Berrabah Bendoukha; Hafida Bendahmane

Inequalities between the sum of powers and the exponential of sum of positive and commuting selfadjoint operators

Archivum Mathematicum, Vol. 47 (2011), No. 4, 257--262

Persistent URL: http://dml.cz/dmlcz/141774

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 47 (2011), 257–262

INEQUALITIES BETWEEN THE SUM OF POWERS AND THE EXPONENTIAL OF SUM OF POSITIVE AND COMMUTING SELFADJOINT OPERATORS

Berrabah Bendoukha and Hafida Bendahmane

Abstract. Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators acting in Hilbert space $\mathcal{H}$ and $\mathcal{B}^+(\mathcal{H})$ the set of all positive selfadjoint elements of $\mathcal{B}(\mathcal{H})$. The aim of this paper is to prove that for every finite sequence $(A_i)_{i=1}^n$ of selfadjoint, commuting elements of $\mathcal{B}^+(\mathcal{H})$ and every natural number $p \geq 1$, the inequality

$$\frac{e^p}{p^p}\Big(\sum_{i=1}^n A_i^p\Big) \leq \exp\Big(\sum_{i=1}^n A_i\Big),$$

holds.

1. Preliminaries and main results

Our starting result in this paper is the following theorem established in [3] for $p = 2$ and extended to case $p \geq 1$ in [2].

Theorem 1.0.1. *Let $(x_i)_{i=1}^n$ be a sequence of nonnegative real numbers. Then for every real $p \geq 1$, inequality*

$$\frac{e^p}{p^p}\sum_{i=1}^n x_i^p \leq \exp\Big(\sum_{i=1}^n x_i\Big), \tag{1.0.1}$$

holds. Equality in (1.0.1) holds if $x_i = p$ for a certain $1 \leq i \leq n$ and $x_j = 0$ for $j \neq i$. So the constant $\frac{e^p}{p^p}$ is the best possible.

Our goal is to obtain a similar result for sequences of positive operators in Hilbert space.

Let $\mathcal{H}$ be a complex Hilbert space with inner scalar product $\langle \cdot, \cdot \rangle$ and $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators acting in Hilbert space $\mathcal{H}$. $I_{\mathcal{H}}$ will denote the unity in $\mathcal{B}(\mathcal{H})$. An element A of $\mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Ax, x\rangle \geq 0$ for all elements $x \in \mathcal{H}$. Let A and B be two positive elements of $\mathcal{B}(\mathcal{H})$. Then $A \geq B$ means that $\langle Ax, x\rangle - \langle Bx, x\rangle \geq 0$ for every $x \in \mathcal{H}$. We need the following properties of positive operators [1, 4].

2010 *Mathematics Subject Classification*: primary 47B60; secondary 47A30.

Key words and phrases: commuting operators, positive selfadjoint operator, spectral representation.

Received September 21, 2009, revised May 2011. Editor V. Müller.

Theorem 1.0.2. *Let $\mathcal{B}^+(\mathcal{H})$ be the set of all positive elements of $\mathcal{B}(\mathcal{H})$. Then,*

a) *$\mathcal{B}^+(\mathcal{H})$ is a closed cone,*
b) *$A, B \in \mathcal{B}^+(\mathcal{H})$ and commute $\Longrightarrow AB \in \mathcal{B}^+(\mathcal{H})$,*
c) *$A \in \mathcal{B}^+(\mathcal{H}) \Longleftrightarrow A = B^2$, (B is a positive selfadjoint operator).*
d) *$A \in \mathcal{B}^+(\mathcal{H})$ and selfadjoint if and only if, A has a spectral representation of the form:*

$$A = \int_m^{M+\epsilon} \lambda dE_\lambda \,, \tag{1.0.2}$$

where, ϵ is any positive real number,

$$0 \leq m = \inf_{\|x\|=1} \langle Ax, x\rangle \leq M = \sup_{\|x\|=1} \langle Ax, x\rangle < +\infty \,.$$

e) *If A is selfadjoint with spectral representation* (1.0.2)*, then for every real function f continuous on $[m, M+\epsilon]$,*

$$f(A) = \int_m^{M+\epsilon} f(\lambda) dE_\lambda \,, \tag{1.0.3}$$

and $f(A) = 0$ (resp. $f(A) \geq 0$) if and only if, $f(\lambda) = 0$ (resp. $f(\lambda) \geq 0$) on $[m, M+\epsilon]$.

Note that m and M in precedent theorem are respectively the smallest and biggest values of the spectrum of A.

Definition 1.0.3. Let $A \in \mathcal{B}^+(\mathcal{H})$. $\exp(A)$ is the element of $\mathcal{B}(\mathcal{H})$ given by formula,

$$\exp(A) = \sum_{k=0}^{+\infty} \frac{A^k}{k!} \,. \tag{1.0.4}$$

It is easy to check that $\exp(z.I_{\mathcal{H}}) = \exp(z) \cdot I_{\mathcal{H}}$ for any complex z. Moreover, if A, B are two commuting elements of $\mathcal{B}(\mathcal{H})$ then,

$$\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A) \,.$$

If A is a selfadjoint element of $\mathcal{B}^+(\mathcal{H})$ with spectral representation (1.0.2) and p a natural number, then according to Theorem 1.0.2, we have representations

$$A^p = \int_m^{M+\epsilon} \lambda^p dE_\lambda \quad \text{and} \quad \exp(A) = \int_m^{M+\epsilon} \exp(\lambda) dE_\lambda \,, \tag{1.0.5}$$

which we will frequently use throughout this paper.

We have the following main results:

Theorem 1.0.4. *Let $A \in \mathcal{B}^+(\mathcal{H})$. Then for every natural $p \geq 1$,*

$$\frac{e^p}{p^p} A^p \leq \exp(A) \,. \tag{1.0.6}$$

Moreover, if $A = p \cdot I_{\mathcal{H}}$ then, we have equality in (1.0.6) *and the constant $\frac{e^p}{p^p}$ is the best possible.*

Theorem 1.0.5. *Let $(A_i)_{i=1}^n$ be a finite sequence of commuting, selfadjoint elements of $\mathcal{B}^+(\mathcal{H})$. Then for every natural $p \geq 1$,*

$$\frac{e^p}{p^p}\sum_{i=1}^n A_i^p \leq \exp\Big(\sum_{i=1}^n A_i\Big). \tag{1.0.7}$$

Moreover, if for a certain $1 \leq i \leq n$, $A_i = p \cdot I_{\mathcal{H}}$ and $A_j = 0$ for $j \neq i$, then, we have equality in (1.0.7) *and the constant $\frac{e^p}{p^p}$ is the best possible.*

Remark 1.0.6. If A_i are roots of polynomial $P_n(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 I_{\mathcal{H}}$ and all operators $A_i - A_j$ $(i \neq j)$ are invertible then, for every natural $p \geq 1$,

$$\frac{e^p}{p^p}\sum_{i=1}^n A_i^p \leq \frac{e^p}{p^p}(-a_{n-1})^p \cdot I_{\mathcal{H}} \leq e^{-a_{n-1}} \cdot I_{\mathcal{H}} \tag{1.0.8}$$

Indeed, as in the scalar case, we have $A_1 + A_2 + \cdots + A_n = -a_{n-1} \cdot I_{\mathcal{H}}$.

Theorem 1.0.7. *Let $p \geq 1$ be a natural number and $(A_i)_{i=1}^n$ a finite sequence of invertible, commuting and selfadjoint elements of $\mathcal{B}^+(\mathcal{H})$ such that $0 < A_i \leq p \cdot I_{\mathcal{H}}$ for every $1 \leq i \leq n$. Then,*

$$\exp\Big(\sum_{i=1}^n A_i\Big) \leq \frac{p^p}{n}e^{np}\Big(\sum_{i=1}^n A_i^{-p}\Big). \tag{1.0.9}$$

Moreover, if $1 \leq i \leq n$, $A_i = p \cdot I_{\mathcal{H}}$ for every $1 \leq i \leq n$ then, we have equality in (1.0.9) *and the constant $\frac{p^p}{n}e^{np}$ is the best possible.*

From Theorems 1.0.5 and 1.0.7, follows,

Corollary 1.0.8. *Let $p \geq 1$ be a natural number and $(A_i)_{i=1}^n$ a finite sequence of invertible, commuting and selfadjoint elements of $\mathcal{B}^+(\mathcal{H})$ such that $0 < A_i \leq p \cdot I_{\mathcal{H}}$ for every $1 \leq i \leq n$. Then,*

$$\frac{e^p}{p^p}\sum_{i=1}^n A_i^p \leq \exp\Big(\sum_{i=1}^n A_i\Big) \leq \frac{p^p}{n}e^{np}\Big(\sum_{i=1}^n A_i^{-p}\Big). \tag{1.0.10}$$

Constants $\frac{p^p}{n}$ and $\frac{p^p}{n}e^{np}$, are the best possible.

2. Proofs of main results

2.1. Theorem 1.0.4.

Proof. Consider in the space $\mathcal{H}$ the selfadjoint operator $B = \exp(A) - e^p p^{-p} A^p$. We need to prove that B is positive. Let

$$A = \int_m^{M+\epsilon} \lambda dE_\lambda$$

be the spectral representation of A. By Theorem 1.0.2,

$$B = \int_m^{M+\epsilon} (e^\lambda - e^p p^{-p} \lambda^p) dE_\lambda\,. \tag{2.1.1}$$

According to [2], we have

$$\lambda \geq 0 \Longrightarrow e^{\lambda} - e^p p^{-p} \lambda^p \geq 0\,. \tag{2.1.2}$$

Hence, operator B is positive. It is easy to check that if $A = p \cdot I_{\mathcal{H}}$ then, we have equality in (1.0.6). Let now α be a constant such that $\alpha A^p \leq \exp(A)$ for all $A \in \mathcal{B}^+(\mathcal{H})$. Setting $A = p \cdot I_{\mathcal{H}}$, we obtain that $\alpha \leq e^p p^{-p}$. This finishes the proof. □

2.2. **Theorem 1.0.5.** To prove this theorem we need the following lemma.

Lemma 2.2.1. *Let $p \geq 1$ be a natural number and $(A_i)_{i=1}^n$ a finite sequence of commuting, selfadjoint elements of $\mathcal{B}^+(\mathcal{H})$. Then,*

$$\sum_{i=1}^n A_i^p \leq \Big(\sum_{i=1}^n A_i\Big)^p\,. \tag{2.2.1}$$

Proof. If $n = 2$ then by Theorem 1.0.2, operator $A_1^k A_2^{p-k}$ is positive for every $0 \leq k \leq p$. By the binomial theorem, we have:

$$(A_1 + A_2)^p = \sum_{k=0}^p \frac{p!}{k!(p-k)!} A_1^k A_2^{p-k} = A_1^p + A_2^p + \sum_{k=1}^{p-1} \frac{p!}{k!(p-k)!} A_1^k A_2^{p-k}\,.$$

Since operator

$$\sum_{k=1}^{p-1} \frac{p!}{k!(p-k)!} A_1^k A_2^{p-k}$$

is positive (as a sum of positive operators), it follows from the last equality that

$$A_1^p + A_2^p \leq (A_1 + A_2)^p\,.$$

Suppose now that

$$\sum_{i=1}^n A_i^p \leq \Big(\sum_{i=1}^n A_i\Big)^p\,.$$

Then,

$$\Big(\sum_{i=1}^{n+1} A_i\Big)^p = \Big(\sum_{i=1}^n A_i + A_{n+1}\Big)^p \geq \Big(\sum_{i=1}^n A_i\Big)^p + A_{n+1}^p$$
$$\geq \sum_{i=1}^n A_i^p + A_{n+1}^p = \sum_{i=1}^{n+1} A_i^p\,.$$

□

Let us now prove Theorem 1.0.5.

Proof. According to lemma and Theorem 1.0.4, we have:

$$e^p p^{-p} \sum_{i=1}^n A_i^p \leq e^p p^{-p} \Big(\sum_{i=1}^n A_i\Big)^p \leq \exp\Big(\sum_{i=1}^n A_i\Big)\,. \tag{2.2.2}$$

On the other hands, it is easy to check that we have equalities in this last formula if we set $A_i = p \cdot I_{\mathcal{H}}$ for a certain i and null operator for others indices. The same argumentation used in the precedent theorem shows that $e^p p^{-p}$ is the best possible constant. $\square$

2.3. **Theorem 1.0.7.**

Proof. Let us firstly remark that invertibility of operators A_i, $(i = 1, 2, \dots, n)$ implies that

$$A_i = \int_{m_i}^{M_i+\epsilon} \lambda dE_\lambda, \quad m_i > 0$$

and

$$A_i^{-1} = \int_{m_i}^{M_i+\epsilon} \lambda^{-1} dE_\lambda \,.$$

Since $A_i \leq p \cdot I_{\mathcal{H}}$, it follows from the spectral representation

$$A_i - p \cdot I_{\mathcal{H}} = \int_{m_i}^{M_i+\epsilon} (\lambda - p) dE_\lambda$$

that for all $1 \leq i \leq n$,

$$\lambda \in [m_i, M_i + \epsilon] \Longrightarrow \lambda \leq p \,.$$

Consequently, for all $1 \leq i \leq n$,

$$A_i^{-1} - p^{-1} \cdot I_{\mathcal{H}} = \int_{m_i}^{M_i+\epsilon} (\lambda^{-1} - p^{-1}) dE_\lambda \Longrightarrow p^{-1} \cdot I_{\mathcal{H}} \leq A_i^{-1}$$

$$\Longrightarrow \frac{n}{p^p} \cdot I_{\mathcal{H}} \leq \sum_{i=1}^{n} A_i^{-p} \,.$$

Selfadjoint operators $A_1^{-1}, A_2^{-1}, \dots, A_n^{-1}$ are bounded, commuting and positive. Using Theorem 1.0.5, we obtain

$$\frac{e^p}{p^p} \sum_{i=1}^{n} A_i^{-p} = \frac{e^p}{p^p} \sum_{i=1}^{n} (A_i^{-1})^p \leq \exp\Big(\sum_{i=1}^{n} A_i^{-1}\Big) \,. \tag{2.3.1}$$

Since, $\exp(A) \leq \exp(B)$ for $A \leq B$ then, we have finally

$$\exp\Big(\sum_{i=1}^{n} A_i\Big) \leq \exp\Big(\sum_{i=1}^{n} p \cdot I_{\mathcal{H}}\Big) = e^{np} \frac{p^p}{n} \frac{n}{p^p} \cdot I_{\mathcal{H}} \leq e^{np} \frac{p^p}{n} \sum_{i=1}^{n} A_i^{-p} \,.$$

It is clear that for equality holds for $A_i = p \cdot I_{\mathcal{H}}$ for every $1 \leq i \leq n$. This finishes the proof. $\square$

References

[1] Akhiezer, N. I., Glasman, I. M., *Theory of linear operators in Hilbert space*, Tech. report, Vyshcha Shkola, Kharkov, 1977, English transl. Pitman (APP), 1981.

[2] Belaidi, B., Farissi, A. El, Latreuch, Z., *Inequalities between sum of the powers and the exponential of sum of nonnegative sequence*, RGMIA Research Collection, 11 (1), Article 6, 2008.

[3] Qi, F., *Inequalities between sum of the squares and the exponential of sum of nonnegative sequence*, J. Inequal. Pure Appl. Math. **8** (3) (2007), 1–5, Art. 78.

[4] Weidman, J., *Linear operators in Hilbert spaces*, New York, Springer, 1980.

Laboratoire de Mathématiques pures et appliquées,
Abdelhmid Ibn Badis Mostaganem University,
B.O. 227, Mostaganem (27000), Algeria
E-mail: bbendoukha@gmail.com bendahmanehafida@yahoo.fr