \widehat{\boldsymbol{\Theta}} \\ \mathbf{Y} \end{pmatrix} \sim \left[\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X} \end{pmatrix}\begin{pmatrix} \boldsymbol{\Theta} \\ \boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \mathbf{W}, & -\mathbf{W}\mathbf{D}' \\ -\mathbf{D}\mathbf{W}, & \boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}' \end{pmatrix}\right]$$

is equivalent to (1). The BLUE of $\begin{pmatrix} \boldsymbol{\Theta} \\ \boldsymbol{\beta} \end{pmatrix}$ is

$$\begin{pmatrix} \widehat{\widehat{\boldsymbol{\Theta}}} \\ \widehat{\boldsymbol{\beta}} \end{pmatrix} = \left[\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}' \end{pmatrix}\begin{pmatrix} \mathbf{W}, & -\mathbf{W}\mathbf{D}' \\ -\mathbf{D}\mathbf{W}, & \boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}' \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X} \end{pmatrix}\right]^{-1}$$

$$\times \begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}' \end{pmatrix}\begin{pmatrix} \mathbf{W}, & -\mathbf{W}\mathbf{D}' \\ -\mathbf{D}\mathbf{W}, & \boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}' \end{pmatrix}^{-1}\begin{pmatrix} \widehat{\boldsymbol{\Theta}} \\ \mathbf{Y}-\mathbf{D}\widehat{\boldsymbol{\Theta}} \end{pmatrix};$$

$$\left[\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}' \end{pmatrix}\begin{pmatrix} \mathbf{W}, & -\mathbf{W}\mathbf{D}' \\ -\mathbf{D}\mathbf{W}, & \boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}' \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X} \end{pmatrix}\right]^{-1}$$

$$= \left[\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}' \end{pmatrix}\begin{pmatrix} \mathbf{W}^{-1}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D}, & \mathbf{D}'\boldsymbol{\Sigma}^{-1} \\ \boldsymbol{\Sigma}^{-1}\mathbf{D}, & \boldsymbol{\Sigma}^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X} \end{pmatrix}\right]^{-1}$$

$$= \begin{pmatrix} \mathbf{W}^{-1}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D}, & \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X} \\ \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}, & \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X} \end{pmatrix}^{-1} = \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix},$$

$$\begin{aligned} \boxed{11} &= \left[\mathbf{W}^{-1}+\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{D}\right]^{-1} = (\mathbf{W}^{-1}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1} \\ &\quad +(\mathbf{W}^{-1}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}(\mathbf{W}^{-1}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}, \\ \boxed{12} &= -\boxed{11}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1} \\ &= -(\mathbf{W}^{-1}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22} = \boxed{21}', \\ \boxed{22} &= \left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X}-\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}(\mathbf{W}^{-1}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\right]^{-1} \\ &= \left[\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\right]^{-1}, \end{aligned}$$

$$\begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}' \end{pmatrix}\begin{pmatrix} \mathbf{W}, & -\mathbf{W}\mathbf{D}' \\ -\mathbf{D}\mathbf{W}, & \boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}' \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{W}^{-1}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D}, & \mathbf{D}'\boldsymbol{\Sigma}^{-1} \\ \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}, & \mathbf{X}'\boldsymbol{\Sigma}^{-1} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \widehat{\widehat{\boldsymbol{\Theta}}} \\ \widehat{\boldsymbol{\beta}} \end{pmatrix} = \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix}\begin{pmatrix} \mathbf{W}^{-1}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D}, & \mathbf{D}'\boldsymbol{\Sigma}^{-1} \\ \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}, & \mathbf{X}'\boldsymbol{\Sigma}^{-1} \end{pmatrix}\begin{pmatrix} \widehat{\boldsymbol{\Theta}} \\ \mathbf{Y}-\mathbf{D}\widehat{\boldsymbol{\Theta}} \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{11}(\mathbf{W}^{-1}\widehat{\boldsymbol{\Theta}}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}) + \boxed{12}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\ \boxed{21}(\mathbf{W}^{-1}\widehat{\boldsymbol{\Theta}}+\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}) + \boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \end{pmatrix},$$

$$
\begin{aligned}
\widehat{\widehat{\boldsymbol{\Theta}}} &= \boxed{11}(\mathbf{W}^{-1}\widehat{\boldsymbol{\Theta}} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}) - (\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&= \big[(\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1} + (\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D} \\
&\quad \times (\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}\big](\mathbf{W}^{-1}\widehat{\boldsymbol{\Theta}} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}) - (\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1} \\
&\quad \times \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&= \big[\mathbf{W} - \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\mathbf{W}\big](\mathbf{W}^{-1}\widehat{\boldsymbol{\Theta}} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}) \\
&\quad + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\boxed{22}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\mathbf{W} \\
&\quad \times (\mathbf{W}^{-1}\widehat{\boldsymbol{\Theta}} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}) - \big[\mathbf{W} - \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\mathbf{W}\big] \\
&\quad \times \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&= \widehat{\boldsymbol{\Theta}} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\big[-\mathbf{D}\widehat{\boldsymbol{\Theta}} + \mathbf{X}\boxed{22}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\widehat{\boldsymbol{\Theta}}\big] \\
&\quad + \mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\big[-\mathbf{D}\mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \mathbf{X}\boxed{22}\mathbf{X}' \\
&\quad \times (\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}\big] - \mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&\quad + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&= \widehat{\boldsymbol{\Theta}} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\big[-\mathbf{D}\widehat{\boldsymbol{\Theta}} - (\mathbf{D}\mathbf{W}\mathbf{D}' + \boldsymbol{\Sigma} - \boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1}\mathbf{Y}\big] \\
&\quad + \mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\boxed{22}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1} \\
&\quad \times \big[\mathbf{D}\widehat{\boldsymbol{\Theta}} + (\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}' - \boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1}\mathbf{Y}\big] - \mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&\quad + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&= \widehat{\boldsymbol{\Theta}} - \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\widehat{\boldsymbol{\Theta}} - \mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{Y} \\
&\quad + \mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\boxed{22}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\widehat{\boldsymbol{\Theta}} \\
&\quad + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\boxed{22}\mathbf{X}\boldsymbol{\Sigma}^{-1}\mathbf{Y} - \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\boxed{22}\mathbf{X}' \\
&\quad \times (\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{Y} - \mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1} \\
&\quad \times \mathbf{D}\mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&= \widehat{\boldsymbol{\Theta}} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\big[\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}} - \mathbf{X}\boxed{22}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1} \\
&\quad \times (\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}})\big] + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} - \mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X} \\
&\quad \times \boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} - \mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \mathbf{W}\mathbf{D}' \\
&\qquad \times(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}(\mathbf{D}\mathbf{W}\mathbf{D}' + \boldsymbol{\Sigma} - \boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} = \mathbf{0},
\end{aligned}
$$

we have

$$
\widehat{\widehat{\boldsymbol{\Theta}}} = \widehat{\boldsymbol{\Theta}} + \mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\big[\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}} - \mathbf{X}\boxed{22}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}(\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}})\big].
$$

The estimator $\widehat{\boldsymbol{\beta}}$ is

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= \boxed{21}(\mathbf{W}^{-1}\widehat{\boldsymbol{\Theta}} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}) + \boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&= -\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}(\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}(\mathbf{W}^{-1}\widehat{\boldsymbol{\Theta}} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}) + \boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\
&= \boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} - \boxed{22}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{DW}(\mathbf{W}^{-1}\widehat{\boldsymbol{\Theta}} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}) \\
&= \boxed{22}\mathbf{X}'\big[\boldsymbol{\Sigma}^{-1}\mathbf{Y} - (\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{D}\widehat{\boldsymbol{\Theta}} - (\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{DWD}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}\big] \\
&= \boxed{22}\mathbf{X}'\big[\boldsymbol{\Sigma}^{-1}\mathbf{Y} - (\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{D}\widehat{\boldsymbol{\Theta}} - \boldsymbol{\Sigma}^{-1}\mathbf{Y} + (\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{Y}\big] \\
&= \boxed{22}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}(\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}}) \\
&= \big[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}\big]^{-1}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}(\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}}).
\end{aligned}$$

The expressions for $\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Theta}}})$ and $\mathrm{Var}(\widehat{\boldsymbol{\beta}})$ are

$$\mathrm{Var}(\widehat{\widehat{\boldsymbol{\Theta}}}) = \boxed{11}, \quad \mathrm{Var}(\widehat{\boldsymbol{\beta}}) = \boxed{22},$$

i.e.

$$\begin{aligned}
\boxed{11} &= (\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1} + (\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\boxed{22}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D} \\
&\quad \times (\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1} \\
&= \mathbf{W} - \mathbf{WD}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{DW} + \mathbf{WD}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}\boxed{22} \\
&\quad \times \mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{DW} \\
&= \mathbf{W} - \mathbf{WD}'\big\{(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1} - (\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X} \\
&\quad \times \big[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}\big]^{-1}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\big\}\mathbf{DW} \\
&= \mathbf{W} - \mathbf{WD}'\big[\mathbf{M}_X(\boldsymbol{\Sigma} + \mathbf{DWD}')\mathbf{M}_X\big]^{+}\mathbf{DW}
\end{aligned}$$

and

$$\boxed{22} = \big[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}\big]^{-1}$$

(cf. also [3, p. 325]). □

Remark 2.2 Since

$$\begin{aligned}
\mathrm{Var}(\widehat{\boldsymbol{\beta}}) &= \big[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}\big]^{-1} = \big\{\mathbf{X}'\big[\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{D} \\
&\quad \times (\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\big]\mathbf{X}\big\}^{-1} \\
&= \big[\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X} - \mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}(\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}\big]^{-1} \\
&= (\mathbf{X}\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1} + (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}\big[\mathbf{W}^{-1} + \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{D} \\
&\quad - \mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}\big]^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1} \\
&= (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1} + (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}\big[\mathbf{W}^{-1} \\
&\quad + \mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{D}\big]^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1},
\end{aligned}$$

the type A uncertainty is given by the matrix $(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}$ and the type B uncertainty by the matrix

$$(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}\big[\mathbf{W}^{-1}+\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}\mathbf{M}_X)^{+}\mathbf{D}\big]^{-1}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}.$$

Remark 2.3 The BLUE $\widehat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ from Lemma 1.1 is the same as the BLUE from the model

$$\mathbf{Y}-\mathbf{D}\widehat{\boldsymbol{\Theta}} \sim_n (\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}'). \tag{2}$$

In many situation the estimator $\widehat{\boldsymbol{\Theta}}$ cannot be corrected and thus the estimator $\widehat{\widehat{\boldsymbol{\Theta}}}$ has been used only exceptionally. Therefore only the estimator $\widehat{\boldsymbol{\beta}}$ will be dealt with in the following text.

Remark 2.4 Another unbiased estimator of the vector $\boldsymbol{\beta}$ is

$$\begin{aligned}\tilde{\boldsymbol{\beta}} &= (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\mathbf{D}\widehat{\boldsymbol{\Theta}}),\\ \mathrm{Var}(\tilde{\boldsymbol{\beta}}) &= (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}.\end{aligned}$$

Obviously $\mathrm{Var}(\tilde{\boldsymbol{\beta}})-\mathrm{Var}(\widehat{\boldsymbol{\beta}})$ is positive semidefinite matrix. In this case the type B uncertainty is given by the matrix

$$(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{D}\mathbf{W}\mathbf{D}'\boldsymbol{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}.$$

In the estimator $\widehat{\boldsymbol{\beta}}$ from Lemma 1.1 the type B uncertainty is suppressed more effectively than in the estimator $\tilde{\boldsymbol{\beta}}$. Nevertheless the estimator $\tilde{\boldsymbol{\beta}}$ is frequently used in practice.

3 The matrix Σ is of the form $\sigma^2\mathbf{V}$

In the following text the symbol $\mathbf{A}^+$ means the Moore–Penrose generalized inverse of the matrix $\mathbf{A}$, i.e. $\mathbf{A}\mathbf{A}^+\mathbf{A}=\mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+=\mathbf{A}^+$, $\mathbf{A}\mathbf{A}^+=(\mathbf{A}\mathbf{A}^+)'$, $\mathbf{A}^+\mathbf{A}=(\mathbf{A}^+\mathbf{A})'$ (in more detail see [10]).

Lemma 3.1 *The unbiased estimator of σ^2 in the model*

$$\mathbf{Y} \sim_n \left[(\mathbf{D},\mathbf{X})\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\beta}\end{pmatrix}, \sigma^2\mathbf{V}\right]$$

is

$$\widehat{\sigma_1^2} = \mathbf{Y}'(\mathbf{M}_{(D,X)}\mathbf{V}\mathbf{M}_{(D,X)})^{+}\mathbf{Y}/[n-r(\mathbf{D},\mathbf{X})].$$

and in the case of normality of the observation vector $\mathbf{Y}$ *it is valid that it is uniformly best unbiased estimator,*

$$\widehat{\sigma_1^2} \sim \sigma^2\frac{\chi^2_{n-r(D,X)}}{n-r(\mathbf{D},\mathbf{X})} \quad \textit{and} \quad \mathrm{Var}(\widehat{\sigma_1^2}) = \frac{2\sigma^4}{n-r(\mathbf{D},\mathbf{X})},$$

where $\chi^2_{n-r(X,D)}$ *is the chi-squared random variable with* $n-r(\mathbf{D},\mathbf{X})$ *degrees of freedom.*

Proof The expression for $\widehat{\sigma_1^2}$ is straightforward transcription of the well known formula

$$\widehat{\sigma_1^2} = \frac{\left[\mathbf{Y} - \widehat{(\mathbf{D},\mathbf{X})\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\beta}\end{pmatrix}}\right]' \mathbf{V}^{-1} \left[\mathbf{Y} - \widehat{(\mathbf{D},\mathbf{X})\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\beta}\end{pmatrix}}\right]}{n - r(\mathbf{D},\mathbf{X})}, \quad n - r(\mathbf{D},\mathbf{X}) > 0.$$

Full proof of Lemma 3.1 is given in [4, p. 81]. □

It is to be said that the estimator of $\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\beta}\end{pmatrix}$ in the model considered need not exist, however the estimator

$$\widehat{(\mathbf{D},\mathbf{X})\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\beta}\end{pmatrix}}$$

always exists. In practice it can be valid that $n - r(\mathbf{D},\mathbf{X}) = 0$ (see the section Numerical example).

The symbol

$$\begin{pmatrix}\mathbf{D}'\\ \mathbf{X}'\end{pmatrix}^-_{m(V)}$$

used in the following remark means the minimum $\mathbf{V}$-norm generalized inverse of the matrix $\begin{pmatrix}\mathbf{D}'\\ \mathbf{X}'\end{pmatrix}$, i.e.

$$\begin{pmatrix}\mathbf{D}'\\ \mathbf{X}'\end{pmatrix}\begin{pmatrix}\mathbf{D}'\\ \mathbf{X}'\end{pmatrix}^-_{m(V)}\begin{pmatrix}\mathbf{D}'\\ \mathbf{X}'\end{pmatrix} = \begin{pmatrix}\mathbf{D}'\\ \mathbf{X}'\end{pmatrix}$$

and

$$\mathbf{V}\begin{pmatrix}\mathbf{D}'\\ \mathbf{X}'\end{pmatrix}^-_{m(V)}\begin{pmatrix}\mathbf{D}'\\ \mathbf{X}'\end{pmatrix} = (\mathbf{D},\mathbf{X})\left[\begin{pmatrix}\mathbf{D}'\\ \mathbf{X}'\end{pmatrix}^-_{m(V)}\right]'\mathbf{V}$$

(in more detail see [10]).

Remark 3.2 Since

$$\left(\mathbf{M}_{(D,X)}\mathbf{V}\mathbf{M}_{(D,X)}\right)^+ =$$
$$= (\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+ - (\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\left[\mathbf{D}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\right]^+\mathbf{D}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+$$

and

$$\mathbf{Y}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{Y} = \tilde{\mathbf{v}}'\mathbf{V}^{-1}\tilde{\mathbf{v}}$$

where

$$\tilde{\mathbf{v}} = \mathbf{Y} - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y},$$

we have

$$\mathbf{Y}'\left(\mathbf{M}_{(D,X)}\mathbf{V}\mathbf{M}_{(D,X)}\right)^+\mathbf{Y} =$$
$$= \mathbf{v}'\mathbf{V}^{-1}\mathbf{v} = \tilde{\mathbf{v}}'\mathbf{V}^{-1}\tilde{\mathbf{v}} - \tilde{\mathbf{v}}'\mathbf{V}^{-1}\mathbf{D}\left[\mathbf{D}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}'\mathbf{V}^{-1}\tilde{\mathbf{v}},\right.$$

where

$$\mathbf{v} = \left\{ \mathbf{I} - (\mathbf{D}, \mathbf{X}) \left[\begin{pmatrix} \mathbf{D}' \\ \mathbf{X}' \end{pmatrix}^-_{m(V)} \right]' \right\} \mathbf{Y}.$$

Lemma 3.3 *The unbiased estimator of σ^2 in the model*

$$\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}} \sim_n (\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V} + \mathbf{DWD}')$$

is

$$\begin{aligned} \widehat{\sigma_2^2} &= \frac{A - B}{\mathrm{Tr}\left[(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}\right]}, \\ A &= (\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}})'(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+(\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}}), \\ B &= \mathrm{Tr}\left[(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{DWD}'\right], \end{aligned}$$

where $\mathbf{\Sigma}_0 = \sigma_0^2\mathbf{V} + \mathbf{DWD}'$ and σ_0^2 is an approximate value of σ^2.

In the case of normality of the vector $\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}}$ it is valid that

$$\mathrm{Var}_{\sigma_0^2}(\widehat{\sigma_2^2}) = \frac{2}{\mathrm{Tr}\left[(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}\right]}.$$

Proof The unbiasedness of the estimator is obvious. In the case of normality it is sufficient to take into account the relationship

$$\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \mathbf{\Sigma}) \quad \Rightarrow \quad \mathrm{Var}(\mathbf{Y}'\mathbf{AY}) = 2\,\mathrm{Tr}(\mathbf{A\Sigma A\Sigma}) + 4\boldsymbol{\mu}'\mathbf{A\Sigma A}\boldsymbol{\mu}$$

($\mathbf{A}$ is any symmetric $n \times n$ matrix). □

In order to suppress the influence of the chosen σ_0^2 on the estimator, an iteration procedure is used (the same is valid for $\widehat{\sigma}_3^2$ from Lemma 3.4).

Lemma 3.4 *Let*

$$\begin{aligned} \mathbf{S}_{V,DWD'} &= \begin{pmatrix} \boxed{aa}, & \boxed{ab} \\ \boxed{ba}, & \boxed{bb} \end{pmatrix}, \\ \boxed{aa} &= \mathrm{Tr}\left[\mathbf{V}(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\right], \\ \boxed{ab} &= \mathrm{Tr}\left[\mathbf{V}(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{DWD}'(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\right] = \boxed{ba}, \\ \boxed{bb} &= \mathrm{Tr}\left[\mathbf{DWD}'(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\mathbf{DWD}'(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+\right] \end{aligned}$$

and

$$\mathbf{S}_{V,DWD'}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the unbiased estimator with a minimum variance at the point σ_0^2 of σ^2 in the model (2) is

$$\widehat{\sigma_3^2} = (\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}})'(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+(\lambda_1\mathbf{V} + \lambda_2\mathbf{DWD}')(\mathbf{M}_X\mathbf{\Sigma}_0\mathbf{M}_X)^+(\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}})$$

and in the case of normality of the vector $\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}}$ *it is valid that*

$$\mathrm{Var}_{\sigma_0^2}(\widehat{\sigma_3^2}) = 2(1,0)\mathbf{S}^{-1}_{V,DWD'}\begin{pmatrix}1\\0\end{pmatrix}.$$

If $\mathbf{S}_{V,DWD'}$ *is singular, the last equation need not have a solution.*

Proof Let $\widehat{\sigma_3^2} = (\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}})'\mathbf{A}(\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}})$, where $\mathbf{A} = \mathbf{A}'$, $\mathbf{AX} = \mathbf{0}$, $\mathrm{Tr}(\mathbf{AV}) = 1$ and $\mathrm{Tr}(\mathbf{DWD'A}) = 0$. The equality $\mathbf{AX} = \mathbf{0}$ implies $\mathbf{A} = \mathbf{M}_X\mathbf{S}\mathbf{M}_X$, where $\mathbf{S}$ is any symmetric matrix. In the case of normality

$$\begin{aligned}
&\mathrm{Var}\big[(\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}})'\mathbf{A}(\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}})\big] = \\
&= 2\,\mathrm{Tr}\big[(\mathbf{M}_X\mathbf{S}\mathbf{M}_X(\sigma^2\mathbf{V}+\mathbf{DWD}')\mathbf{M}_X\mathbf{S}\mathbf{M}_X(\sigma^2\mathbf{V}+\mathbf{DWD}')\big] \\
&= 2\sigma^4\,\mathrm{Tr}(\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X) + 4\sigma^2\,\mathrm{Tr}(\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X) \\
&\quad + 2\,\mathrm{Tr}(\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X).
\end{aligned}$$

In order to minimize $\mathrm{Var}\big[(\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}})'\mathbf{A}(\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}})\big]$, under the constraints $\mathrm{Tr}(\mathbf{AV}) = 1$ and $\mathrm{Tr}(\mathbf{ADWD}') = 0$, by a suitable choice of the matrix $\mathbf{A}$, we use the auxiliary Lagrange function

$$\begin{aligned}
\Phi(\mathbf{S}) &= \sigma^4\,\mathrm{Tr}(\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X) + 2\sigma^2\,\mathrm{Tr}(\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X) \\
&\quad + \mathrm{Tr}(\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X) - 2\lambda_1\big[\mathrm{Tr}(\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X) - 1\big] \\
&\quad - 2\lambda_2\,\mathrm{Tr}(\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial\Phi(\mathbf{S})}{\partial\mathbf{S}} &= 4\sigma^4\mathbf{M}_X\mathbf{V}\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X + 2\sigma^2(2\mathbf{M}_X\mathbf{V}\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X \\
&\quad + 2\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X) + 4\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X \\
&\quad - 2\sigma^4\,\mathrm{Diag}(\mathbf{M}_X\mathbf{V}\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X) - 2\sigma^2\big[\mathrm{Diag}(\mathbf{M}_X\mathbf{V}\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X) \\
&\quad -2\,\mathrm{Diag}(\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X)\big] -2\,\mathrm{Diag}(\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X) \\
&\quad - 2\lambda_1\big[2\mathbf{M}_X\mathbf{V}\mathbf{M}_X - \mathrm{Diag}(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)\big] \\
&\quad - 2\lambda_2\big[2\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X - \mathrm{Diag}(\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X)\big].
\end{aligned}$$

Thus we obtain the system of equations

$$\begin{aligned}
&\sigma^4\mathbf{M}_X\mathbf{V}\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X + \sigma^2\mathbf{M}_X\mathbf{V}\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X \\
&+ \sigma^2\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{V}\mathbf{M}_X + \mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X\mathbf{S}\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X \\
&- \lambda_1\mathbf{M}_X\mathbf{V}\mathbf{M}_X - \lambda_2\mathbf{M}_X\mathbf{DWD}'\mathbf{M}_X = \mathbf{0}
\end{aligned}$$

$$\mathbf{A} = \mathbf{M}_X\mathbf{S}\mathbf{M}_X, \quad \mathrm{Tr}(\mathbf{AV}) = 1, \quad \mathrm{Tr}(\mathbf{ADWD}') = 0,$$

i.e.

$$\begin{aligned}\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X\mathbf{S}\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X &= \lambda_1\mathbf{M}_X\mathbf{V}\mathbf{M}_X + \lambda_2\mathbf{M}_X\mathbf{D}\mathbf{W}\mathbf{D}'\mathbf{M}_X\\
\Rightarrow \mathbf{M}_X\mathbf{S}\mathbf{M}_X = \mathbf{A} &= \lambda_1(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\\
&+ \lambda_2(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+,\end{aligned}$$

$$\mathrm{Tr}\left[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}\right]\lambda_1 + \mathrm{Tr}\left[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}\right]\lambda_2 = 1,$$

$$\mathrm{Tr}\left[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'\right]\lambda_1 + \mathrm{Tr}\left[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'\right]\lambda_2 = 0,$$

$$\mathbf{S}_{V,DWD'}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}.$$

In the case of normality of the vector $\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}}$ it is valid that

$$\begin{aligned}&\mathrm{Var}_{\sigma_0^2}\Big\{(\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}})'\big[\lambda_1(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+ + \lambda_2(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\\
&\quad\times \mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\big](\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}})\Big\}\\
&= 2\,\mathrm{Tr}\Big\{\big[\lambda_1(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\\
&+\lambda_2(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\big]\boldsymbol{\Sigma}_0[\lambda_1(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\\
&+ \lambda_2(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\big]\boldsymbol{\Sigma}_0\Big\}\\
&= 2\Big\{\lambda_1^2\,\mathrm{Tr}\left[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\Sigma}_0(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\Sigma}_0\right]\\
&+ 2\lambda_1\lambda_2\,\mathrm{Tr}\big[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\Sigma}_0(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}\\
&\times (\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\Sigma}_0\big] + \lambda_2^2\,\mathrm{Tr}\big[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\Sigma}_0\\
&\times (\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\Sigma}_0\big]\Big\}\\
&= 2(\lambda_1,\lambda_2)\mathbf{S}_{V,DWD'}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = 2(1,0)\mathbf{S}^+_{V,DWD'}\begin{pmatrix}1\\ 0\end{pmatrix}.\end{aligned}$$

□

Remark 3.5 Since

$$\begin{aligned}&\frac{1}{\mathrm{Tr}\left[(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}\right]} < (1,0)\mathbf{S}^+_{V,DWD'}\begin{pmatrix}1\\ 0\end{pmatrix}\\
&= \left\{\mathrm{Tr}\left[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\right]\right.\\
&\qquad\left. - \frac{\left\{\mathrm{Tr}\left[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\right]\right\}^2}{\mathrm{Tr}\left[\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\right]}\right\}^{-1},\end{aligned}$$

the estimator $\widehat{\sigma_2^2}$ is obviously better than the estimator $\widehat{\sigma_3^2}$.

Lemma 3.6 *In the model (2) the estimator*

$$\widehat{\sigma_4^2} = \frac{(\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}})'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+(\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}}) - \mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'\right]}{n-r(\mathbf{X})}$$

is unbiased and in the case of normality of the vector $\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}}$ *it is valid that*

$$\mathrm{Var}(\widehat{\sigma}_4^2) = \frac{2\sigma^4}{n-r(\mathbf{X})} + \frac{4\sigma^2\,\mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{V}\right]}{[n-r(\mathbf{X})]^2} + \frac{2\,\mathrm{Tr}\left[\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\right]}{[n-r(\mathbf{X})]^2}.$$

Proof

$$E\Big\{(\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}})'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+(\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}}) - \mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'\right]\Big\}$$
$$= \mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+(\sigma^2\mathbf{V}+\mathbf{D}\mathbf{W}\mathbf{D}')\right] - \mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'\right]$$
$$= \sigma^2\,\mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{V}\right] = \sigma^2\left[n-r(\mathbf{X})\right].$$

$$\mathrm{Var}(\widehat{\sigma_4^2}) = \frac{2}{[n-r(\mathbf{X})]^2}\,\mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+(\sigma^2\mathbf{V}+\mathbf{D}\mathbf{W}\mathbf{D}')(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\right.$$
$$\left.\times(\sigma^2\mathbf{V}+\mathbf{D}\mathbf{W}\mathbf{D}')\right] = \frac{2}{[n-r(\mathbf{X})]^2}\Big\{\sigma^4\,\mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{V}\right]$$
$$+2\sigma^2\,\mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'\right] + \mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'\right.$$
$$\left.\times(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'\right]\Big\}$$
$$= \frac{2\sigma^4}{n-r(\mathbf{X})} + \frac{4\sigma^2\,\mathrm{Tr}\left[(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{V}\right]}{[n-r(\mathbf{X})]^2} + \frac{2\,\mathrm{Tr}\left[\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X\mathbf{V}\mathbf{M}_X)^+\right]}{[n-r(\mathbf{X})]^2}.$$

□

4 Insensitivity region for the variance of the estimator of $\mathbf{h}'\boldsymbol{\beta}$

In this section the notation $\sigma^2 = \vartheta$ will be used. Let $\mathbf{h}$ be any k-dimensional vector. The aim is to find the neighbourhood $\mathcal{N}_{h'\beta}$ of the value ϑ_0 with the property

$$\vartheta \in \mathcal{N}_{h'\beta} \Rightarrow \sqrt{\mathrm{Var}_{\vartheta_0}\left[\mathbf{h}'\widehat{\boldsymbol{\beta}}(\vartheta)\right]} \le (1+\varepsilon)\sqrt{\mathrm{Var}_{\vartheta_0}\left[\mathbf{h}'\widehat{\boldsymbol{\beta}}(\vartheta_0)\right]}.$$

The neighbourhood $\mathcal{N}_{h'\beta}$ is called the insensitivity region.

Theorem 4.1 *Let the observation vector* $\mathbf{Y}-\mathbf{D}\widehat{\mathbf{\Theta}}$ *in the model (2) be normally distributed. Then the insensitivity region* $\mathcal{N}_{h'\beta}$ *for the variance of the estimator* *of* $\mathbf{h}'\boldsymbol{\beta}$ *in the model (2) is*

$$\mathcal{N}_{h'\beta} = \Bigg\{\vartheta\colon |\vartheta - \vartheta_0|$$

$$\leq \sqrt{\frac{2\varepsilon\mathbf{h}'(\mathbf{X}\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{h}}{\mathbf{h}'(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}\boldsymbol{\Sigma}_0^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{h}}}\Bigg\}$$

Proof Let us consider the relationships

$$\Phi(\vartheta) = \mathbf{h}'\big[\mathbf{X}'(\vartheta\mathbf{V} + \mathbf{DWD}')^{-1}\mathbf{X}\big]^{-1}\mathbf{X}'(\vartheta\mathbf{V} + \mathbf{DWD}')^{-1}(\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}}),$$

$$\frac{\partial\Phi(\vartheta)}{\partial\vartheta}\Big|_{\vartheta=\vartheta_0} = -\mathbf{h}'(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}\boldsymbol{\Sigma}_0^{-1}\mathbf{v},$$

where

$$\boldsymbol{\Sigma}_0 = \vartheta_0\mathbf{V} + \mathbf{DWD}', \quad \mathbf{v} = \big[\mathbf{I} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\big](\mathbf{Y} - \mathbf{D}\widehat{\boldsymbol{\Theta}}).$$

Let the symbol $\mathbf{M}_X^{\Sigma_0^{-1}}$ denote the expression $\mathbf{I} - \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}$.

Since $\mathbf{v}$ and $\widehat{\boldsymbol{\beta}}$ are stochastically independent it is valid that

$$\mathrm{Var}_{\vartheta_0}\left[\mathbf{h}'\widehat{\boldsymbol{\beta}}(\vartheta_0 + \delta\vartheta)\right] \approx$$

$$\approx \mathbf{h}'(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{h} + \mathbf{h}'(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}\boldsymbol{\Sigma}_0^{-1}\mathbf{M}_X^{\Sigma_0^{-1}}$$

$$\times \left[\boldsymbol{\Sigma}_0 - \mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\right]\left(\mathbf{M}_X^{\Sigma_0^{-1}}\right)'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}\boldsymbol{\Sigma}_0^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{h}$$

$$= \mathbf{h}'(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{h}$$

$$+ (\delta\vartheta)^2\mathbf{h}'(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}\boldsymbol{\Sigma}_0^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{h}.$$

If ε^2 is neglected in the expression $(1+\varepsilon)^2$, then the proof of the statement can be easily finished. □

5 Linearization regions for the bias of the estimators of $\boldsymbol{\beta}$ and σ^2

In this section the model

$$\begin{pmatrix}\widehat{\boldsymbol{\Theta}}\\ \mathbf{Y}\end{pmatrix} \sim_{l+n} \left[\begin{pmatrix}\boldsymbol{\Theta}\\ \mathbf{f}(\boldsymbol{\Theta},\boldsymbol{\beta})\end{pmatrix}, \begin{pmatrix}\mathbf{W}, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}\end{pmatrix}\right] \tag{3}$$

will be considered. The function $\mathbf{f}(\cdot,\cdot\cdot)$ is assumed to be of the form

$$\mathbf{f}(\boldsymbol{\Theta},\boldsymbol{\beta}) = \mathbf{f}_0 + \mathbf{D}\delta\boldsymbol{\Theta} + \mathbf{X}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta},\delta\boldsymbol{\beta}), \quad \mathbf{f}_0 = \mathbf{f}(\boldsymbol{\Theta}^{(0)},\boldsymbol{\beta}^{(0)}),$$

$$\mathbf{D} = \frac{\partial \mathbf{f}(\boldsymbol{\Theta},\boldsymbol{\beta})}{\partial \boldsymbol{\Theta}'}\Big|_{\Theta=\Theta^{(0)},\beta=\beta^{(0)}}, \quad \mathbf{X} = \frac{\partial \mathbf{f}(\boldsymbol{\Theta},\boldsymbol{\beta})}{\partial \boldsymbol{\beta}'}\Big|_{\Theta=\Theta^{(0)},\beta=\beta^{(0)}},$$

$$\boldsymbol{\kappa}(\delta\boldsymbol{\Theta},\delta\boldsymbol{\beta}) = (\kappa_1(\delta\boldsymbol{\Theta},\delta\boldsymbol{\beta}),\ldots,\kappa_n(\delta\boldsymbol{\Theta},\delta\boldsymbol{\beta}))',$$

$$\kappa_i(\delta\boldsymbol{\Theta},\delta\boldsymbol{\beta}) = (\delta\boldsymbol{\Theta}',\delta\boldsymbol{\beta}')\frac{\partial^2 f_i(\boldsymbol{\Theta},\boldsymbol{\beta})}{\partial\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\beta}\end{pmatrix}\partial(\boldsymbol{\Theta}',\boldsymbol{\beta}')}\Bigg|_{\Theta=\Theta^{(0)},\beta=\beta^{(0)}}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix},$$

where $\boldsymbol{\Theta}^{(0)},\boldsymbol{\beta}^{(0)}$ are approximate values of the parameters $\boldsymbol{\Theta},\boldsymbol{\beta}$. The estimator $\widehat{\boldsymbol{\Theta}}$ is considered in the form $\widehat{\boldsymbol{\Theta}} = \boldsymbol{\Theta}^{(0)} + \widehat{\delta\boldsymbol{\Theta}}$.

Thus the model (3) can be rewritten as

$$\begin{pmatrix}\widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}^{(0)}\\ \mathbf{Y} - \mathbf{f}_0\end{pmatrix} \sim_{l+n} \left[\begin{pmatrix}\mathbf{I}, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}\end{pmatrix}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} + \begin{pmatrix}\mathbf{0}\\ \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta},\delta\boldsymbol{\beta})\end{pmatrix}, \begin{pmatrix}\mathbf{W}, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}\end{pmatrix}\right]$$

where $\delta\boldsymbol{\Theta} = \boldsymbol{\Theta} - \boldsymbol{\Theta}^{(0)}, \delta\boldsymbol{\beta} = \boldsymbol{\beta} - \boldsymbol{\beta}^{(0)}$. The linearized version of the model (3) is

$$\begin{pmatrix}\widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}^{(0)}\\ \mathbf{Y} - \mathbf{f}_0\end{pmatrix} \sim_{l+n} \left[\begin{pmatrix}\mathbf{I}, & \mathbf{0}\\ \mathbf{D}, & \mathbf{X}\end{pmatrix}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix}, \begin{pmatrix}\mathbf{W}, & \mathbf{0}\\ \mathbf{0}, & \boldsymbol{\Sigma}\end{pmatrix}\right]. \tag{4}$$

It can be proved (in more detail see [3]) that the BLUE of $\delta\boldsymbol{\beta}$ in the model (4) is the BLUE of $\delta\boldsymbol{\beta}$ in the model

$$\mathbf{Y} - \mathbf{f}_0 - \mathbf{D}\widehat{\delta\boldsymbol{\Theta}} \sim_n (\mathbf{X}\delta\boldsymbol{\beta}, \boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}').$$

It is utilized in the following lemma.

Lemma 5.1 *The bias of the BLUE*

$$\widehat{\delta\boldsymbol{\beta}} = \left[\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\right]^{-1}\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}(\mathbf{Y} - \mathbf{f}_0 - \mathbf{D}\widehat{\delta\boldsymbol{\Theta}})$$

in the model

$$\mathbf{Y} - \mathbf{f}_0 - \mathbf{D}\widehat{\delta\boldsymbol{\Theta}} \sim_n (\mathbf{X}\delta\boldsymbol{\beta}, \boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}').$$

is

$$\mathbf{b} = E(\widehat{\delta\boldsymbol{\beta}}) - \delta\boldsymbol{\beta} = \frac{1}{2}\left[\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\right]^{-1}\mathbf{X}'(\boldsymbol{\Sigma}+\mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta},\delta\boldsymbol{\beta})$$

in the model

$$\mathbf{Y} - \mathbf{f}_0 - \mathbf{D}\widehat{\delta\boldsymbol{\Theta}} \sim_n \left(\mathbf{X}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta},\delta\boldsymbol{\beta}), \boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}'\right).$$

Proof It is a consequence of the relationships

$$\mathbf{b} = E(\widehat{\delta\boldsymbol{\beta}}) - \delta\boldsymbol{\beta}, \qquad E(\mathbf{Y} - \mathbf{f}_0 - \mathbf{D}\widehat{\delta\boldsymbol{\Theta}}) = \mathbf{X}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta}).$$

□

Definition 5.2 The measure of nonlinearity for the bias of the estimator of $\boldsymbol{\beta}$ is

$$C_b(\boldsymbol{\Theta}^{(0)}, \boldsymbol{\beta}^{(0)}) = \sup\left\{ \frac{\sqrt{\mathbf{b}'\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{Xb}}}{(\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix}} : \begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \in R^{l+k} \right\}.$$

Here $\mathbf{U}$ is the covariance matrix of the vector $\begin{pmatrix}\widehat{\delta\boldsymbol{\Theta}}\\ \widehat{\delta\boldsymbol{\beta}}\end{pmatrix}$, i.e.

$$\mathbf{U} = \begin{pmatrix} \boxed{\text{aa}}, & \boxed{\text{ab}} \\ \boxed{\text{ba}}, & \boxed{\text{bb}} \end{pmatrix},$$

$$\begin{aligned}
\boxed{\text{aa}} &= \mathbf{W},\\
\boxed{\text{ab}} &= -\mathbf{WD}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}\big[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}\big]^{-1},\\
\boxed{\text{ba}} &= -\big[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}\big]^{-1}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{DW},\\
\boxed{\text{bb}} &= \big[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{X}\big]^{-1}.
\end{aligned}$$

The measure of nonlinearity enables us to construct so called linearization region $\mathcal{L}_b$. It is neighbourhood of the point $\boldsymbol{\beta}_0$ with the property

$$\boldsymbol{\beta}_0 + \delta\boldsymbol{\beta} \in \mathcal{L}_b \Rightarrow \sqrt{\mathbf{b}'\big[\,\text{Var}(\widehat{\delta\boldsymbol{\beta}})\big]\mathbf{b}} \le \varepsilon.$$

Lemma 5.3 *The linearization region for the bias of the estimator of the parameter $\boldsymbol{\beta}$ in the model (3) is*

$$\mathcal{L}_b = \left\{ \begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} : (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \le \frac{\varepsilon}{C_b(\boldsymbol{\Theta}^{(0)}, \boldsymbol{\beta}^{(0)})} \right\},$$

i.e.

$$\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \in \mathcal{L}_b \Rightarrow \sqrt{\mathbf{b}'\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{Xb}} \le \varepsilon$$

Proof With respect to Definition 5.2 it is valid that the inequalities

$$\sqrt{\mathbf{b}'\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{DWD}')^{-1}\mathbf{Xb}} \le (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} C_b(\boldsymbol{\Theta}^{(0)}, \boldsymbol{\beta}^{(0)}) \le \varepsilon$$

imply the statement of the lemma. □

Definition 5.4 Let $\mathbf{M}_{(D,X)} \neq \mathbf{0}$. Then the measure of nonlinearity for the bias of the estimator $\widehat{\sigma_1^2}$ is

$$C_{1,\sigma^2}^{(int)} = \sup\left\{ \frac{\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})\left(\mathbf{M}_{(D,X)}\mathbf{V}\mathbf{M}_{(D,X)}\right)^{+}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})}}{(\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix}} : \begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \in R^{l+k} \right\}.$$

Theorem 5.5 *Let in the model (3)* $\boldsymbol{\Sigma} = \sigma^2\mathbf{V}$. *Then in the case of normality of the vector* $\mathbf{Y}$ *the linearization region for the bias of the estimator* $\widehat{\sigma_1^2}$ *of* σ^2 *is*

$$\mathcal{L}_{1,\sigma^2} = \left\{ (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \leq \sigma\frac{\sqrt{8[n - r(\mathbf{D}, \mathbf{X})]\varepsilon}}{C_{1,\sigma^2}^{(int)}} \right\},$$

i.e.

$$\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \in \mathcal{L}_{1,\sigma^2} \Rightarrow \left|\sqrt{E(\widehat{\sigma_1^2})} - \sigma\right| \leq \varepsilon\sigma.$$

Proof Since $\widehat{\sigma_1^2} \sim \sigma^2\frac{\chi_f^2(\delta)}{f}$, $f = n - r(\mathbf{D}, \mathbf{X})$, (in more detail see [6]) where the parameter noncentrality

$$\delta = \frac{1}{4\sigma^2}\boldsymbol{\kappa}'(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})\left(\mathbf{M}_{(D,X)}\mathbf{V}\mathbf{M}_{(D,X)}\right)^{+}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta}),$$

it is valid that

$$E(\widehat{\sigma_1^2}) = \frac{\sigma^2}{n - r(\mathbf{D}, \mathbf{X})}\Big[n - r(\mathbf{D}, \mathbf{X})$$
$$+ \frac{1}{4\sigma^2}\boldsymbol{\kappa}'(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})\left(\mathbf{M}_{(D,X)}\mathbf{V}\mathbf{M}_{(D,X)}\right)^{+}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})\Big].$$

The requirement $\sqrt{E(\widehat{\sigma_1^2})} - \sigma \leq \varepsilon\sigma$ leads, with respect to Definition 5.4, to the relationships

$$\sqrt{\boldsymbol{\kappa}'(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})\left(\mathbf{M}_{(D,X)}\mathbf{V}\mathbf{M}_{(D,X)}\right)^{+}\boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})} \leq (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} C_{1,\sigma^2}^{(int)}$$
$$\leq \sqrt{8\sigma^2[n - r(\mathbf{D}, \mathbf{X})]}.$$

Now it is obvious how to finish the proof. □

Now, let us consider the estimator $\widehat{\sigma_2^2}$.

Definition 5.6 In the model (3) the measure of nonlinearity for the bias of the estimator $\widehat{\sigma_2^2}$ is

$$C_{2,\sigma^2} = \sup\left\{ \frac{\sqrt{\boldsymbol{\kappa}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^{+}\boldsymbol{\kappa}}}{(\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix}} : \begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \in R^{l+k} \right\}.$$

Theorem 5.7 *The linearization region for the bias of the estimator $\widehat{\sigma_2^2}$ in the model (3) is*

$$\mathcal{L}_{2,\sigma^2} = \left\{ \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} : (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1} \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \right.$$

$$\left. \leq \frac{\sigma}{C_{2,\sigma^2}} \sqrt{2\varepsilon \operatorname{Tr}\left[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\right]} \right\},$$

i.e.

$$\begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \in \mathcal{L}_{2,\sigma^2} \Rightarrow \left| \sqrt{E(\widehat{\sigma_2^2})} - \sigma \right| \leq \varepsilon\sigma.$$

Proof The mean value of the estimator $\widehat{\sigma_2^2}$ in the model (3) is

$$E(\widehat{\sigma_2^2}) = \sigma^2 + \frac{\boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\kappa}(\delta\boldsymbol{\Theta}, \delta\boldsymbol{\beta})}{\operatorname{Tr}\left[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\right]},$$

thus

$$\sqrt{E(\widehat{\sigma_2^2})} = \sigma\sqrt{1 + \frac{\boldsymbol{\kappa}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\kappa}}{\operatorname{Tr}\left[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\right]}}$$

$$\approx \sigma + \frac{1}{2\sigma}\frac{\boldsymbol{\kappa}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\kappa}}{\operatorname{Tr}\left[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\right]}.$$

With respect to Definition 5.6 we have

$$\sqrt{\boldsymbol{\kappa}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\kappa}} \leq (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1} \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} C_{2,\sigma^2}$$

$$\leq \sigma\sqrt{2\varepsilon \operatorname{Tr}\left[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\right]}$$

and this implies the statement of the theorem. □

6 Numerical example

Let, in the plane, four points A_1, A_2, A_3, A_4 be given by their coordinates, i.e. $A_i \begin{pmatrix} \Theta_{2i-1} \\ \Theta_{2i} \end{pmatrix}$, $i = 1, 2, 3, 4$,

$$A_1 \begin{pmatrix} 201.31 \text{ m} \\ 210.80 \text{ m} \end{pmatrix}, \quad A_2 \begin{pmatrix} 406.73 \text{ m} \\ 863.45 \text{ m} \end{pmatrix}, \quad A_3 \begin{pmatrix} 1050.47 \text{ m} \\ 216.66 \text{ m} \end{pmatrix}, \quad A_4 \begin{pmatrix} 630.17 \text{ m} \\ 28.29 \text{ m} \end{pmatrix}.$$

The coordinates are estimated and their estimator is

$$\widehat{\boldsymbol{\Theta}} = \begin{pmatrix} \widehat{\Theta}_1 \\ \vdots \\ \widehat{\Theta}_8 \end{pmatrix} \sim N_8(\boldsymbol{\Theta}, \mathbf{W}), \qquad \mathbf{W} = (0.1 \text{ m})^2\mathbf{I}_8.$$

Coordinates $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ of a point P must be estimated by measured distances d_i

$$d_i = E(Y_i) = \sqrt{(\Theta_{2i-1} - \beta_1)^2 + (\Theta_{2i} - \beta_2)^2}, \quad i = 1, \dots, 4,$$

$$\mathbf{f}(\boldsymbol{\Theta}, \boldsymbol{\beta}) = \begin{pmatrix} d_1 \\ \vdots \\ d_4 \end{pmatrix},$$

where the approximate coordinates $\boldsymbol{\beta}_0$ are

$$P \begin{pmatrix} 503.1 \text{ m} \\ 431.9 \text{ m} \end{pmatrix},$$

$$\mathbf{Y} \sim N_4(\mathbf{f}(\boldsymbol{\Theta}, \boldsymbol{\beta}), (0.01m)^2 \mathbf{I}_4).$$

The linearized models of this measurement are

$$\mathbf{Y} - \mathbf{f}_0 \sim N_4 \left[(\mathbf{D}, \mathbf{X}) \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix}, (0.01 \text{ m})^2 \mathbf{I}_4 \right]$$

and

$$\mathbf{Y} - \mathbf{f}_0 - \mathbf{D}\widehat{\delta\boldsymbol{\Theta}} \sim N_4 \left[\mathbf{X}\delta\boldsymbol{\beta}, (0.01 \text{ m})^2 \mathbf{I}_4 + \mathbf{D}(0.1 \text{ m})^2 \mathbf{I}_8 \mathbf{D}' \right],$$

respectively, where $\mathbf{f}_0 = \mathbf{f}(\boldsymbol{\Theta}_0, \boldsymbol{\beta}_0)$, $\boldsymbol{\Theta}_0 = \widehat{\boldsymbol{\Theta}}$, i.e. $\mathbf{f}_0 - \mathbf{D}\widehat{\boldsymbol{\Theta}} \approx \mathbf{f}(\widehat{\boldsymbol{\Theta}}, \boldsymbol{\beta}_0)$. Here

$$\mathbf{f}_0 = (f_{0,1}, \dots, f_{0,4})' = (374.12, 442.18, 588.17, 423.14)',$$

$$f_{0,i} = \sqrt{\left(\widehat{\Theta}_{2i-1} - \beta_1^{(0)}\right)^2 + \left(\widehat{\Theta}_{2i} - \beta_2^{(0)}\right)^2}, \qquad \mathbf{D} = \frac{\partial \mathbf{f}(\boldsymbol{\Theta}, \boldsymbol{\beta})}{\partial \boldsymbol{\Theta}'}\Big|_{\Theta = \widehat{\Theta}, \beta = \beta^{(0)}},$$

$$\frac{\partial f_{0,i}}{\partial \Theta_{2i-1}} = \frac{\widehat{\Theta}_{2i-1} - \beta_1^{(0)}}{f_{0,i}}, \quad \frac{\partial f_{0,i}}{\partial \Theta_{2i}} = \frac{\widehat{\Theta}_{2i} - \beta_2^{(0)}}{f_{0,i}},$$

$$\frac{\partial f_{0,i}}{\partial \beta_1} = -\frac{\widehat{\Theta}_{2i-1} - \beta_1^{(0)}}{f_{0,i}}, \quad \frac{\partial f_{0,i}}{\partial \beta_2} = -\frac{\widehat{\Theta}_{2i} - \beta_2^{(0)}}{f_{0,i}},$$

$$\mathbf{D} = \begin{pmatrix} \frac{\partial f_{0,1}}{\partial \Theta_1}, & \frac{\partial f_{0,1}}{\partial \Theta_2}, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & \frac{\partial f_{0,2}}{\partial \Theta_3}, & \frac{\partial f_{0,2}}{\partial \Theta_4}, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & \frac{\partial f_{0,3}}{\partial \Theta_5}, & \frac{\partial f_{0,3}}{\partial \Theta_6}, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & \frac{\partial f_{0,4}}{\partial \Theta_7}, & \frac{\partial f_{0,4}}{\partial \Theta_8} \end{pmatrix}, \qquad \mathbf{D}\mathbf{D}' = \mathbf{I}_4,$$

$$\mathbf{X} = \begin{pmatrix} \frac{\partial f_{0,1}}{\partial \beta_1}, & \frac{\partial f_{0,1}}{\partial \beta_2} \\ \frac{\partial f_{0,2}}{\partial \beta_1}, & \frac{\partial f_{0,2}}{\partial \beta_2} \\ \frac{\partial f_{0,3}}{\partial \beta_1}, & \frac{\partial f_{0,3}}{\partial \beta_2} \\ \frac{\partial f_{0,4}}{\partial \beta_1}, & \frac{\partial f_{0,4}}{\partial \beta_2} \end{pmatrix} = \begin{pmatrix} 0.806\,68, & 0.590\,99 \\ 0.217\,94, & -0.975\,96 \\ -0.930\,63, & 0.365\,95 \\ -0.300\,30, & 0.953\,84 \end{pmatrix},$$

$$\boldsymbol{\Sigma}_0 = (0.01 \text{ m})^2 \mathbf{I}_4 + (0.1 \text{ m})^2 \mathbf{D}\mathbf{D}' = 0.0101 \mathbf{I}_4.$$

The estimator $\widehat{\sigma_1^2}$ does not exist, since $\mathbf{M}_{(D,X)} = \mathbf{0}$.

The estimator $\widehat{\sigma_2^2}$ is

$$\widehat{\sigma_2^2} = \frac{A - B}{\mathrm{Tr}\left[(\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V} (\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V}\right]},$$
$$A = (\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}})'(\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V} (\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ (\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}}),$$
$$B = \mathrm{Tr}\left[(\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V} (\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{D}\mathbf{W}\mathbf{D}'\right],$$

and its dispersion is

$$\mathrm{Var}_{\sigma_0^2}(\widehat{\sigma_2^2}) = \frac{2}{\mathrm{Tr}\left[(\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V} (\mathbf{M}_X \mathbf{\Sigma}_0 \mathbf{M}_X)^+ \mathbf{V}\right]} = 1.020\,01 \times 10^{-4}.$$

It is to be remarked that the realization of $\widehat{\sigma^2}$ can be negative mainly in the case that σ^2 is small in comparison with a precision characterized by $\mathbf{W}$.

The estimator $\widehat{\sigma_3^2}$ in this case does not exist.

The estimator $\widehat{\sigma_4^2}$ is

$$\widehat{\sigma_4^2} = \frac{(\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}})'(\mathbf{M}_X \mathbf{V} \mathbf{M}_X)^+ (\mathbf{Y} - \mathbf{D}\widehat{\mathbf{\Theta}}) - \mathrm{Tr}\left[(\mathbf{M}_X \mathbf{V} \mathbf{M}_X)^+ \mathbf{D}\mathbf{W}\mathbf{D}'\right]}{n - r(\mathbf{X})},$$

$$\mathrm{Var}(\sigma_4^2) = \frac{2\sigma^4}{n - r(\mathbf{X})} + \frac{4\sigma^2\,\mathrm{Tr}\left[(\mathbf{M}_X \mathbf{V} \mathbf{M}_X)^+ \mathbf{D}\mathbf{W}\mathbf{D}' (\mathbf{M}_X \mathbf{V} \mathbf{M}_X)^+ \mathbf{V}\right]}{[n - r(\mathbf{X})]^2}$$
$$+ \frac{2\,\mathrm{Tr}\left[\mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X \mathbf{V} \mathbf{M}_X)^+ \mathbf{D}\mathbf{W}\mathbf{D}'(\mathbf{M}_X \mathbf{V} \mathbf{M}_X)^+\right]}{[n - r(\mathbf{X})]^2} = 1.020\,01 \times 10^{-4}.$$

It is to be remarked that a realization of $\widehat{\sigma_4^2}$ can be negative as in the case of $\widehat{\sigma_2^2}$.

The accuracy of different estimators of σ^2 is the same, however it is not sufficiently good. For example in the case of $\widehat{\sigma}_2^2$ it is approximately valid that

$$\frac{\sqrt{\mathrm{Var}(\sqrt{\widehat{\sigma}_2^2})}}{\sigma} = \sqrt{\left(\frac{1}{2\sqrt{\sigma_2^2}}\right)^2 \mathrm{Var}(\widehat{\sigma}_2^2)} = \frac{1}{2 \times 10^{-2}} \sqrt{1.020\,01 \times 10^{-4}} = 0.505.$$

Thus the relative standard deviation is 50.5% what is rather large number. However nothing better can be expected because of the poor precision of the first stage measurement ($\mathbf{W}$).

It is of some interest to determine the estimates $\widehat{\sigma_2}$ and $\widehat{\sigma_4}$ in our example. If the errors of measurement are $(0.022\ \mathrm{m}, 0.032\ \mathrm{m}, 0.007\ \mathrm{m}, -0.083\ \mathrm{m})$, then $\widehat{\sigma_2} = \widehat{\sigma_3} = 0.021m$.

Insensitivity regions Let

$$W_h = \mathbf{h}'(\mathbf{X}'\mathbf{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}_0^{-1}\mathbf{V}(\mathbf{M}_X \mathbf{\Sigma}_0^{-1} \mathbf{M}_X)^+ \mathbf{V}\mathbf{\Sigma}_0^{-1}\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}_0^{-1}\mathbf{X})^{-1}\mathbf{h}.$$

If $\mathbf{h} = (1,0)'$, then $\mathbf{W}_{(1,0)} = \mathbf{0}$ and also for $\mathbf{h} = (0,1)'$ $\mathbf{W}_{(0,1)} = \mathbf{0}$.

Thus the estimators of β_1 and β_2 are not sensitive on the small changes of the value σ^2.

Linearization regions

$$\mathbf{U} = \begin{pmatrix} \boxed{\text{aa}}, & \boxed{\text{ab}} \\ \boxed{\text{ba}}, & \boxed{\text{bb}} \end{pmatrix},$$

$$\boxed{\text{aa}} = \mathbf{W} = (0.1)^2 \mathbf{I}_8,$$

$$\boxed{\text{ab}} = -\mathbf{W}\mathbf{D}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{F}\left[\mathbf{F}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{F}\right]^{-1}$$

$$= \begin{pmatrix} 0.004\ 533, & 0.002\ 734 \\ 0.003\ 321, & 0.002\ 003 \\ 0.000\ 091, & -0.000\ 893 \\ -0.000\ 409, & 0.003\ 998 \\ 0.005\ 089, & -0.000\ 664 \\ -0.002\ 001, & 0.000\ 261 \\ 0.000\ 287, & -0.001\ 177 \\ -0.000\ 911, & 0.003\ 738 \end{pmatrix},$$

$$\boxed{\text{ba}} = -\left[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\right]^{-1}\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{D}\mathbf{W} = \boxed{\text{ab}}',$$

$$\boxed{\text{bb}} = \left[\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\right]^{-1} = \begin{pmatrix} 0.006\ 319, & 0.000\ 978 \\ 0.000\ 978, & 0.004\ 457 \end{pmatrix},$$

$$C_b(\boldsymbol{\Theta}^{(0)}, \boldsymbol{\beta}^{(0)}) = \sup\left\{ \frac{\sqrt{\mathbf{b}'\mathbf{X}'(\boldsymbol{\Sigma} + \mathbf{D}\mathbf{W}\mathbf{D}')^{-1}\mathbf{X}\mathbf{b}}}{(\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix}} : \begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \in R^{l+k} \right\}$$
$$= 0.000\ 119\ 76.$$

If $\varepsilon = 0.1$, then

$$\mathcal{L}_b = \left\{ \begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} : (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \leq \frac{\varepsilon}{C_b(\boldsymbol{\Theta}^{(0)}, \boldsymbol{\beta}^{(0)})} \right\}$$

is the ellipsoid with the semiaxes equal to

$a_1 = 0.268\,\text{m}$, $a_2 = 0.292\,\text{m}$, $a_3 = 0.346\,\text{m}$, $a_4 = 0.346\,\text{m}$, $a_5 = 0.346\,\text{m}$, $a_6 = 0.346\,\text{m}$, $a_7 = 0.346\,\text{m}$, $a_8 = 0.346\,\text{m}$, $a_9 = 5.487\,\text{m}$, $a_{10} = 6.530\,\text{m}$.

$$C_{2,\sigma^2} = \sup\left\{ \frac{\sqrt{\boldsymbol{\kappa}'(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\boldsymbol{\kappa}}}{(\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1}\begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix}} : \begin{pmatrix}\delta\boldsymbol{\Theta}\\ \delta\boldsymbol{\beta}\end{pmatrix} \in R^{l+k} \right\}$$
$$= 0.003\ 776.$$

If $\varepsilon = 0.1$, then

$$\mathcal{L}_{2,\sigma^2} = \left\{ \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} : (\delta\boldsymbol{\Theta}', \delta\boldsymbol{\beta}')\mathbf{U}^{-1} \begin{pmatrix} \delta\boldsymbol{\Theta} \\ \delta\boldsymbol{\beta} \end{pmatrix} \right.$$

$$\left. \leq \frac{\sigma}{C_{2,\sigma^2}} \sqrt{2\varepsilon \operatorname{Tr}\left[\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\mathbf{V}(\mathbf{M}_X\boldsymbol{\Sigma}_0\mathbf{M}_X)^+\right]} \right\}$$

is the ellipsoid with the semiaxes

$$a_1 = 1.504\,\text{m}, \quad a_2 = 1.642\,\text{m}, \quad a_3 = 1.943\,\text{m} \quad a_4 = 1.943\,\text{m}, \quad a_5 = 1.943\,\text{m},$$
$$a_6 = 1.943\text{m}, \quad a_7 = 1.943\text{m}, \quad a_8 = 1.943\text{m}, \quad a_9 = 30.813\text{m}, \quad a_{10} = 36.668\text{m}.$$

The linearization regions $\mathcal{L}_b$ and $\mathcal{L}_{2,\sigma^2}$ are sufficiently large with respect to requirements of geodetical practice.

References

[1] Guide to the Expression of Uncertainty in Measurement. *International Organization for Standardization*, ISO/IEC Guide 98:1993, Switzerland, 1993.

[2] Korbašová, M., Marek, J.: *Connecting measurements in surveying and its problem.* In: INGEO 2004, 3rd international Conference on Engineering Surveying, November 11–13, 2004, *Slovak University of Technology*, Bratislava, Slovakia, 2004.

[3] Kubáček, L., Kubáčková, L., Volaufová, J.: Statistical Models with Linear Structures. *Veda*, Bratislava, 1995.

[4] Kubáček, L., Kubáčková, L.: Statistics and Metrology. *Vyd. University Palackého*, Olomouc, 2008, (in Czech).

[5] Kubáček, L., Marek, J.: *Partial optimum estimator in two stage regression model with constraints and a problem of equivalence.* Math. Slovaca **55** (2005), 477–494.

[6] Kubáček, L., Tesaříková, E.: Weakly Nonlinear Regression Models. *Vyd. Univerzity Palackého*, Olomouc, 2008.

[7] Marek, J.: *Estimation in connecting measurements.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **42** (2003), 69–86.

[8] Marek, J.: *Estimation in connecting measurements with constraints of type II.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **43** (2004), 119–131.

[9] Marek, J., Fišerová, E.: Statistical analysis of geodetical measurements. In: ROBUST 2004, sborník 13. letní školy, Třešť, 7.–11. 6. 2004, *JČMF*, Praha, 2004, 253–260.

[10] Rao, C. R., Mitra, S. K.: Generalized Inverse of Matrices and its Applications. *Wiley*, New York–London–Sydney–Toronto, 1971.