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A NOTE ON TRANSITIVELY D-SPACES

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Abstract. In this note, we show that if for any transitive neighborhood assignment φ for X there is a point-countable refinement $\mathscr{F}$ such that for any non-closed subset A of X there is some $V \in \mathscr{F}$ such that $|V \cap A| \geqslant \omega$, then X is transitively D. As a corollary, if X is a sequential space and has a point-countable wcs^*-network then X is transitively D, and hence if X is a Hausdorff k-space and has a point-countable k-network, then X is transitively D. We prove that if X is a countably compact sequential space and has a point-countable wcs^*-network, then X is compact. We point out that every discretely Lindelöf space is transitively D. Let (X, τ) be a space and let $(X, \mathscr{T})$ be a butterfly space over (X, τ). If (X, τ) is Fréchet and has a point-countable wcs^*-network (or is a hereditarily meta-Lindelöf space), then $(X, \mathscr{T})$ is a transitively D-space.

Keywords: transitively D, sequential, discretely Lindelöf, wcs^*-network

MSC 2010: 54F99, 54G99

1. Introduction

The notion of a D-space was first investigated by van Douwen and Pfeffer in [9]. A *neighborhood assignment* for a space X is a function φ from X to the topology of the space X such that $x \in \varphi(x)$ for any $x \in X$. A space X is called a *D-space* if for any neighborhood assignment φ for X there exists a closed discrete subspace D of X such that $X = \bigcup\{\varphi(d) \colon d \in D\}$ (cf. [9]). Many classes of spaces are known to be D-spaces (cf. [1], [2], [6], [8], [11], and [17]).

In [4], it was proved that if a space X has a point-countable base then X is a D-space. In [7] and [22], it was proved that if a space X has a point-countable weak base then X is a D-space. In [24], Peng proved that a space X is a D-space if X is

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sequential and has a point-countable cs^*-network. Recall that a family $\mathscr{F}$ of subsets of a space X is a *cs^*-network* of X, if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ which converges to a point x and any open set U which contains x, there is some $F \in \mathscr{F}$ such that $x \in F \subseteq U$ and $|\{n\colon x_n \in F\}| = \omega$ (cf. [19]). A subset U of a space X is called *sequentially open* if each sequence in X converging to a point in U is eventually in U. A space X is called *sequential* if every sequentially open subset of X is open in X (cf. [29]). Every first countable space is a sequential space. In 2007, Peng proved that if a regular space X is sequential and has a point-countable k-network then X is a D-space (cf. [25]). Recall that a family $\mathscr{F}$ of a space X is a *k-network* for X if for any open set U in X and any compact set $C \subseteq U$, there exists a finite $\mathscr{F}' \subseteq \mathscr{F}$ such that $C \subseteq \bigcup \mathscr{F}' \subseteq U$. In [31], Tkachuk proved that every monotonically monolithic space is hereditarily a D-space. Inspired by Tkachuk's idea, Peng introduced the concept of weakly monotonically monolithic spaces and proved that every weakly monotonically monolithic space is a D-space (cf. [26]). Many known conclusions on D-spaces can be obtained by this conclusion, and every regular sequential space which has a point-countable wcs^*-network is a D-space. Recall that a family $\mathscr{F}$ of subsets of a space X is a *wcs^*-network* of X, if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ which converges to a point x and any open set U which contains x, there is some $F \in \mathscr{F}$ such that $F \subseteq U$ and $|\{n\colon x_n \in F\}| = \omega$ (cf. [19]). If $\mathscr{F}$ is a k-network of X, then $\mathscr{F}$ is a *wcs^*-network* of X.

In 2008, Peng introduced the notion of linear D-spaces (cf. [23]). The linear D-spaces are called transitively D-spaces in [21], since the concept of linearly D has a proper meaning in [16]. A *transitive neighborhood assignment* for a space X is a function φ from X to the topology of the space X such that $x \in \varphi(x)$ and for each $y \in \varphi(x)$ we have $\varphi(y) \subseteq \varphi(x)$ for any $x \in X$. A space X is called *transitively D*, if for any transitive neighborhood assignment φ for X there exists a closed discrete subspace D of X such that $X = \bigcup\{\varphi(d)\colon d \in D\}$. By the definitions, we know that every D-space is transitively D. It was proved that every meta-Lindelöf space is transitively D (cf. [21]). Gruenhage proved that every submeta-Lindelöf space is transitively D (cf. [13]). In [27], Peng proved that every weak $\overline{\delta\theta}$-refinable space is transitively D.

In [18, page 28], it was proved that there is a non-regular first countable space X which has a point-countable wcs^*-network and X is not meta-Lindelöf. Thus we have the following question: Is every sequential T_1-space which has a point-countable wcs^*-network a D-space or a transitively D-space? In this note, we show that every sequential T_1-space which has a point-countable wcs^*-network is a transitively D-space, and hence if X is a Hausdorff k-space and has a point-countable k-network, then X is transitively D. To obtain these conclusions, we prove that if for any transitive neighborhood assignment φ for X there is a point-countable refinement $\mathscr{F}$

such that for any non-closed subset A of X there is some $V \in \mathscr{F}$ such that $|V \cap A| \geqslant \omega$, then X is transitively D. We also point out that every discretely Lindelöf space is transitively D.

Let (X, τ) be a topological space. If $\mathscr{T}$ is a finer topology on X such that each $p \in X$ has a neighborhood base consisting of sets B such that $B \setminus \{p\}$ is open in the topology τ, then $(X, \mathscr{T})$ is called a *butterfly space over* (X, τ). In [15], it was pointed out that many examples in the area of generalized metrizable spaces are butterfly topologies over separable metric spaces (e.g., the tangent disc space and the Sorgenfrey line), and every such butterfly space is subparacompact. Thus by the conclusion which appears in [13], we know that every butterfly space over a separable metrizable space is a transitively D-space. In the last part of this note, we discuss the transitive D-property of a butterfly space over (X, τ).

All the spaces in this note are assumed to be T_1-spaces. The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. In notation and terminology we will follow [10] and [12].

2. Main results

Recall that a space X is a *meta-Lindelöf* space if for any open cover $\mathscr{U}$ of X there is a point-countable open refinement.

Proposition 1. *If X is a meta-Lindelöf space, then for any transitive neighborhood assignment φ for X there is a point-countable refinement $\mathscr{F}$ such that if $A \subseteq X$ is not closed then there is some $F \in \mathscr{F}$ such that $|F \cap A| \geqslant \omega$.*

Proof. Let φ be any transitive neighborhood assignment for X. Thus φ has a point-countable open refinement $\mathscr{F}$. The family $\mathscr{F}$ satisfies the conditions of the theorem. □

Proposition 2. *If a space X is sequential and has a point-countable wcs^*-network, then for any transitive neighborhood assignment φ for X there is a point-countable refinement $\mathscr{F}$ such that if $A \subseteq X$ is not closed then there is some $F \in \mathscr{F}$ such that $|F \cap A| \geqslant \omega$.*

Proof. Let φ be any transitive neighborhood assignment for X and let $\mathscr{F}^*$ be a point-countable wcs^*-network of X. If $\mathscr{F} = \{F\colon F \in \mathscr{F}^*$ and there is some $x \in X$ such that $F \subseteq \varphi(x)\}$, then $\mathscr{F}$ is a point-countable refinement of φ. If $A \subseteq X$ is not closed, then there is a sequence $\{x_n\colon n \in \mathbb{N}\} \subseteq A$, which converges to some point $x \in \overline{A} \setminus A$ by sequential property of X. Thus there is some $F \in \mathscr{F}^*$ such that $F \subseteq \varphi(x)$ and $|\{n\colon x_n \in F\}| = \omega$, and hence $F \in \mathscr{F}$ and $|F \cap A| = \omega$. □

Example 3 ([30, Example 78]). Let $\mathscr{T}$ be the usual Euclidean topology of $\mathbb{R}^2$. Let $S_1 = \{(x, y)\colon\ x, y \in \mathbb{R},\ y > 0\}$, $L = \{(x, 0)\colon\ x \in \mathbb{R}\}$ and $X = S_1 \cup L$. Let $\mathscr{T}^* = \{\mathscr{T}|X\} \cup \{\{x\} \cup (S_1 \cap U)\colon\ x \in L,\ x \in U$ and $U \in \mathscr{T}\}$, where $\mathscr{T}|X = \{U \cap X\colon\ U \in \mathscr{T}\}$. The space $(X, \mathscr{T}^*)$ is a non-regular T_2-space.

In [18, page 28], it is pointed out that the space X which appears in Example 3 has a locally countable k-network but it has no point-countable cs^*-network. Thus the space X in Example 3 is a first countable space and has a point-countable wcs^*-network, but it has no point-countable cs^*-network. The space in Example 3 is a D-space, since the subspace L is closed discrete in X and S_1 is a D-space.

The proof of the following Theorem 4 is analogous to the proof of Theorem 3 in [21].

Theorem 4. *Let X be a space. If for any transitive neighborhood assignment φ for X there is a point-countable refinement $\mathscr{F}$ such that if $A \subseteq X$ is not closed then there is some $V \in \mathscr{F}$ such that $|V \cap A| \geqslant \omega$, then X is transitively D.*

P r o o f. Let $\varphi = \{\varphi(x)\colon\ x \in X\}$ be any transitive neighborhood assignment for X. We let $\mathscr{F}$ be a point-countable refinement of φ such that if $A \subseteq X$ is not closed then there is some $V \in \mathscr{F}$ such that $|V \cap A| \geqslant \omega$.

We can assume $X = \{x_\alpha\colon\ \alpha < \gamma\}$, where γ is a cardinal. Suppose for each $\alpha < \beta$, we have chosen a closed discrete subspace D_α satisfying the following conditions:

(1) $x_\alpha \in \bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\eta\colon\ \eta \leqslant \alpha\}\}$;
(2) $\bigcup\{D_\eta\colon\ \eta < \alpha\}$ is a closed discrete subspace of X;
(3) $D_\alpha \cap (\bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\eta\colon\ \eta < \alpha\}\}) = \emptyset$;
(4) If $V \in \mathscr{F}$ and $V \cap D_\eta \neq \emptyset$ for some $\eta \leqslant \alpha$, then $|V \cap D_\eta| < \omega$ and $V \cap D_\beta = \emptyset$ if $\eta < \beta \leqslant \alpha$.

Before we construct D_β, let us show that $D'_\beta = \bigcup\{D_\alpha\colon\ \alpha < \beta\}$ is a closed discrete subspace of X.

By the condition (2), we know that for any $\alpha < \beta$, $D'_\alpha = \bigcup\{D_\eta\colon\ \eta < \alpha\}$ is a closed discrete subspace of X. In what follows, we show that the set D'_β is a closed discrete subspace of X.

First, assume β is a limit ordinal. If $x \in \bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\alpha\colon\ \alpha < \beta\}\}$, then let α_x be the smallest ordinal such that $x \in \bigcup\{\varphi(d)\colon\ d \in D_{\alpha_x}\}$. By the condition (2), we know that $\bigcup\{D_\eta\colon\ \eta < \alpha_x\}$ is a closed discrete subspace of X and D_{α_x} is also a closed discrete subspace of X. So we let $V_x = ((\bigcup\{\varphi(d)\colon\ d \in D_{\alpha_x}\}) \setminus \bigcup\{D_\eta\colon\ \eta < \alpha_x\}) \cap V'_x$, where V'_x is an open set of X such that $x \in V'_x$ and $|V'_x \cap D_{\alpha_x}| \leqslant 1$. So $x \in V_x$, V_x is an open set of X, and $|V_x \cap D'_\beta| \leqslant 1$. Thus $x \notin \overline{D'_\beta} \setminus D'_\beta$. So we have proved that $\overline{D'_\beta} \setminus D'_\beta \subseteq X \setminus \bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\alpha\colon\ \alpha < \beta\}\}$.

Suppose D'_β is not closed in X. There is some $V \in \mathscr{F}$ such that $|V \cap D'_\beta| \geqslant \omega$. Let η be the smallest ordinal such that $V \cap D_\eta \neq \emptyset$. Then by the condition (4) we know that $|V \cap D_\eta| < \omega$ and $V \cap D_{\eta'} = \emptyset$ if $\eta < \eta' < \beta$. Thus $|V \cap D'_\beta| < \omega$; this contradiction shows that D'_β is closed. So D'_β is a closed discrete subspace of X if β is a limit ordinal.

Now assume that $\beta = \alpha + 1$ for some α. So $D'_\beta = \bigcup\{D_\eta\colon \eta < \beta\} = \bigcup\{D_\eta\colon \eta \leqslant \alpha\} = (\bigcup\{D_\eta\colon \eta < \alpha\}) \cup D_\alpha$. By the condition (2), we know that $D'_\alpha = \bigcup\{D_\eta\colon \eta < \alpha\}$ and D_α are closed discrete subspaces of X. So D'_β is a closed discrete subspace of X.

We let $U_\beta = \bigcup\{\varphi(d)\colon d \in \bigcup\{D_\eta\colon \eta < \beta\}\}$. Now we will construct D_β.

If $x_\beta \in U_\beta$, then we let $D_\beta = \emptyset$. So we assume that $x_\beta \notin U_\beta$. If $\mathscr{V}'_{x_\beta} = \{V\colon x_\beta \in V$ and $V \in \mathscr{F}\}$, then $\mathscr{V}'_{x_\beta}$ is a countable subfamily of $\mathscr{F}$. Since $\mathscr{F}$ is a refinement of φ, there is some $x_V \in X$ such that $V \subseteq \varphi(x_V)$ for each $V \in \mathscr{V}'_{x_\beta}$. If $\mathscr{V}_{x_\beta} = \{\varphi(x_V)\colon V \in \mathscr{V}'_{x_\beta}\}$, then $\mathscr{V}_{x_\beta}$ is a countable subfamily of φ. Enumerate $\mathscr{V}_{x_\beta}$ by prime numbers.

If E is an element of the neighborhood assignment $\varphi = \{\varphi(x)\colon x \in X\}$, then there is some $x \in X$ such that $E = \varphi(x)$. The point x is called a *center point* of the set E. If $E = \varphi(x) = \varphi(y)$ for any two points x and y, then the points x and y are all center points of the set E.

Let $y_1 = x_\beta$ and $\mathscr{V}_{y_1} = \mathscr{V}_{x_\beta}$. If $x_V \in \varphi(x_\beta)$ for each $V \in \mathscr{V}'_{x_\beta}$, then we let $y_2 = x_\eta$, where $\eta = \min\{\eta'\colon x_{\eta'} \notin U_\beta \cup \varphi(x_\beta)\}$. Now assume that there is some $V \in \mathscr{V}'_{x_\beta}$ such that $V \subset \varphi(x_V)$ and $x_V \notin \varphi(x_\beta)$. Since $x_\beta \in V \subset \varphi(x_V)$ and $x_\beta \notin U_\beta$, we have $x_V \notin U_\beta$. Thus $\varphi(x_V) \in \mathscr{V}_{y_1}$ and $x_V \notin \varphi(x_\beta) \cup U_\beta$. We take the first member E of $\mathscr{V}_{y_1}$ such that a center point z of E does not belong to $\varphi(y_1)$. Thus $E = \varphi(z)$ and $z \notin \varphi(y_1)$. Since $E \in \mathscr{V}_{y_1}$, there is some $V' \in \mathscr{V}'_{x_\beta}$ such that $E = \varphi(x_{V'})$. Since the neighborhood assignment φ is transitive and $z \notin \varphi(y_1)$, we have $x_{V'} \notin \varphi(y_1)$.

Denote $y_2 = x_{V'}$. Thus $\varphi(y_2) = \varphi(x_{V'})$. The family $\mathscr{V}'_{y_2} = \{V\colon y_2 \in V$ and $V \in \mathscr{F} \setminus \mathscr{V}'_{y_1}\}$ is a countable family, where $\mathscr{V}'_{y_1} = \mathscr{V}'_{y_\beta}$. If $\mathscr{V}'_{y_2} = \emptyset$, then we let $\mathscr{V}_{y_2} = \{\emptyset\}$. If $\mathscr{V}'_{y_2} \neq \emptyset$, then for each $V \in \mathscr{V}'_{y_2}$ there is some $x_V \in X$ such that $V \subseteq \varphi(x_V)$. If $\mathscr{V}_{y_2} = \{\varphi(x_V)\colon V \in \mathscr{V}'_{y_2}\}$, then $\mathscr{V}_{y_2}$ is a countable subfamily of φ. We enumerate $\mathscr{V}_{y_2}$ by the squares of prime numbers.

Suppose we have finished n steps. We have $\varphi(y_1), \ldots, \varphi(y_n)$, and families $\mathscr{V}'_{y_i} \subset \mathscr{F}$, $\mathscr{V}_{y_i} \subset \varphi$, such that if $j < i$ and $V \in \mathscr{V}'_{y_j}$ then $V \notin \mathscr{V}'_{y_i}$ for each $i \leqslant n$. If $\bigcup\{\varphi(y_i)\colon i \leqslant n\} \cup U_\beta = X$, then stop the induction, and let $D_\beta = \{y_i\colon i \leqslant n\}$. So we assume $\bigcup\{\varphi(y_i)\colon i \leqslant n\} \cup U_\beta \neq X$. If the center points of members of $\bigcup\{\mathscr{V}_{y_i}\colon i \leqslant n\}$ are all contained in $\bigcup\{\varphi(y_i)\colon i \leqslant n\} \cup U_\beta$, then $\bigcup(\bigcup\{\mathscr{V}_{y_i}\colon i \leqslant n\}) \subseteq \bigcup\{\varphi(y_i)\colon i \leqslant n\} \cup U_\beta$, since the neighborhood assignment φ is transitive. In this case, we let $\min\{\eta'\colon x_{\eta'} \notin (\bigcup\{\varphi(y_i)\colon i \leqslant n\} \cup U_\beta)\} = \eta$, and denote x_η by y_{n+1}.

Now assume that there is some $E' \in \bigcup\{\mathscr{V}_{y_i}\colon i \leqslant n\}$ such that a center point of E' does not belong to $\bigcup\{\varphi(y_i)\colon i \leqslant n\} \cup U_\beta$. We have that every center point of E' does not belong to $\bigcup\{\varphi(y_i)\colon\ i \leqslant n\} \cup U_\beta$. We take the first member E of $\bigcup\{\mathscr{V}_{y_i}\colon\ i \leqslant n\}$ such that a center point of E does not belong to $\bigcup\{\varphi(y_i)\colon\ i \leqslant n\} \cup U_\beta$. Thus there are some $i \leqslant n$ and some $V \in \mathscr{V}'_{y_i}$ such that $E = \varphi(x_V)$ and $x_V \notin \bigcup\{\varphi(y_i)\colon\ i \leqslant n\} \cup U_\beta$. We let $y_{n+1} = x_V$. In this case, $\varphi(y_{n+1}) = \varphi(x_V)$.

Let $\mathscr{V}'_{y_{n+1}} = \{V\colon\ V \in \mathscr{F} \setminus \bigcup\{\mathscr{V}'_{y_i}\colon\ i \leqslant n\}$ and $y_{n+1} \in V\}$. If $\mathscr{V}'_{y_{n+1}} = \emptyset$, then we let $\mathscr{V}_{y_{n+1}} = \{\emptyset\}$. If $\mathscr{V}'_{y_{n+1}} \neq \emptyset$, then for each $V \in \mathscr{V}'_{y_{n+1}}$ there is some $x_V \in X$ such that $V \subseteq \varphi(x_V)$. If $\mathscr{V}_{y_{n+1}} = \{\varphi(x_V)\colon\ V \in \mathscr{V}'_{y_{n+1}}\}$, then $\mathscr{V}_{y_{n+1}}$ is a countable subfamily of φ. We enumerate it by the $(n+1)$st powers of prime numbers.

In this way, we obtain the set $D_\beta = \{y_n\colon\ n \in \mathbb{N}\}$. We have $D_\beta \cap U_\beta = \emptyset$. Let us show that the set D_β is a closed discrete subspace of X.

For any $y \in \bigcup\{\varphi(d)\colon\ d \in D_\beta\} \cup U_\beta$, we know that there is an open set V_y of X such that $y \in V_y$ and $|V_y \cap D_\beta| \leqslant 1$. Thus $y \notin \overline{D_\beta} \setminus D_\beta$.

Suppose $\overline{D_\beta} \setminus D_\beta \neq \emptyset$. Thus there is some $V \in \mathscr{F}$ such that $|V \cap D_\beta| \geqslant \omega$ and $V \subseteq \varphi(x)$ for some $x \in X$. If $m = \min\{n\colon\ y_n \in V$ and $n \in \mathbb{N}\}$, then $\mathscr{V}_{y_m} = \{E_{p^m}\colon\ p$ is a prime number$\}$. Since the set $V \in \mathscr{V}'_{y_m} \setminus \bigcup\{\mathscr{V}'_{y_n}\colon\ n < m\}$, there is some $x_V \in X$ such that $V \subseteq \varphi(x_V)$ and $\varphi(x_V) \in \mathscr{V}_{y_m}$. Thus $\varphi(x_V) = E_{q^m}$ for some prime number q. Since $y_m \in V \subset \varphi(x_V)$ and $y_m \notin U_\beta$, the point $x_V \notin U_\beta$. Since there is some $n' \in \mathbb{N}$ such that $q^m < 2^{n'}$ and $n' > m$, the set $\mathscr{A} = \{p^i\colon\ p^i < q^m,\ i \leqslant n'$ and p is a prime number$\}$ is a finite set. Thus there is some $n > n'$ such that $l = \min\{s\colon\ E_s \in \bigcup\{\mathscr{V}_{y_i}\colon\ i \leqslant n\}$ and a center point of E_s is not contained in $\bigcup\{\varphi(y_i)\colon\ i \leqslant n\} \cup U_\beta\} \geqslant q^m$. If $l = q^m$, then the point $x_V \notin \bigcup\{\varphi(y_i)\colon\ i \leqslant n\} \cup U_\beta$, since $E_{q^m} = \varphi(x_V)$ and the point x_V is a center point of E_{q^m}. Thus $y_{n+1} = x_V$, so $V \subseteq \varphi(x_V) = \varphi(y_{n+1})$. If $l > q^m$, then the point $x_V \in \bigcup\{\varphi(y_i)\colon\ i \leqslant n\} \cup U_\beta$. Since $x_V \notin U_\beta$, there is some $i \leqslant n$ such that $x_V \in \varphi(y_i)$. Thus $V \subseteq \varphi(x_V) \subseteq \varphi(y_i)$. Thus $|V \cap D_\beta| < \omega$. This contradicts $|V \cap D_\beta| \geqslant \omega$. Thus D_β is a closed discrete subspace of X.

From the above discussion, we know that the family $\{D_\eta\colon\ \eta \leqslant \beta\}$ satisfies the conditions (1), (2), (3), and (4).

If $D = \bigcup\{D_\alpha\colon\ \alpha < \gamma\}$, then we can easily see that $X = \bigcup\{\varphi(d)\colon\ d \in D\}$ and D is a closed discrete subspace of X. So X is transitively D. $\square$

By Proposition 2 and Theorem 4, we have the following theorem.

Theorem 5. *If a space X is sequential and has a point-countable wcs^*-network, then X is transitively D.*

A space X is a *k-space* if and only if $A \subseteq X$ is closed in X whenever $A \cap C$ is relatively closed in C for each $C \in \mathscr{C}$, where $\mathscr{C}$ is the family of all compact sets of X (cf. [12]).

By conclusions which appear in [14], we know that if X is a compact Hausdorff space and has a point-countable k-network then X is metrizable. So we know that if X is a Hausdorff k-space and has a point-countable k-network then X is sequential. Thus we have the following corollary.

Corollary 6. *If X is a Hausdorff k-space and has a point-countable k-network, then X is transitively D.*

By Proposition 1 and Theorem 4, we have the following corollary.

Corollary 7 ([21, Theorem 3]). *Every meta-Lindelöf space is transitively D.*

In [16], Guo and Junnila introduced the concept of linearly D. A family $\mathscr{U}$ of subsets of X is *monotone* if $\mathscr{U}$ is linearly ordered by inclusion. A neighborhood assignment φ for X is *monotone* provided that $\{\varphi(x)\colon x \in X\}$ is a monotone family. A space X is *linearly D* provided that for every monotone neighborhood assignment φ for X there exists a closed discrete subspace D of X such that $X = \bigcup\{\varphi(d)\colon d \in D\}$ (cf. [16]). By conclusions which appears in [15], we know that a space X is linearly Lindelöf if and only if X is linearly D and has countable extent. Recall that a space X is *linearly Lindelöf* if every increasing open cover $\{U_\alpha\colon \alpha \in \kappa\}$ has a countable subcover (by increasing, we mean that $\alpha < \beta < \kappa$ implies $U_\alpha \subseteq U_\beta$).

Lemma 8 ([21, Theorem 2]). *A transitively D-space is linearly D.*

Theorem 9. *If X is a countably compact sequential space which has a point-countable wcs*-network, then X is compact.*

P r o o f. By Theorem 5 and Lemma 8, we know that X is linearly D. Thus X is countably compact linearly D, and hence X is countably compact linearly Lindelöf. Thus X is compact. □

A space X is *discretely Lindelöf*, if the closure of every discrete subspace of X is Lindelöf (cf. [20]). The notion of a discretely Lindelöf space is called strongly discretely Lindelöf in [3] and [5].

Lemma 10 ([28, Lemma 2.1]). *Let X be a space. If φ is a neighborhood assignment for X, then there is a discrete subspace A of X, and an open family $\{V(x)\colon x \in A\}$ such that $X = \bigcup\{\varphi(x)\colon x \in \overline{A}\}$, and $X \setminus \bigcup\{V(x)\colon x \in A\} = \overline{A} \setminus A$, $V(x) \cap A = \{x\}$ and $x \in V(x) \subseteq \varphi(x)$ for each $x \in A$.*

Lemma 11 ([21, Theorem 21]**.** *If for any transitive neighborhood assignment* φ *for* X *there is a closed transitively* D*-subspace* $F \subseteq X$ *such that* $X = \bigcup\{\varphi(d) \colon d \in F\}$*, then* X *is transitively* D*.*

Theorem 12. *Let* X *be a space. If the closure of every discrete subspace of* X *is transitively* D*, then* X *is transitively* D*.*

Proof. Let φ be any transitive neighborhood assignment for X. By Lemma 10, there is a discrete subspace D of X such that $X = \bigcup\{\varphi(x) \colon x \in \overline{D}\}$. The set D is a discrete subspace of X, and hence $\overline{D}$ is transitively D. Thus X is transitively D by Lemma 11. □

By Corollary 7, we know that every Lindelöf space is transitively D ([23, Corollary 2]). So we have:

Corollary 13. *If* X *is a discretely Lindelöf space, then* X *is transitively* D*.*

We do not know if every discretely Lindelöf Tychonoff space is Lindelöf (cf. [5] and [20]). Arhangel'skii proved the following lemma from which it follows that every discretely Lindelöf space is linearly Lindelöf.

Lemma 14 ([3, Lemma 6])**.** *If* X *is a discretely Lindelöf space, then every open cover whose cardinality does not have countable cofinality has a subcover of strictly smaller cardinality.*

We do not know if there exists a transitively D-space which is not a D-space. The following comments should be useful.

Comment 15. If a space X is discretely Lindelöf, then X is a linearly Lindelöf transitively D-space by Lemma 14 and Corollary 13. Suppose a space X is a discretely Lindelöf space which is not Lindelöf, the space X should not be a D-space, since every linearly Lindelöf D-space is Lindelöf.

Comment 16. If a space X is sequential and has a point-countable wcs^*-network, then X is transitively D by Theorem 5. Suppose we have a sequential space which has a point-countable k-network and is not a D-space, then such a space would be a transitively D-space which is not a D-space.

In what follows, we will discuss the transitive D-property of a butterfly space over (X, τ).

Proposition 17. *If* φ *is a neighborhood assignment for a space* X*, then the set* $D = \{y\colon\ y \in X$ *and* $y \notin \varphi(x)$ *for each* $x \in X \setminus \{y\}\}$ *is a closed discrete subspace of* X*.*

P r o o f. If $y \in D$, then $\varphi(y) \cap D = \{y\}$. If $y \notin D$, then $\varphi(y) \cap D = \emptyset$. Thus D is a closed discrete subspace of X. □

Theorem 18. *If* φ *is a neighborhood assignment for a space* X*, then there is a closed discrete subspace* $D \subseteq X$ *such that either* $X = \bigcup\{\varphi(d)\colon\ d \in D\}$ *or for each* $x \in X \setminus \bigcup\{\varphi(d)\colon\ d \in D\}$ *there is some* $y \in (X \setminus \bigcup\{\varphi(d)\colon\ d \in D\}) \setminus \{x\}$ *such that* $x \in \varphi(y)$ *and* $\varphi(x) \cap D = \emptyset$*.*

P r o o f. Let $X = F_0$ and $D_0 = \{x\colon\ x \in F_0$ and $x \notin \varphi(y)$ for each $y \in F_0 \setminus \{x\}\}$. The set D_0 is a closed discrete subspace of X by Proposition 17.

Let α be an ordinal. Suppose we have a closed discrete subspace D_β and a closed subspace F_β of X for each $\beta < \alpha$ such that the following conditions hold:

(1) $D_\beta \subseteq F_\beta$ and for each $d \in D_\beta$ we have $d \notin \varphi(x)$ for each $x \in F_\beta \setminus \{d\}$;
(2) $F_\beta = X \setminus \bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\gamma\colon\ \gamma < \beta\}\}$;
(3) The set $\bigcup\{D_\gamma\colon\ \gamma < \beta\}$ is a closed discrete subspace of X.

If $F_\alpha = X \setminus \bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\beta\colon\ \beta < \alpha\}\}$, then F_α is closed in X. Before we construct D_α, we prove that the set $\bigcup\{D_\beta\colon\ \beta < \alpha\}$ is a closed discrete subspace of X.

(1) Suppose $\alpha = \gamma + 1$ for some ordinal γ. We know that $\bigcup\{D_\beta\colon\ \beta < \gamma\}$ and D_γ are closed discrete subspaces of X. Thus the set $\bigcup\{D_\beta\colon\ \beta < \alpha\} = \bigcup\{D_\beta\colon\ \beta < \gamma\} \cup D_\gamma$ is a closed discrete subspace of X.

(2) Suppose α is a limit ordinal. We know that $\bigcup\{D_\gamma\colon\ \gamma < \beta\}$ is a closed discrete subspace of X for each $\beta < \alpha$. For each $x \in \bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\beta\colon\ \beta < \alpha\}\}$ we let β_x be the smallest ordinal such that $x \in \varphi(d)$ for some $d \in D_{\beta_x}$. Thus $\varphi(d) \cap D_\gamma = \emptyset$ if $\beta_x + 1 \leqslant \gamma < \alpha$. By induction, we know that $\bigcup\{D_\gamma\colon\ \gamma < \beta_x\}$ and D_{β_x} are closed discrete subspaces of X. Thus there is an open neighborhood V_x of x such that $|V_x \cap (\bigcup\{D_\gamma\colon\ \gamma \leqslant \beta_x\})| \leqslant 1$. Thus $|(V_x \cap \varphi(d)) \cap (\bigcup\{D_\beta\colon\ \beta < \alpha\})| \leqslant 1$. For each $x \in X \setminus \bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\beta\colon\ \beta < \alpha\}\}$ and for each $\beta < \alpha$, we have $D_\beta \subseteq F_\beta$ and $x \in \bigcap\limits_{\beta<\alpha} F_\beta$. Thus $\varphi(x) \cap D_\beta = \emptyset$. So we have proved that $\bigcup\{D_\beta\colon\ \beta < \alpha\}$ is a closed discrete subspace of X.

If $D_\alpha = \{x\colon\ x \in F_\alpha$ and $x \notin \varphi(d)$ for each $d \in F_\alpha \setminus \{x\}\}$, then $D_\alpha \subseteq F_\alpha$ and D_α is a closed discrete subspace of F_α by Proposition 17 and hence D_α is a closed discrete subspace of X. If $x \in F_\alpha \setminus \bigcup\{\varphi(d)\colon\ d \in D_\alpha\}$, then $\varphi(x) \cap D_\alpha = \emptyset$.

In this way, we have some ordinal Λ such that either $X = \bigcup\{\varphi(d)\colon\ d \in D_\beta$ and $\beta < \Lambda\}$ or $F_\Lambda = X \setminus \bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\beta\colon\ \beta < \Lambda\}\} \neq \emptyset$, and for each $x \in F_\Lambda$ there is some $y \in F_\Lambda \setminus \{x\}$ such that $x \in \varphi(y)$. If $X = \bigcup\{\varphi(d)\colon\ d \in D_\beta$ and $\beta < \Lambda\}$, then the set $\bigcup\{D_\beta\colon\ \beta < \Lambda\}$ is a closed discrete subspace of X, since $\bigcup\{D_\beta\colon\ \beta < \alpha\}$ is a closed discrete subspace of X, $D_\alpha \subseteq F_\alpha$ and $F_\alpha = X \setminus \bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\beta\colon\ \beta < \alpha\}\}$ for each $\alpha < \Lambda$. If $F_\Lambda = X \setminus \bigcup\{\varphi(d)\colon\ d \in \bigcup\{D_\beta\colon\ \beta < \Lambda\}\} \neq \emptyset$, then $\bigcup\{D_\beta\colon\ \beta < \Lambda\}$ is closed discrete and for each $x \in F_\Lambda$ there is some $y \in F_\Lambda \setminus \{x\}$ such that $x \in \varphi(y)$ and $\varphi(x) \cap D_\beta = \emptyset$ for each $\beta < \Lambda$. □

Recall that if $(X, \mathscr{T})$ is a butterfly space over (X, τ) then $\tau \subseteq \mathscr{T}$.

Theorem 19. *Let $(X, \mathscr{T})$ be a butterfly space over (X, τ). If (X, τ) is a hereditarily meta-Lindelöf space, then $(X, \mathscr{T})$ is a transitively D-space.*

P r o o f. Let φ be any transitive neighborhood assignment for the space $(X, \mathscr{T})$. For each $x \in X$, let $\mathscr{B}(x)$ be a neighborhood base of x in $(X, \mathscr{T})$, consisting of sets B such that $B \setminus \{x\} \in \tau$ for each $B \in \mathscr{B}(x)$. Let $\varphi'(x) \in \mathscr{B}(x)$ and $\varphi'(x) \subseteq \varphi(x)$ for each $x \in X$. Thus $\varphi' = \{\varphi'(x)\colon\ x \in X\}$ is a neighborhood assignment for the space $(X, \mathscr{T})$. Thus there is a closed discrete (in $(X, \mathscr{T})$) subspace $D \subseteq X$ such that either $X = \bigcup\{\varphi'(d)\colon\ d \in D\}$ or for each $x \in X \setminus \bigcup\{\varphi'(d)\colon\ d \in D\}$ there is some $y \in (X \setminus \bigcup\{\varphi'(d)\colon\ d \in D\}) \setminus \{x\}$ such that $x \in \varphi'(y)$ and $\varphi'(x) \cap D = \emptyset$ by Theorem 18.

We assume that $F = X \setminus \bigcup\{\varphi'(d)\colon\ d \in D\} \neq \emptyset$. For each $x \in F$, there is some $x_y \in F \setminus \{x\}$ such that $x \in \varphi'(x_y)$. If $\varphi_1(x) = \varphi'(x_y) \setminus \{x_y\}$, then $\varphi_1(x) \in \tau$. Thus $\varphi_1 = \{\varphi_1(x) \cap F\colon\ x \in F\}$ is a neighborhood assignment for the subspace $(F, \tau|F)$. Since $(F, \tau|F)$ is meta-Lindelöf, the cover φ_1 has a point-countable open refinement $\mathscr{W}$ such that $W \in \tau|F \subseteq \mathscr{T}|F$ for each $W \in \mathscr{W}$. For each $W \in \mathscr{W}$, there is some $x \in F$ such that $W \subseteq \varphi_1(x) \subseteq \varphi'(x_y) \subseteq \varphi(x_y)$. Thus $\mathscr{W}$ is a point-countable open (in $(F, \mathscr{T}|F)$) refinement of $\{\varphi(x) \cap F\colon\ x \in F\}$. Thus by Proposition 1 and Theorem 4, there is a closed discrete subspace (in $(F, \mathscr{T}|F)$) $D_F \subseteq F$ such that $F \subseteq \bigcup\{\varphi(d)\colon\ d \in D_F\}$. For each $x \in F$, there is an open set $V_x \in \mathscr{T}$ such that $x \in V_x$ and $|V_x \cap D_F| \leqslant 1$. If $O_x = V_x \cap \varphi'(x)$, then $O_x \in \mathscr{T}$, $x \in O_x$, and $|O_x \cap (D \cup D_F)| \leqslant 1$. Thus $D \cup D_F$ is a closed discrete subspace of X and $X = \bigcup\{\varphi(d)\colon\ d \in D \cup D_F\}$. So $(X, \mathscr{T})$ is transitively D. □

Corollary 20. *If (M, τ) is a metrizable space and $(M, \mathscr{T})$ is a butterfly space over (M, τ), then $(M, \mathscr{T})$ is transitively D.*

In [13], it was proved that every submeta-Lindelöf space is transitively D. Recall that a space X is *submeta-Lindelöf* if any open cover $\mathscr{U}$ of X has an open refinement $\mathscr{V} = \bigcup\{\mathscr{V}_n\colon\ n \in \mathbb{N}\}$ such that $\bigcup \mathscr{V}_n = X$ for each $n \in \mathbb{N}$, and for each $x \in X$ there is some $n \in \mathbb{N}$ such that $|\{V\colon\ x \in V$ and $V \in \mathscr{V}_n\}| \leqslant \omega$. By the proof of the conclusion in [13], we have the following lemma.

Lemma 21 (cf. [13])**.** *Let X be a space and let φ be any transitive neighborhood assignment for X. If φ has an open refinement $\mathscr{V} = \bigcup\{\mathscr{V}_n\colon n \in \mathbb{N}\}$ such that $\bigcup \mathscr{V}_n = X$ for each $n \in \mathbb{N}$, and for each $x \in X$ there is some $n \in \mathbb{N}$ such that $|\{V\colon x \in V \text{ and } V \in \mathscr{V}_n\}| \leqslant \omega$, then there is a closed discrete subspace $D \subseteq X$ such that $X = \bigcup\{\varphi(d)\colon d \in D\}$.*

By Lemma 21 and the proof of Theorem 19, we have:

Theorem 22. *Let $(X, \mathscr{T})$ be a butterfly space over (X, τ). If (X, τ) is a hereditarily submeta-Lindelöf space, then $(X, \mathscr{T})$ is a transitively D-space.*

Recall that a space X is a *Fréchet* space if whenever $A \subseteq X$ and $x \in \overline{A}$ there is a sequence $\{x_n\colon n \in \mathbb{N}\} \subseteq A$ such that x is a limit point of the sequence $\{x_n\}_{n\in\mathbb{N}}$. We know every subspace of a Fréchet space is Fréchet.

Theorem 23. *Let $(X, \mathscr{T})$ be a butterfly space over (X, τ). If (X, τ) is Fréchet and has a point-countable wcs***-network, then $(X, \mathscr{T})$ is a transitively D-space.*

P r o o f. Let φ be any transitive neighborhood assignment for the space $(X, \mathscr{T})$. For each $x \in X$, let $\mathscr{B}(x)$ be a neighborhood base of x in $(X, \mathscr{T})$, consisting of sets B such that $B \setminus \{x\} \in \tau$ for each $B \in \mathscr{B}(x)$. Let $\varphi'(x) \in \mathscr{B}(x)$ and $\varphi'(x) \subseteq \varphi(x)$ for each $x \in X$. Thus $\varphi' = \{\varphi'(x)\colon x \in X\}$ is a neighborhood assignment for the space $(X, \mathscr{T})$. Thus there is a closed discrete (in $(X, \mathscr{T})$) subspace $D \subseteq X$ such that either $X = \bigcup\{\varphi'(d)\colon d \in D\}$ or for each $x \in X \setminus \bigcup\{\varphi'(d)\colon d \in D\}$ there is some $x_y \in (X \setminus \bigcup\{\varphi'(d)\colon d \in D\}) \setminus \{x\}$ such that $x \in \varphi'(x_y)$ and $\varphi'(x) \cap D = \emptyset$ by Theorem 18.

We assume $F = X \setminus \bigcup\{\varphi'(d)\colon d \in D\} \neq \emptyset$. For each $x \in F$, there is some $x_y \in F \setminus \{x\}$ such that $x \in \varphi'(x_y)$. We let $\varphi_1(x) = \varphi'(x_y) \setminus \{x_y\}$ for $x \in F$. Let $\mathscr{W}^*$ be a point-countable *wcs**-network of (X, τ) and let $\mathscr{W} = \{W \cap F\colon W \in \mathscr{W}^*$ and there is some $x \in F$ such that $W \subseteq \varphi_1(x)\}$.

If $A \subseteq F$ and A is not closed in $(F, \mathscr{T}|F)$, then A is not closed in $(F, \tau|F)$. Thus there is some $x \in F \cap (\overline{A}^{(\tau)} \setminus A)$ and a sequence $\{x_n\colon n \in \mathbb{N}\} \subseteq A$ such that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to the point x. Thus there is some $W \in \mathscr{W}^*$ such that $|\{n\colon x_n \in W\}| = \omega$ and $W \subseteq \varphi_1(x)$. So $W \subseteq \varphi_1(x) \subseteq \varphi'(x_y) \subseteq \varphi(x_y)$ for some $x_y \in F$. So $W \cap F \in \mathscr{W}$, and hence the family $\mathscr{W}$ satisfies the conditions of Theorem 4. Thus there is closed discrete subspace (in $(F, \mathscr{T}|F)$) $D_F \subseteq F$ such that $F \subseteq \bigcup\{\varphi(d)\colon d \in D_F\}$. For each $x \in F$, there is some $V_x \in \mathscr{T}$ such that $|V_x \cap D_F| \leqslant 1$ and $x \in V_x$. If $O_x = V_x \cap \varphi'(x)$ then $x \in O_x$, $O_x \in \mathscr{T}$, and $|O_x \cap (D \cup D_F)| \leqslant 1$. Thus $D \cup D_F$ is closed discrete in $(X, \mathscr{T})$ and $X = \bigcup\{\varphi(d)\colon d \in D \cup D_F\}$. Thus $(X, \mathscr{T})$ is transitively D. □

By Corollary 6, we know that if X is a Hausdorff k-space which has a point-countable k-network then X is transitively D. By Theorem 3 which appears in [25], we know that if a regular space X is sequential and has a point-countable k-network then X is a D-space. Thus we have the following problem:

Problem 24. Is there a non-D Hausdorff k-space which has a point-countable k-network?

Problem 25. Is there a non-D-space which is transitively D?

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References

[1] *A. V. Arhangel'skii*: D-spaces and covering properties. Topology Appl. *146–147* (2005), 437–449.

[2] *A. V. Arhangel'skii*: D-spaces and finite unions. Proc. Am. Math. Soc. *132* (2004), 2163–2170.

[3] *A. V. Arhangel'skii*: A generic theorem in the theory of cardinal invariants of topological spaces. Commentat. Math. Univ. Carol. *36* (1995), 303–325.

[4] *A. V. Arhangel'skii, R. Z. Buzyakova*: Addition theorems and D-spaces. Commentat. Math. Univ. Carol. *43* (2002), 653–663.

[5] *A. V. Arhangel'skii, R. Z. Buzyakova*: On linearly Lindelöf and strongly discretely Lindelöf spaces. Proc. Am. Math. Soc. *127* (1999), 2449–2458.

[6] *C. R. Borges, A. C. Wehrly*: A study of D-spaces. Topol. Proc. *16* (1991), 7–15.

[7] *D. K. Burke*: Weak-bases and D-spaces. Commentat. Math. Univ. Carol. *48* (2007), 281–289.

[8] *R. Z. Buzyakova*: On D-property of strong Σ-spaces. Commentat. Math. Univ. Carol. *43* (2002), 493–495.

[9] *E. K. van Douwen, W. F. Pfeffer*: Some properties of the Sorgenfrey line and related spaces. Pac. J. Math. *81* (1979), 371–377.

[10] *R. Engelking*: General Topology. Sigma Series in Pure Mathematics, Vol. 6, revised ed. Heldermann, Berlin, 1989.

[11] *G. Gruenhage*: A note on D-spaces. Topology Appl. *153* (2006), 2229–2240.

[12] *G. Gruenhage*: Generalized Metric Spaces. In: Handbook of Set-Theoretic Topology (K. Kunen, J. E. Vaughan, eds.). North-Holland, Amsterdam, 1984, pp. 423–501.

[13] *G. Gruenhage*: Submeta-Lindelöf implies transitively D. Preprint.

[14] *G. Gruenhage, E. Michael, Y. Tanaka*: Spaces determined by point-countable covers. Pac. J. Math. *113* (1984), 303–332.

[15] *G. Gruenhage*: A survey of D-spaces. Contemporary Mathematics 533 (L. Babinkostova, ed.). American Mathematical Society, Providence, 2011, pp. 13–28.

[16] *H. F. Guo, H. Junnila*: On spaces which are linearly D. Topology Appl. *157* (2010), 102–107.

[17] *S. Lin*: A note on D-spaces. Commentat. Math. Univ. Carol. *47* (2006), 313–316.

[18] *S. Lin*: Point-Countable Covers and Sequence-Covering Mappings. Chinese Science Press, Beijing, 2002. (In Chinese.)

[19] *S. Lin, Y. Tanaka*: Point-countable k-networks, closed maps and related results. Topology Appl. *59* (1994), 79–86.
[20] *E. Pearl*: Linearly Lindelöf problems. In: Open Problems in Topology II (E. Pearl, ed.). Elsevier, Amsterdam, 2007, pp. 225–231.
[21] *L.-X. Peng*: On spaces which are D, linearly D and transitively D. Topology Appl. *157* (2010), 378–384.
[22] *L.-X. Peng*: The D-property of some Lindelöf spaces and related conclusions. Topology Appl. *154* (2007), 469–475.
[23] *L.-X. Peng*: On linear neighborhood assignments and dually discrete spaces. Topology Appl. *155* (2008), 1867–1874.
[24] *L.-X. Peng*: A special point-countable family that makes a space to be a D-space. Adv. Math. (China) *37* (2008), 724–728.
[25] *L.-X. Peng*: A note on D-spaces and infinite unions. Topology Appl. *154* (2007), 2223–2227.
[26] *L.-X. Peng*: On weakly monotonically monolithic spaces. Commentat. Math. Univ. Carol. *51* (2010), 133–142.
[27] *L.-X. Peng*: A note on transitively D and D-spaces. Houston J. Math. To appear.
[28] *L.-X. Peng, Franklin D. Tall*: A note on linearly Lindelöf spaces and dual properties. Topol. Proc. *32* (2008), 227–237.
[29] *F. Siwiec*: On defining a space by a weak base. Pac. J. Math. *52* (1974), 233–245.
[30] *L. A. Steen, Jr. J. A. Seebach*: Counterexamples in Topology. 2nd edition. Springer, New York-Heidelberg-Berlin, 1978.
[31] *V. V. Tkachuk*: Monolithic spaces and D-spaces revisited. Topology Appl. *156* (2009), 840–846.

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