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TRIPLE AUTOMORPHISMS OF SIMPLE LIE ALGEBRAS

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Abstract. An invertible linear map φ on a Lie algebra L is called a triple automorphism of it if $\varphi([x,[y,z]]) = [\varphi(x),[\varphi(y),\varphi(z)]]$ for $\forall\, x,y,z \in L$. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank l defined over an algebraically closed field F of characteristic zero, $\mathfrak{p}$ an arbitrary parabolic subalgebra of $\mathfrak{g}$. It is shown in this paper that an invertible linear map φ on $\mathfrak{p}$ is a triple automorphism if and only if either φ itself is an automorphism of $\mathfrak{p}$ or it is the composition of an automorphism of $\mathfrak{p}$ and an extremal map of order 2.

Keywords: simple Lie algebras, parabolic subalgebras, triple automorphisms of Lie algebras

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1. Introduction

For an associative algebra $\mathfrak{A}$ defined over a field F, a linear map φ on it is called *a Lie triple derivation* if $\varphi([x,[y,z]]) = [\varphi(x),[y,z]] + [x,[\varphi(y),z]] + [x,[y,\varphi(z)]]$ for any $x,y,z \in \mathfrak{A}$, where the bracket operation is defined as $[x,y] = xy - yx$. The set of all Lie triple derivations of $\mathfrak{A}$, which we denote by $\mathrm{TDer}(\mathfrak{A})$, forms a Lie algebra relative to the ordinary bracket operation $[\varphi_1,\varphi_2] = \varphi_1\varphi_2 - \varphi_2\varphi_1$, and contains the set $\mathrm{Der}(\mathfrak{A})$ of all Lie derivations of $\mathfrak{A}$ as its subalgebra. The concept, Lie triple derivation, was first introduced by C. Robert Miers [1] for Von Neumann algebras and was recently extensively studied by other authors for more general operator algebras. For example, Lie triple derivations of TUHF algebras were determined by P. Ji and L. Wang [2]; Lie triple derivations of nest algebras were described by J. H. Zhang, et al., [3]; and by F. Lu [4], respectively. More recently, H. T. Wang and Q. G. Li [5] transferred this concept to Lie algebras and they described triple derivations

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for a nilpotent linear Lie algebra over a commutative ring. In fact, the derivation algebra, $\mathrm{Der}(\mathfrak{A})$, of an algebra $\mathfrak{A}$ has a close relation with the automorphism group, $\mathrm{Aut}(\mathfrak{A})$, of $\mathfrak{A}$. In our view, $\mathrm{Der}(\mathfrak{A})$ is just a linearization of $\mathrm{Aut}(\mathfrak{A})$, and conversely, $\mathrm{Aut}(\mathfrak{A})$ is just a group object of $\mathrm{Der}(\mathfrak{A})$. From this point of view, a similar concept, which can be viewed as the group object of $\mathrm{TDer}(\mathfrak{A})$, is now introduced naturally. An invertible linear map φ on $\mathfrak{A}$ is called a *Lie triple automorphism* if $\varphi([x,[y,z]]) = [\varphi(x),[\varphi(y),\varphi(z)]]$ for any $x,y,z \in \mathfrak{A}$. For an abstract Lie algebra L over F, we define the so-called *triple automorphism* similarly. An invertible linear map φ on L is called a *triple automorphism* if $\varphi([x,[y,z]]) = [\varphi(x),[\varphi(y),\varphi(z)]]$ for any $x,y,z \in L$. It is clear that the product of two such maps and the inverse of such map are also such maps. So all triple automorphisms of L form a group under composition of maps, which is denoted by $\mathrm{TAut}(L)$. Note that every automorphism of L is obviously a triple automorphism of it. However, the converse of this assertion may be false. As a simple example, we consider the radical of a simple Lie algebra of type A_2, consisting of all 3×3 strictly upper triangular matrices. One will see that the invertible linear map, permuting E_{12} and E_{13}, fixing E_{23}, and extending linearly, is a triple automorphism of it, but fails to be an automorphism. To give a more interesting example we consider the general linear Lie algebra $\mathrm{gl}(n,F)$ consisting of all $n \times n$ matrices over F. Define ω to be the map on $\mathrm{gl}(n,F)$ sending any matrix $(a_{i,j}) \in \mathrm{gl}(n,F)$ to $((-1)^{j-i+1}a_{i,j})$. Then it is not difficult to verify that ω is a triple automorphism of $\mathrm{gl}(n,F)$, but it fails to be an automorphism. The above two examples show that it may be interesting to study how much $\mathrm{TAut}(L)$ differs from $\mathrm{Aut}(L)$ for a given Lie algebra L. In this article, we wish to answer this question for L an arbitrary parabolic subalgebra of a finite-dimensional simple Lie algebra over F.

2. Notation and some elementary results

We follow the notation of [6]. Let F be an algebraically closed field of characteristic zero, $\mathfrak{g}$ a finite-dimensional simple Lie algebra of rank l over F, $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$, $\Phi \subseteq \mathfrak{h}^*$ the corresponding root system of $\mathfrak{g}$, Δ a fixed base of Φ and Φ^+ (resp., Φ^-) the set of positive (resp., negative) roots relative to Δ. The roots in Δ are called *simple*. Actually, Δ defines a partial order on Φ in such a way that $\beta \prec \alpha$ iff $\alpha - \beta$ is a sum of simple roots or $\beta = \alpha$. For $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha \in \Phi$, the integer $\sum_{\alpha\in\Delta} k_\alpha$ is called the *height* of β and denoted by $\mathrm{ht}\,\beta$. By θ we denote the unique maximal root in Φ. We denote by $\ker\alpha$, for $\alpha \in \Phi$, the kernel of α in $\mathfrak{h}$. For each $\alpha \in \Phi^+$, let e_α be a non-zero element of $\mathfrak{g}_\alpha$, then there is a unique element $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $e_\alpha, e_{-\alpha}, h_\alpha = [e_\alpha, e_{-\alpha}]$ span a three-dimensional simple subalgebra of $\mathfrak{g}$

isomorphic to sl$(2,F)$ via $e_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The set $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ forms a basis of $\mathfrak{g}$. If $\alpha, \beta, \alpha+\beta \in \Phi$, since $[e_\alpha, e_\beta]$ is a scalar multiple of $e_{\alpha+\beta}$, then we define $N_{\alpha,\beta}$ by $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$, which we call the *structure constants* of $\mathfrak{g}$. We can choose a basis $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ of $\mathfrak{g}$ such that all structure constants of $\mathfrak{g}$ are integers, which we call a *Chevalley basis* of $\mathfrak{g}$. In the remainder of this paper, the set $\{h_\alpha, e_\beta, e_{-\beta} \mid \alpha \in \Delta, \beta \in \Phi^+\}$ will always denote a Chevalley basis of $\mathfrak{g}$. For the base Δ of Φ, let $\mathfrak{d}_\Delta = \{d_\alpha \mid \alpha \in \Delta\}$ be the dual basis of $\mathfrak{h}$ relative to Δ. Namely, $\beta(d_\alpha)$ takes the value 0 when $\beta \neq \alpha \in \Delta$ and takes the value 1 when $\beta = \alpha \in \Delta$. A symmetric bilinear form $(\ ,\ )$ is defined on the l-dimensional real vector space spanned by Φ, which is dual to the Killing form on $\mathfrak{g}$. For $\alpha, \beta \in \Phi$, let $\langle \beta, \alpha \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$. If $\alpha \neq \pm\beta$, let p, q be the greatest non-negative integers for which $\beta - p\alpha, \beta + q\alpha \in \Phi$, then $\langle \beta, \alpha \rangle = p - q$, and $N_{\alpha,\beta} = \pm(p+1)$. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called *parabolic* if it includes some Borel subalgebra. For each subset π of Δ, let $\Phi_\pi = \mathbb{Z}\pi \cap \Phi$, $\Phi_\pi^- = \Phi_\pi \cap \Phi^-$. Define $\mathfrak{p}_\pi$ to be the subalgebra of $\mathfrak{g}$ generated by all $\mathfrak{g}_\alpha$, $\alpha \in \Delta$ or $\alpha \in -\pi$, along with $\mathfrak{h}$. If $\pi = \emptyset$, then $\mathfrak{p}_\pi$ is a Borel subalgebra of $\mathfrak{g}$, which is denoted by $\mathfrak{b}$. It is well known that every parabolic subalgebra of $\mathfrak{g}$ is conjugate under an inner automorphism to one of the $\mathfrak{p}_\pi$, thus in order to determine the triple automorphisms of an arbitrary parabolic subalgebra of $\mathfrak{g}$, it suffices to determine those of $\mathfrak{p}_\pi$. We now introduce several types of standard triple automorphisms for $\mathfrak{p}_\pi$, and later on we will use them to build every triple automorphism of $\mathfrak{p}_\pi$.

(i) If $x \in \mathfrak{p}_\pi$ is ad-nilpotent, then the map $\exp(\operatorname{ad} x)$ is an automorphism of $\mathfrak{p}_\pi$. We denote by $\operatorname{Int}(\mathfrak{p}_\pi)$ the group generated by all such elements. Each element in it is called an *inner automorphism* of $\mathfrak{p}_\pi$. For $\alpha \in \Phi^+ \cup \Phi_\pi^-$ and $t \in F$, te_α is obviously ad-nilpotent in $\mathfrak{p}_\pi$, so the map $\sigma_\alpha(t) = \exp(\operatorname{ad} te_\alpha)$ is an inner automorphism of $\mathfrak{p}_\pi$.

(ii) Let ϱ be a symmetry (nontrivial or trivial) of the Dynkin diagram of Φ, or equivalently, $\langle \varrho(\alpha), \varrho(\beta) \rangle = \langle \alpha, \beta \rangle$ for any $\alpha, \beta \in \Delta$. If $\alpha \in \pi$ implies that $\varrho(\alpha) \in \pi$, then ϱ can be extended to an automorphism $\bar{\varrho}$ of $\Phi^+ \cup \Phi_\pi^-$ by

$$\sum_{\alpha \in \Delta} k_\alpha \alpha \mapsto \sum_{\alpha \in \Delta} k_\alpha \varrho(\alpha).$$

Using $\bar{\varrho}$ we can define an automorphism φ_ϱ of $\mathfrak{p}_\pi$ in the following way:

$$\sum_{\alpha \in \Delta} a_\alpha h_\alpha + \sum_{\alpha \in \Phi^+ \cup \Phi_\pi^-} b_\alpha e_\alpha \mapsto \sum_{\alpha \in \Delta} a_\alpha h_{\varrho(\alpha)} + \sum_{\alpha \in \Phi^+ \cup \Phi_\pi^-} b_\alpha r_\alpha e_{\bar{\varrho}(\alpha)}, \quad a_\alpha, b_\alpha \in F,$$

where $r_\alpha = \pm 1$ and $r_\alpha r_{-\alpha} = 1$ if $\alpha \in \Phi_\pi$; and $N_{\bar{\varrho}(\alpha), \bar{\varrho}(\beta)} r_\alpha r_\beta = N_{\alpha,\beta} r_{\alpha+\beta}$ if $\alpha + \beta$ is also a root. φ_ϱ is called a *graph automorphism* of $\mathfrak{p}_\pi$.

(iii) Let $P = \mathbb{Z}\Phi$ be the set of all $\mathbb{Z}$-linear combinations of the elements of Φ. It is a free abelian group of rank l and has Δ as a basis. A homomorphism χ from the additive group P into the multiplicative group F^* of non-zero elements of F is called a *character* of P. Each character χ of P gives rise to an automorphism φ_χ of $\mathfrak{p}_\pi$, by

$$h + \sum_{\alpha\in\Phi^+\cup\Phi_\pi^-} a_\alpha e_\alpha \mapsto h + \sum_{\alpha\in\Phi^+\cup\Phi_\pi^-} a_\alpha \chi(\alpha) e_\alpha, \quad h \in \mathfrak{h},\ a_\alpha \in F.$$

φ_χ is called a *diagonal automorphism* of $\mathfrak{p}_\pi$.

(iv) Define $\omega\colon\ \mathfrak{p}_\pi \to \mathfrak{p}_\pi$ by

$$h + \sum_{\alpha\in\Phi^+\cup\Phi_\pi^-} a_\alpha e_\alpha \mapsto -h + \sum_{\alpha\in\Phi^+\cup\Phi_\pi^-} (-1)^{(1+ht\ \alpha)} a_\alpha e_\alpha, \quad h \in \mathfrak{h},\ a_\alpha \in F.$$

Obviously ω^2 is just the identity map, thus ω is invertible. It is not difficult to verify that the following equalities hold, where $h, d \in \mathfrak{h}$, $\beta, \gamma \in \Phi^+ \cup \Phi_\pi^-$:

(1) $\varphi([h,[e_\beta,e_\gamma]]) = ([\varphi(h),[\varphi(e_\beta),\varphi(e_\gamma)]])$;
(2) $\varphi([e_\beta,[h,e_\gamma]]) = ([\varphi(e_\beta),[\varphi(h),\varphi(e_\gamma)]])$;
(3) $\varphi([h,[d,e_\gamma]]) = ([\varphi(h),[\varphi(d),\varphi(e_\gamma)]])$;
(4) $\varphi([e_\gamma,[h,d]]) = ([\varphi(e_\gamma),[\varphi(h),\varphi(d)]])$.

Since $\mathfrak{h}$ along with $e_\beta, \beta \in \Phi^+ \cup \Phi_\pi^-$ span $\mathfrak{p}_\pi$, we conclude that ω is exactly a triple automorphism of $\mathfrak{p}_\pi$, which we call an *extremal triple automorphism* of $\mathfrak{p}_\pi$. But it fails to be an automorphism of $\mathfrak{p}_\pi$, since $\omega([d_\alpha, e_\alpha]) = e_\alpha \neq [\omega(d_\alpha), \omega(e_\alpha)] = -e_\alpha$ for $\alpha \in \Delta$.

In this article we will show that any $\varphi \in \mathrm{TAut}(\mathfrak{p}_\pi)$ is just a composition of those standard maps. To prove this statement, we need some preliminary results.

Lemma 2.1. *Let* $\varphi \in \mathrm{TAut}(\mathfrak{p}_\pi)$, *then* $[x,y] = 0 \Leftrightarrow [\varphi(x),\varphi(y)] = 0$.

Proof. Suppose $[x,y]=0$, then $[z,[x,y]]=0$ for every $z \in \mathfrak{p}_\pi$. Applying φ we have that $[\varphi(z),[\varphi(x),\varphi(y)]]=0$ for every $z \in \mathfrak{p}_\pi$. Since $\varphi(z)$, $z \in \mathfrak{p}_\pi$ exhaust $\mathfrak{p}_\pi$, the element $[\varphi(x),\varphi(y)]$ must belong to the center of $\mathfrak{p}_\pi$, so it must be zero. Considering φ^{-1}, we get the converse result. $\square$

For $\beta \in \Phi^+ \cup \Phi_\pi^-$, let

$$X_\beta = \{\alpha \in \Phi^+ \cup \Phi_\pi^- \mid \alpha + \beta \in \Phi \cup \{0\}\};$$
$$Y_\beta = \{\alpha \in \Phi^+ \cup \Phi_\pi^- \mid \alpha + \beta \notin \Phi \cup \{0\}\}.$$

Lemma 2.2. *Let $h \in \mathfrak{h}$, $\beta \in \Phi^+ \cup \Phi_\pi^-$. If $\alpha(h) = 0$ for all $\alpha \in Y_\beta$, then $h = 0$.*

Proof. Let $\Delta_1 = \Delta \cap Y_\beta$ and $\Delta_2 = \Delta \cap X_\beta$. Then $\alpha(h) = 0$ for each $\alpha \in \Delta_1$. For $\alpha \in \Delta_2$, let k be the maximal positive integer such that $\alpha + k\beta \in \Phi \cup \{0\}$, i.e., $\alpha+k\beta \in \Phi\cup\{0\}$ and $\alpha+(k+1)\beta \notin \Phi\cup\{0\}$. Then one will see that $\alpha+k\beta \in \Phi^+\cup\Phi_\pi^-$. Thus $(\alpha+k\beta)(h) = 0$ (by assumption). It follows that $\alpha(h) = 0$ (note that $\beta(h) = 0$). Therefore, $\alpha(h) = 0$ for all $\alpha \in \Delta$, forcing $h = 0$. □

For later use, we need to extend the definition of $\varphi \in \mathrm{TAut}(\mathfrak{p}_\pi)$ to the whole $\mathfrak{g}$ by sending e_β to zero for each $\beta \in \Phi^- \setminus \Phi_\pi^-$ and extending linearly. For brevity, the extension of φ is also denoted by φ.

Lemma 2.3. *Let $\varphi \in \mathrm{TAut}(\mathfrak{p}_\pi)$, $d \in \mathfrak{h}$.*
- (i) *Each $\varphi(e_\beta)$, $\beta \in \Phi^+ \cup \Phi_\pi^-$, is an eigenvector of $\varphi(d)$. In particular, $\varphi(d)$ is semisimple;*
- (ii) *φ sends a semisimple element to a semisimple one;*
- (iii) *φ sends a Cartan subalgebra to a Cartan subalgebra;*
- (iv) *There exists $\sigma \in \mathrm{Int}(\mathfrak{p}_\pi)$ such that $\sigma \cdot \varphi$ stabilizes $\mathfrak{h}$;*
- (v) *If φ stabilizes $\mathfrak{h}$, then for each $\alpha \in \Phi^+ \cup \Phi_\pi^-$, there exists $\beta \in \Phi^+ \cup \Phi_\pi^-$ such that $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_\beta$.*

Proof. For (i), if $\beta(d) = 0$, Lemma 2.1 shows that $[\varphi(d), \varphi(e_\beta)] = 0$, the assertion holds. Now assume that $\beta(d) \neq 0$. Because $\varphi(\mathfrak{h})$ and $\varphi(e_\gamma), \gamma \in \Phi^+ \cup \Phi_\pi^-$, span $\mathfrak{p}_\pi$, we may assume that

$$[\varphi(d), \varphi(e_\beta)] = \varphi(t_\beta) + \sum_{\gamma\in\Phi^+\cup\Phi_\pi^-} b_\gamma \varphi(e_\gamma), \quad t_\beta \in \mathfrak{h},\ b_\gamma \in F.$$

If there exists some $\beta_0 \in \Phi^+ \cup \Phi_\pi^-$, distinct from $\pm\beta$, such that $b_{\beta_0} \neq 0$, we choose $h \in \mathfrak{h}$ such that $\beta(h) = 0$ and $\beta_0(h) \neq 0$. Considering the equality

$$[\varphi(h), [\varphi(d), \varphi(e_\beta)]] = \left[\varphi(h), \varphi(t_\beta) + \sum_{\gamma\in\Phi^+\cup\Phi_\pi^-} b_\gamma \varphi(e_\gamma)\right],$$

we find that the left-hand side of the equality is just zero since $[h, [d, e_\beta]] = 0$. Thus

$$\left[\varphi(h), \varphi(t_\beta) + \sum_{\gamma\in\Phi^+\cup\Phi_\pi^-} b_\gamma \varphi(e_\gamma)\right] = 0,$$

which leads to

$$\left[h, t_\beta + \sum_{\gamma\in\Phi^+\cup\Phi_\pi^-} b_\gamma e_\gamma\right] = \sum_{\gamma\in\Phi^+\cup\Phi_\pi^-} b_\gamma \gamma(h) e_\gamma = 0,$$

which is absurd since $b_{\beta_0}\beta_0(h) \neq 0$. So $[\varphi(d), \varphi(e_\beta)] = \varphi(t_\beta) + b_\beta\varphi(e_\beta) + b_{-\beta}\varphi(e_{-\beta})$. It follows from $[e_\beta, [d, e_\beta]] = 0$ that

$$[\varphi(e_\beta), [\varphi(d), \varphi(e_\beta)]] = [\varphi(e_\beta), \varphi(t_\beta + b_\beta e_\beta + b_{-\beta}e_{-\beta})] = 0.$$

This implies that $[e_\beta, t_\beta + b_\beta e_\beta + b_{-\beta}e_{-\beta}] = 0$, forcing $b_{-\beta} = 0$. So $[\varphi(d), \varphi(e_\beta)] = \varphi(t_\beta) + b_\beta\varphi(e_\beta)$. For $\alpha \in Y_\beta$, by $[e_\alpha, [d, e_\beta]] = 0$ we have that

$$[\varphi(e_\alpha), [\varphi(d), \varphi(e_\beta)]] = [\varphi(e_\alpha), \varphi(t_\beta + b_\beta e_\beta)] = 0.$$

This implies that $[e_\alpha, t_\beta + b_\beta e_\beta] = -\alpha(t_\beta)e_\alpha = 0$. So $\alpha(t_\beta) = 0$ for each $\alpha \in Y_\beta$, forcing $t_\beta = 0$ (recall Lemma 2.2). Hence $[\varphi(d), \varphi(e_\beta)] = b_\beta\varphi(e_\beta)$. Obviously, $[\varphi(d), \varphi(\mathfrak{h})] = 0$. Since $\varphi(\mathfrak{h})$ along with all e_β, $\beta \in \Phi^+ \cup \Phi_\pi^-$, span $\mathfrak{p}_\pi$, we conclude that $\varphi(d)$ is semisimple.

For (ii), let h be an arbitrary semisimple element in $\mathfrak{p}_\pi$, then h is contained in a Cartan subalgebra. Since each Cartan subalgebra is conjugate under an inner automorphism to $\mathfrak{h}$, we can find some $\sigma \in \mathrm{Int}(\mathfrak{p}_\pi)$, $d \in \mathfrak{h}$ such that $h = \sigma(d)$. Applying (i) to $\varphi(h) = (\varphi\sigma)(d)$ we complete the proof.

For (iii), let C be a Cartan subalgebra of $\mathfrak{p}_\pi$: C is nilpotent and it equals its normalizer. Since each Cartan subalgebra is conjugate under an inner automorphism to $\mathfrak{h}$, we may directly assume that C is just $\mathfrak{h}$ itself. Obviously, $\varphi(\mathfrak{h})$ is nilpotent (note that it is abelian). Assume that $x \in \mathfrak{p}_\pi$ normalize $\varphi(\mathfrak{h})$, and write x as $x = \varphi(t) + \sum_{\beta \in \Phi^+ \cup \Phi_\pi^-} b_\beta\varphi(e_\beta)$, $t \in \mathfrak{h}$. Choose $d_0 \in \mathfrak{h}$ such that $\beta(d_0) \neq 0$ for all $\beta \in \Phi$, and assume that $[\varphi(d_0), \varphi(e_\beta)] = a_\beta\varphi(e_\beta)$, $\forall\, \beta \in \Phi^+ \cup \Phi_\pi^-$ (recall (i)). Then $[x, \varphi(d_0)] = -\sum_{\beta \in \Phi^+ \cup \Phi_\pi^-} a_\beta b_\beta\varphi(e_\beta) \in \varphi(\mathfrak{h})$. It follows that $b_\beta = 0$ for all $\beta \in \Phi^+ \cup \Phi_\pi^-$ (note that each a_β is nonzero). So $x = \varphi(t) \in \varphi(\mathfrak{h})$. This says that $\varphi(\mathfrak{h})$ equals its normalizer. So $\varphi(\mathfrak{h})$ is also a Cartan subalgebra.

(iv) is obvious.

For (v), assume that $\varphi(e_\alpha) = h + \sum_{\beta \in \Phi^+ \cup \Phi_\pi^-} b_\beta e_\beta$, $h \in \mathfrak{h}$. There exists at least one root in $\Phi^+ \cup \Phi_\pi^-$, say β_0, such that $b_{\beta_0} \neq 0$. Choose $h_0 \in \mathfrak{h}$ such that $\beta_0(h_0) \neq 0$ and such that $\beta(h_0) \neq \beta_0(h_0)$ for each other $\beta \in \Phi^+ \cup \Phi_\pi^-$ distinct from β_0. On the one hand

$$[h_0, \varphi(e_\alpha)] = \sum_{\beta \in \Phi^+ \cup \Phi_\pi^-} \beta(h_0) b_\beta e_\beta,$$

on the other hand, it is just a nonzero scalar multiple of $\varphi(e_\alpha)$ (thanks to (i)). So we have that $h = 0$, and except for b_{β_0}, each other b_β is just zero. Hence $\varphi(e_\alpha) = b_{\beta_0} e_{\beta_0} \in \mathfrak{g}_{\beta_0}$. This implies that $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_{\beta_0}$. $\square$

Lemma 2.3 shows that if $\varphi \in \mathrm{TAut}(\mathfrak{p}_\pi)$ stabilizes $\mathfrak{h}$, then φ induces a permutation ϱ_φ on $\Phi^+ \cup \Phi_\pi^-$ in such a way that $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_{\varrho_\varphi(\alpha)}$.

Lemma 2.4. *Suppose that $\varphi \in \mathrm{TAut}(\mathfrak{p}_\pi)$ stabilizes $\mathfrak{h}$, and suppose that $\alpha + \beta$ is a root for $\alpha, \beta \in \Phi^+ \cup \Phi_\pi^-$. Then*
(i) *$\varrho_\varphi(\alpha) + \varrho_\varphi(\beta)$ is also a root, and it coincides with $\varrho_\varphi(\alpha + \beta)$.*
(ii) *$\varphi([\mathfrak{g}_\alpha, \mathfrak{g}_\beta]) = [\varphi(\mathfrak{g}_\alpha), \varphi(\mathfrak{g}_\beta)]$.*

P r o o f. Applying φ to $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta} = [\mathfrak{h}, [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]]$, we have that

$$\begin{aligned}\varphi([\mathfrak{g}_\alpha, \mathfrak{g}_\beta]) = \varphi(\mathfrak{g}_{\alpha+\beta}) &= [\mathfrak{h}, [\varphi(\mathfrak{g}_\alpha), \varphi(\mathfrak{g}_\beta)]] \\ &= [\varphi(\mathfrak{g}_\alpha), \varphi(\mathfrak{g}_\beta)].\end{aligned}$$

Thus (ii) holds. The equality $\mathfrak{g}_{\varrho_\varphi(\alpha+\beta)} = [\mathfrak{g}_{\varrho_\varphi(\alpha)}, \mathfrak{g}_{\varrho_\varphi(\beta)}]$ shows that $\varrho_\varphi(\alpha) + \varrho_\varphi(\beta)$ is also a root, and it coincides with $\varrho_\varphi(\alpha + \beta)$. $\square$

Lemma 2.5. *If $\varphi \in \mathrm{TAut}(\mathfrak{p}_\pi)$ stabilizes $\mathfrak{h}$, then there exists $\sigma \in \mathrm{Int}(\mathfrak{p}_\pi)$, stabilizing $\mathfrak{h}$, such that the permutation on $\Phi^+ \cup \Phi_\pi^-$ induced by $\varphi \cdot \sigma$ stabilizes Φ^+ and Φ_π^-, respectively.*

P r o o f. Obviously, if ϱ_φ stabilizes Φ^+, then it stabilizes Φ_π^-. So we only need to show that ϱ_φ stabilizes Φ^+. It is not difficult to see (by Lemma 2.4) that:
- For $\alpha \in \Delta$, if $\varrho_\varphi(\alpha) \in \Phi_\pi^-$, then $\alpha \in \pi$ and $\varrho_\varphi(-\alpha) = -\varrho_\varphi(\alpha)$.
- If $\varrho_\varphi(\alpha) \in \Phi^+$ for all $\alpha \in \Delta$, then $\varrho_\varphi(\beta) \in \Phi^+$ for all $\beta \in \Phi^+$.

Now let $N(\varrho_\varphi)$ be the number of positive roots sent by ϱ_φ into Φ_π^-. We will give the remainder of the proof by induction on $N(\varrho_\varphi)$. If $N(\varrho_\varphi) = 0$, then the assertion already holds (choose σ to be the identity map). Now assume the assertion holds for $N(\varrho_\varphi) = m - 1$ $(1 \leqslant m \leqslant |\Phi^+|)$. For the case that $N(\varrho_\varphi) = m$, there exists at least one $\gamma \in \Delta$ such that $\varrho_\varphi(\gamma) \in \Phi_\pi^-$. Say that $\alpha \in \Delta$ is such a simple root. By the above discussion, we know $\alpha \in \pi$ and $\varrho_\varphi(-\alpha) = -\varrho_\varphi(\alpha) \in \Phi^+$. Take $w_\alpha = \sigma_\alpha(1)\sigma_{-\alpha}(-1)\sigma_\alpha(1) \in \mathrm{Int}(\mathfrak{p}_\pi)$. One will see that w_α stabilizes $\mathfrak{h}$, ϱ_{w_α} sends α to $-\alpha$ and permutes the set $\Phi^+ \setminus \{\alpha\}$. Thus $(\varphi \cdot w_\alpha)(\mathfrak{g}_\alpha) = \varphi(\mathfrak{g}_{-\alpha}) = \mathfrak{g}_{\varrho_\varphi(-\alpha)}$, where $\varrho_\varphi(-\alpha) \in \Phi^+$. Denote $\varphi \cdot w_\alpha$ by φ_1. Then we have that $N(\varrho_{\varphi_1}) = N(\varrho_\varphi) - 1$. By induction assumption, we can find $\sigma_1 \in \mathrm{Int}(\mathfrak{p}_\pi)$, stabilizing $\mathfrak{h}$, such that $(\varrho_{\varphi_1 \cdot \sigma_1})(\beta) \in \Phi^+$ for all $\beta \in \Phi^+$. Finally, choosing $\sigma = w_\alpha \cdot \sigma_1$, we complete the proof. $\square$

The main result of this paper is as follows.

Theorem 2.6. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank l over an algebraically closed field F of characteristic zero, $\mathfrak{p}_\pi$ a parabolic subalgebra of $\mathfrak{g}$, φ a triple automorphism on $\mathfrak{p}_\pi$. Then either φ itself is an automorphism of $\mathfrak{p}_\pi$, or it takes the form $\varphi = \sigma \cdot \omega$, where σ is an automorphism and ω is the extremal triple automorphism.*

Proof. Let $\varphi \in \mathrm{TAut}(\mathfrak{p}_\pi)$. By Lemma 2.3, we can find some $\sigma_1 \in \mathrm{Int}(\mathfrak{p}_\pi)$ such that $(\sigma_1 \cdot \varphi)(\mathfrak{h}) = \mathfrak{h}$. By Lemma 2.5, we can choose $\sigma_2 \in \mathrm{Int}(\mathfrak{p}_\pi)$ such that the permutation on $\Phi^+ \cup \Phi_\pi^-$ induced by $\sigma_1 \cdot \varphi \cdot \sigma_2$ stabilizes Φ^+ and Φ_π^-, respectively. Denote $\sigma_1 \cdot \varphi \cdot \sigma_2$ by φ_1 and denote the permutation induced by φ_1 by ϱ (for brevity). Then ϱ clearly permutes Δ (recall Lemma 2.4). Now we shall show that $\langle \varrho(\alpha), \varrho(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Delta$. If $(\alpha, \beta) = 0$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$. It follows that $[\mathfrak{g}_{\varrho(\alpha)}, \mathfrak{g}_{\varrho(\beta)}] = 0$, which leads to $(\varrho(\alpha), \varrho(\beta)) = 0$. So the assertion holds. Now suppose $\langle \alpha, \beta \rangle = -k < 0$. Then $\alpha + k\beta$, denoted by γ, is a positive root, but $\alpha + (k+1)\beta$ fails to be a root. Lemma 2.4 shows that $\varrho(\gamma) = \varrho(\alpha) + k\varrho(\beta)$ is also a positive root. By $[\mathfrak{g}_\beta, \mathfrak{g}_\gamma] = 0$, we have that $[\mathfrak{g}_{\varrho(\beta)}, \mathfrak{g}_{\varrho(\gamma)}] = 0$. Thus $\varrho(\beta) + \varrho(\gamma)$ is not a root. This shows that $\langle \varrho(\alpha), \varrho(\beta) \rangle = -k$. Now we see that ϱ is just a symmetry of the Dynkin diagram of Φ. For each $\alpha \in \pi$, it is easy to see that $\varrho(\alpha) \in \pi$. Using ϱ we construct the graph automorphism φ_ϱ of $\mathfrak{p}_\pi$. Then $(\varphi_\varrho)^{-1} \cdot \varphi_1$ stabilizes each $\mathfrak{g}_\alpha$, $\alpha \in \Delta$. Furthermore, one will see, by Lemma 2.4, that $(\varphi_\varrho)^{-1} \cdot \varphi_1$ stabilizes each $\mathfrak{g}_\beta$, $\beta \in \Phi^+$. Denote $(\varphi_\varrho)^{-1} \cdot \varphi_1$ by φ_2.

For any given $\alpha \in \Delta$, since $Fd_\alpha = \bigcap\limits_{\beta \in \Delta \setminus \{\alpha\}} C_{\mathfrak{h}}(\mathfrak{g}_\beta)$ and φ_2 stabilizes $C_{\mathfrak{h}}(\mathfrak{g}_\beta)$ for $\beta \in \Delta$, we have that $\varphi_2(Fd_\alpha) = Fd_\alpha$ for any $\alpha \in \Delta$. Now suppose $\varphi_2(d_\alpha) = c_\alpha d_\alpha$ for $\alpha \in \Delta$. We wish to show that all c_α, $\alpha \in \Delta$ take a common value. Write θ as the linear combination of the simple roots: $\theta = \sum\limits_{\alpha \in \Delta} k_\alpha \alpha$, where all k_α are positive integers. We know that $C_{\mathfrak{h}}(\mathfrak{g}_\theta) = \Big\{ \sum\limits_{\alpha \in \Delta} x_\alpha d_\alpha \in \mathfrak{h} \mid \sum\limits_{\alpha \in \Delta} k_\alpha x_\alpha = 0 \Big\}$, which is an $l-1$ dimensional subspace of $\mathfrak{h}$. If $\sum\limits_{\alpha \in \Delta} k_\alpha x_\alpha = 0$, then $\sum\limits_{\alpha \in \Delta} x_\alpha d_\alpha \in C_{\mathfrak{h}}(\mathfrak{g}_\theta)$. Thus $\sum\limits_{\alpha \in \Delta} c_\alpha x_\alpha d_\alpha = \varphi_2\Big(\sum\limits_{\alpha \in \Delta} x_\alpha d_\alpha\Big) \in \varphi_2(C_{\mathfrak{h}}(\mathfrak{g}_\theta)) = C_{\mathfrak{h}}(\mathfrak{g}_\theta)$, which implies that $\sum\limits_{\alpha \in \Delta} c_\alpha k_\alpha x_\alpha = 0$. So the equation $\sum\limits_{\alpha \in \Delta} k_\alpha x_\alpha = 0$, and the equations

$$\begin{cases} \sum\limits_{\alpha \in \Delta} k_\alpha x_\alpha = 0, \\ \sum\limits_{\alpha \in \Delta} c_\alpha k_\alpha x_\alpha = 0 \end{cases}$$

have the same solutions. So all $c_\alpha k_\alpha / k_\alpha$ $(= c_\alpha)$ are equal for $\alpha \in \Delta$. Now we denote the common value by c. Let α be a fixing simple root. By $[d_\alpha, [d_\alpha, e_\alpha]] = e_\alpha$, we have that $[cd_\alpha, [cd_\alpha, \varphi(e_\alpha)]] = \varphi(e_\alpha)$, which implies that $c^2 = 1$, namely $c = 1$ or -1.

If $c = 1$, then φ_2 fixes each element in $\mathfrak{h}$. If $c = -1$, then φ_2 sends each element h in $\mathfrak{h}$ to $-h$. Considering $\omega^\delta \cdot \varphi_2$, one easily sees that it fixes each element in $\mathfrak{h}$, where δ equals 1 when $c = -1$, and equals 0 when $c = 1$. Denote $\omega^\delta \cdot \varphi_2$ by φ_3.

Now suppose that $\varphi_3(e_\alpha) = b_\alpha e_\alpha$ for $\alpha \in \Delta$, and define

$$\chi\colon\ P = \mathbb{Z}\Phi \to F^*, \quad \sum_{\alpha\in\Delta} k_\alpha \alpha \mapsto \prod_{\alpha\in\Delta} b_\alpha^{k_\alpha}.$$

Then χ is an F-character of P. Using it we construct the diagonal automorphism φ_χ of $\mathfrak{p}_\pi$. Then $\varphi_\chi^{-1} \cdot \varphi_3$ will further fix each e_α for $\alpha \in \Delta$. Denote $\varphi_\chi^{-1} \cdot \varphi_3$ by φ_4. Now we use induction on $\operatorname{ht}\beta$ to show that $\varphi_4(e_\beta) = e_\beta$ for all $\beta \in \Phi^+$. If $\operatorname{ht}\beta = 1$, the assertion already holds. Assume the assertion holds for $\beta \in \Phi^+$ with $\operatorname{ht}\beta = k$ ($1 \leqslant k < \operatorname{ht}\theta$), and consider the root $\gamma \in \Phi^+$ with height $k+1$. Find $\alpha \in \Delta$ such that $\gamma - \alpha \in \Phi^+$. Denote $\gamma - \alpha$ by β and choose $h \in \mathfrak{h}$ such that $\gamma(h) = N_{\beta,\alpha}^{-1}$. Then by applying φ_4 to $[h, [e_\beta, e_\alpha]] = e_\gamma$ we know that φ_4 also fixes e_γ. So φ_4 fixes all e_β for $\beta \in \Phi^+$. For $\alpha \in \pi$, applying φ_4 to $[e_\alpha, [e_{-\alpha}, e_\alpha]] = 2e_\alpha$, we conclude that φ_4 further fixes $e_{-\alpha}$. Based on this fact, one can easily see that φ_4 fixes each $e_{-\beta}$ for $-\beta \in \Phi_\pi^-$. So φ_4 is just the identity map on $\mathfrak{p}_\pi$. Finally we see that $\varphi_\chi^{-1} \cdot \omega^\delta \cdot \varphi_\varrho^{-1} \cdot \sigma_1 \cdot \varphi \cdot \sigma_2 = I_{\mathfrak{p}_\pi}$. So

$$\varphi = \sigma_1^{-1} \cdot \varphi_\varrho \cdot \omega^\delta \cdot \varphi_\chi \cdot \sigma_2^{-1}.$$

If $\delta = 0$ then φ itself is an automorphism, otherwise it is the composition of an automorphism and the extremal triple automorphism ω. This completes the proof. $\square$

Corollary 2.7. *The automorphism group* $\operatorname{Aut}(\mathfrak{p}_\pi)$ *of* $\mathfrak{p}_\pi$ *is normal in* $\operatorname{TAut}(\mathfrak{p}_\pi)$ *and the index* $[\operatorname{TAut}(\mathfrak{p}_\pi), \operatorname{Aut}(\mathfrak{p}_\pi)]$ *is* 2*. The square of an arbitrary triple automorphism of* $\mathfrak{p}_\pi$ *is an automorphism.*

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