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ORDER BOUNDED ORTHOSYMMETRIC BILINEAR OPERATOR

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Abstract. It is proved by an order theoretical and purely algebraic method that any order bounded orthosymmetric bilinear operator $b\colon E\times E\rightarrow F$ where E and F are Archimedean vector lattices is symmetric. This leads to a new and short proof of the commutativity of Archimedean almost f-algebras.

Keywords: vector lattice, positive bilinear operator, orthosymmetric bilinear operator, lattice bimorphism

MSC 2010: 06F25, 46A40, 47A65

1. Introduction

In [10] G. Buskes and A. van Rooij introduced the class of orthosymmetric bilinear operators on vector lattices. It is only recently that the class of such operators have been getting more attention, see [8], [11]. A number of important properties of such operators was revealed. In particular, Buskes and van Rooij in [10] proved that any positive orthosymmetric bilinear operator is symmetric. However, the disadvantage of this approach is that the proof is not intrinsic, i.e., does not take place in the vector lattice itself and makes use of analytic means. The same authors in [10] proved that every positive orthosymmetric bilinear operator defined on a sublattice of an f-algebra can be factored through a positive linear operator and the algebra multiplication. These results gave rise to the concept of the square of a vector lattice, developed in [11]. Recently, G. Buskes and A. G. Kusraev in [8] proved that all orthosymmetric order bounded bilinear operators from $E\times E$ to the relatively uniformly complete vector lattice F can be represented as compositions of order bounded linear operators from $E^{\odot}$ the square of E to F with the canonical bimorphism. Since we wish to avoid representation in this paper, we refer the reader to the proof of Theorem 5.8 in [13] for purely algebraic proof of some analytic techniques

(Lemma 1.6 in [8]) used by Buskes and Kusraev in the proof of the above result. We also refer to [13] for a purely algebraic approach based on the tensor product used by the same authors in the construction of the square of a vector lattice. The present paper is largely motived by that work of Buskes and Kusraev [8]. In fact it could have been entitled "A look at order bounded orthosymmetric operators from an algebraic point of view". Indeed, our main purpose in this paper is to prove that any order bounded orthosymmetric bilinear operator is symmetric. All our results, as well as their proofs, are purely algebraic and do not use any analytic tools. In this sense, we provide not only new results but also new techniques, which we think are useful additions to the literature.

The paper is organized as follows. The main purpose of the first section is to fix the notion and terminology and give a brief outline of some useful results which are of particular importance to this paper. The main results are discussed in the second section.

We use [1], [4], [5], [6], [7], [12], [14], [15], [16], [17] as a starting point and we refer the reader to these standard monographs for terminology, notation and properties not explained or proved in this paper.

2. Preliminaries

A lattice ordered group (briefly an ℓ-group) G is called *Archimedean* if for each nonzero $x \in G$ the set $\{nx\colon\ n = \pm 1, \pm 2, \ldots\}$ has no upper bound in G. *In order to avoid unnecessary repetition we will assume throughout that all ℓ-groups under consideration are Archimedean.* An ℓ-group G which is simultaneously a ring with the property that $xy \in G^+$ for all $x, y \in G^+$ (equivalently, $|xy| \leqslant |x||y|$ for all $x, y \in G$) (where G^+ is the positive cone of G) is called a *lattice ordered ring* (briefly, an *ℓ-ring*). If in addition, G is a real vector lattice, then G is called an *ℓ-algebra*.

An ℓ-algebra A is said to be an *f-algebra* if $x \wedge y = 0$ and $z \in A^+$ implies $xz \wedge y = zx \wedge y = 0$. An *almost f-algebra* A is an ℓ-algebra with the additional property that $x \wedge y = 0$ in A implies $xy = 0$. Both the f-algebras and the almost f-algebras are automatically commutative and have positive squares. More about almost f-algebras can be found in [3].

Next, we discuss linear operators on vector lattices. Let E and F be vector lattices with positive cones E^+ and F^+, respectively, and let T be a linear operator from E into F. One says that T is *order bounded* if for each $x \in E^+$ there exists $y \in F^+$ such that $|T(z)| \leqslant y$ in F whenever $|z| \leqslant x$ in E. The linear operator T is said to be *positive* if $T(E^+) \subset F^+$. The linear operator T is called a *lattice homomorphism* (or *Riesz homomorphism*) whenever $x \wedge y = 0$ implies $T(x) \wedge T(y) = 0$. Obviously, every lattice homomorphism is positive and then order bounded. The set $\mathcal{L}_b(E)$ of all order bounded linear operators on E is an ordered vector space with respect to pointwise operations and order. The positive cone of $\mathcal{L}_b(E)$ is the subset of all positive linear operators. We end this section with some definitions and notation of the classes of bilinear operators on products of Archimedean vector lattices. Let E, F, and G be Archimedean vector lattices. A bilinear operator $b\colon\ E \times F \to G$ is called *positive* if $b(x,y) \geqslant 0$ for all $0 \leqslant x \in E$ and $0 \leqslant y \in F$, and *regular* if it can be represented as the difference of two positive bilinear operators. For any positive bilinear operator b we have $|b(x,y)| \leqslant b(|x|,|y|)$ for all $x \in E$, $y \in F$. A bilinear operator $b\colon\ E \times F \to G$ is said to be a *lattice bimorphism* whenever the partial operators

$$\begin{aligned} b(x,\cdot)\colon\ F &\to G, \\ y &\mapsto b(x,y) \end{aligned}$$

and

$$\begin{aligned} b(\cdot,y)\colon\ E &\to G \\ x &\mapsto b(x,y) \end{aligned}$$

are lattice homomorphisms for every $x \in E^+$ and $y \in F^+$. Evidently, every lattice bimorphism is positive. For a positive bilinear operator b the following assertions are equivalent:

(1) b is a lattice bimorphism;
(2) $|b(x,y)| = b(|x|,|y|)$ for all $x \in E$ and $y \in F$.

A bilinear operator $b\colon\ E \times E \to G$ is called *orthosymmetric* if $|x| \wedge |y| = 0$ implies $b(x,y) = 0$ for arbitrary $x, y \in E$. The difference of two positive orthosymmetric bilinear operators is called *orthoregular*. Recall also that b is said to be *symmetric* if $b(x,y) = b(y,x)$.

3. Main results

We plunge into the matter by the following basic proposition, which turns out to be useful for later purposes.

Proposition 1. *Let E be a vector space and $b\colon\ E \times E \to \mathbb{R}$ a bilinear operator such that $b(x,y) = 0$ if and only if $b(y,x) = 0$ for $x, y \in E$. Then*

$$b(x,y) = b(y,x) \qquad \text{for all } x, y \in E$$

or

$$b(x,y) = -\,b(y,x) \quad \text{for all } x, y \in E.$$

P r o o f. We distinguish two cases:

First case: if $b(x,x)=0$ for all $x\in E$ then $b(x+y,x+y)=0$ for all $x,y\in E$. Thus $b(x,x)+b(x,y)+b(y,x)+b(y,y)=0$. On the other hand, $b(x,x)=b(y,y)=0$. Finally, $b(x,y)=-b(y,x)$ for all $x,y\in E$.

Second case: if there exists $a\in E$ such that $b(a,a)\neq 0$, we claim that $b(x,y)=b(y,x)$ for all $x,y\in E$. The proof proceeds in two steps.

Step 1: We show that $b(a,x)=b(x,a)$ for all $x\in E$. For every $\lambda\in\mathbb{R}$, we have $b(a,x+\lambda a)=b(a,x)+\lambda b(a,a)$. We derive that for $\lambda=-b(a,x)/b(a,a)$ we get $b(a,x+\lambda a)=0$. It follows from the hypothesis that $b(x+\lambda a,a)=0$. Therefore, $b(a,x+\lambda a)=b(x+\lambda a,a)=0$ and thus $b(a,x)+\lambda b(a,a)=b(x,a)+\lambda b(a,a)$ and therefore,

$$b(a,x)=b(x,a)\quad\text{for all } x\in E.$$

Step 2: We claim that $b(x,y)=b(y,x)$ for all $x,y\in E$. If $b(x,a)\neq 0$, then for $\lambda=-b(x,y)/b(x,a)$ we get $b(x,y+\lambda a)=0$ and thus $b(y+\lambda a,x)=0$. So $b(x,y+\lambda a)=b(y+\lambda a,x)=0$, therefore $b(x,y)+\lambda b(x,a)=b(y,x)+\lambda b(a,x)$. Moreover, step 1 yields that $b(x,y)=b(y,x)$. Similarly, if $b(y,a)\neq 0$ we obtain $b(y,x)=b(x,y)$. Now, $b(x,a)=b(y,a)=0$. So, for all $\lambda\in\mathbb{R}$, we have $b(x+a,y+\lambda a)=b(x,y)+\lambda b(a,a)$. If we put $\lambda=-b(x,y)/b(a,a)$, then $b(x+a,y+\lambda a)=0$. By hypothesis, we obtain that $b(y+\lambda a,x+a)$. This shows that $b(x+a,y+\lambda a)=b(y+\lambda a,x+a)=0$. We get that $b(x,y)+\lambda b(a,a)=b(y,x)+\lambda b(a,a)$ and so

$$b(x,y)=b(y,x)$$

for all $x,y\in E$, and the proof is completed. □

The next result is deduced from the preceding proposition by classical means. The details follow.

Theorem 2. *Let E be an Archimedean vector lattice. Then any orthosymmetric lattice bimorphism from $E\times E$ to $\mathbb{R}$ is symmetric.*

P r o o f. Let $b\colon E\times E\to\mathbb{R}$ be an orthosymmetric lattice bimorphism. In order to apply Proposition 1, we show that $b(x,y)=0$ if and only if $b(y,x)=0$ for all $x,y\in E$.

Suppose first that $x,y\in E^+$ are such that $b(x,y)=0$. Now by hypothesis and $(x-x\wedge y)\wedge(y-x\wedge y)=0$ it follows that $b(y-x\wedge y,x-x\wedge y)=0$. So, we can write

$$b(y,x)=b(y,x\wedge y)+b(x\wedge y,x)-b(x\wedge y,x\wedge y). \tag{3.1}$$

Observe now that $0 \leqslant b(x \wedge y, x \wedge y) \leqslant b(x,y) = 0$, so $b(x \wedge y, x \wedge y) = 0$. The fact that b is orthosymmetric and positive implies $b(z,z) \geqslant 0$ for all $z \in E$. This yields that $b(\lambda y + x \wedge y, \lambda y + x \wedge y) \geqslant 0$ for all $\lambda \in \mathbb{R}$. This implies $\lambda^2 b(y,y) + \lambda(b(y, x \wedge y) + b(x \wedge y, y)) + b(x \wedge y, x \wedge y) \geqslant 0$ for all $\lambda \in \mathbb{R}$. From the "negative discriminant" inequality, it follows that $(b(y, x \wedge y) + b(x \wedge y, y))^2 \leqslant 4b(y,y)b(x \wedge y, x \wedge y)$. On the other hand, $b(x \wedge y, x \wedge y) = 0$, so we have $b(y, x \wedge y) + b(x \wedge y, y) = 0$. Now from the fact that $b(y, x \wedge y) \geqslant 0$ and $b(x \wedge y, y) \geqslant 0$ (because $x, y \in E^+$) it follows that $b(y, x \wedge y) = b(x \wedge y, y) = 0$.

Similarly $b(x, x \wedge y) + b(x \wedge y, x) = 0$. And thus $b(x, x \wedge y) = b(x \wedge y, x) = 0$. Finally, via (3.1) we obtain

$$b(y,x) = 0.$$

Assume now that $x, y \in E$ such that $b(x,y) = 0$. By virtue of the fact that $|b(x,y)| = b(|x|,|y|)$, we obtain $b(|x|,|y|) = 0$. So by the first case, $b(|y|,|x|) = 0$ and thus $b(y,x) = 0$. Consequently,

$$b(x,y) = 0 \quad \text{if and only if} \quad b(y,x) = 0$$

for all $x, y \in E$. According to the preceding proposition $b(x,y) = b(y,x)$ for all $x, y \in E$ or $b(x,y) = -b(y,x)$ for all $x, y \in E$. However, since b is positive (it is a lattice bimorphism) we have in the latter case $0 \leqslant b(x,y) = -b(y,x) \leqslant 0$ for all $x, y \in E^+$. This implies that b is zero on E. Hence, we have

$$b(x,y) = b(y,x) \quad \text{for all } x, y \in E,$$

which completes the proof of the theorem. □

We are now in position to prove the first main result of the present work.

Theorem 3. *Let E and F be Archimedean vector lattices. Then any orthosymmetric lattice bimorphism b from $E \times E$ to F is symmetric.*

Proof. Let $x, y \in E^+$, consider the vector sublattice E_0 of E generated by x and y, the vector sublattice F_0 of F generated by $b(x,y)$, $b(y,x)$, $b(x,x)$ and $b(y,y)$. By [9, 1.2 (ii)], F_0 is a slender vector sublattice of F. Now by Theorem 2.2 in [9], the set $H(F_0)$ of all real-valued lattice homomorphisms on F_0 separates the points of F_0, that is, if $a \in F_0$ and $w(a) = 0$ for all $w \in H(F_0)$ then $a = 0$. On the other hand, for all $w \in H(F_0)$ the bilinear operator $w \circ b_{/E_o \times E_0} \colon E_o \times E_0 \to \mathbb{R}$ is an orthosymmetric lattice bimorphism. So, by the preceding theorem, $w \circ b_{/E_o \times E_0}$ is symmetric. And thus

$$w(b(x,y)) = w(b(y,x))$$

for all $w \in H(F_0)$. Consequently,

$$b(x,y) = b(y,x)$$

for all $x, y \in E^+$. The general case is deduced by linearity since every $x, y \in E$ are of the form $x = x^+ - x^-$ and $y = y^+ - y^-$, where $x^+, x^-, y^+, y^- \in E^+$. This completes the proof of the theorem. $\square$

For an arbitrary Archimedean vector lattice E there exist a vector lattice $E^{\odot}$ (unique up to isomorphism) and a lattice bimorphism $\odot\colon (x,y) \to x \odot y$ from $E \times E$ to $E^{\odot}$ such that the following assertions hold:

(1) if b is a symmetric lattice bimorphism from $E \times E$ to some vector lattice F then there is a unique lattice homomorphism $\Phi_b\colon E^{\odot} \to F$ with $b = \Phi_b \odot$;

(2) given an arbitrary $u \in E^{\odot}$, there is $e_0 \in E^+$ such that, for every $\varepsilon > 0$, one can choose $x_1, \dots, x_n, y_1, \dots, y_n \in E$ with

$$\left| u - \sum_{i=1}^{n} x_i \odot y_i \right| \leqslant \varepsilon e_0 \odot e_0;$$

(3) for any $x, y \in E$ we have $x \odot y = 0$ if and only if $|x| \wedge |y| = 0$;

(4) given an element $0 < u \in E^{\odot}$, there exists $e \in E^+$ with $0 < e_0 \odot e_0 \leqslant u$.

The vector lattice $E^{\odot}$ (or the pair $(E^{\odot}, \odot)$) uniquely (up to lattice isomorphism) determined by an arbitrary Archimedean vector lattice E is called the square of E. The lattice bimorphism $\odot\colon E \times E \to E^{\odot}$ is called the canonical bimorphism. The construction of $E^{\odot}$ was first introduced in [9] as follows. Denote by J the smallest relatively uniformly closed order ideal in Fremlin's tensor product $E \mathbin{\overline{\otimes}} E$ containing the set $\{x \otimes y\colon x, y \in E, x \perp y\}$. Define $E^{\odot} = E \otimes E/J$ and $\odot = \varphi \otimes$ where φ: $E \mathbin{\overline{\otimes}} E \to E^{\odot}$ is the quotient homomorphism. Then $E^{\odot}$ is an Archimedean vector lattice and $\odot$ is a lattice bimorphism. Observe that $\odot$ is orthosymmetric. Indeed, if $x \perp y$ then $x \otimes y \in J = \ker \varphi$, thus $x \odot y = \varphi(x \otimes y)$. At this point $\odot$ is an orthosymmetric lattice bimorphism. Consequently, by the preceding theorem, $\odot$ is symmetrical, so that

$$x \odot y = y \odot x$$

for all $x, y \in E$.

We have gathered now all of the ingredients for the proof of the central theorem of this paper, which states that any order bounded orthosymmetric bilinear operator $E \times E \to F$, where E and F are Archimedean vector lattices, is symmetric. The details follow.

Theorem 4. *Let E and F are Archimedean vector lattices. Then every order bounded orthosymmetric bilinear operator $b\colon E \times E \to F$ is symmetric.*

P r o o f. By virtue of the fact that $F \subset F^{ru}$, the uniform completion of F, we can assume without loss of generality that F is relatively uniformly complete. Now according to Theorem 3.4 in [8] there exists a unique order bounded linear operator $\Phi_b\colon E^{\odot} \to F$ such that

$$b(x,y) = \Phi_b(x \odot y)$$

for all $x, y \in E$. On the other hand, we have already mentioned before the preceding theorem that $\odot$ is symmetric. So, we can write

$$b(x,y) = \Phi_b(x \odot y) = \Phi_b(y \odot x) = b(y,x).$$

Thus b is symmetric, which is the desired result. $\square$

In particular, any orthoregular bilinear operator from $E \times E$ to F, where E and F are Archimedean vector lattices, is symmetric.

Now, we give a short historical note about the following application. The commutativity of almost f-algebras has been established by many authors. This result has been proved by Basly and Triki in [2], some years later by Bernau and Huijsmans in [3], and more recently, in [10], by Buskes and van Rooij. Note that except for Bernau and Huijsmans, these authors rely on analytical means and the proof does not take place in the almost f-algebra itself. Note also the disadvantage of Bernau and Huijsman's approach that the proof is long and quite involved. In the final paragraph of this paper, we intend to make some contributions to this area. We give a new proof of the commutativity of almost f-algebras that uses purely algebraic and order theoretical means and does not involve any representation theorems. Interestingly, it deals with positive orthosymmetric maps rather than algebra multiplications and it does not make use of associativity.

Corollary 5. *Any Archimedean almost f-algebra is commutative.*

P r o o f. Let A be an Archimedean vector lattice, and assume that A is an almost f-algebra under $*$. Then the bilinear operator

$$\begin{aligned} b\colon\ & A \times A \to A \\ & (x,y) \mapsto x * y \end{aligned}$$

is a positive orthosymmetric operator, and by the preceding theorem b is symmetric which implies that

$$x * y = y * x$$

for all $x, y \in E$ and we are done. $\square$

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