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ORDER REDUCTION OF THE EULER-LAGRANGE EQUATIONS  
OF HIGHER ORDER INVARIANT VARIATIONAL  
PROBLEMS ON FRAME BUNDLES

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*Abstract.* Let  $\mu: FX \rightarrow X$  be a principal bundle of frames with the structure group  $\mathrm{Gl}_n(\mathbb{R})$ . It is shown that the variational problem, defined by  $\mathrm{Gl}_n(\mathbb{R})$ -invariant Lagrangian on  $J^r FX$ , can be equivalently studied on the associated space of connections with some compatibility condition, which gives us order reduction of the corresponding Euler-Lagrange equations.

*Keywords:* Frame bundle, Euler-Lagrange equations, invariant Lagrangian, Euler-Poincaré reduction

*MSC 2010:* 53C05, 53C10, 58A20, 58E30

1. INTRODUCTION

In the Lagrange theory of the calculus of variations on fibred manifolds, it is well known that the Euler-Lagrange equations are employed for finding the extremal values of the corresponding variational function. In general, for variational problems defined by Lagrangians of order  $r$ , the order of the Euler-Lagrange equations is  $2r$ . In the case when an underlying manifold has the structure of a principal fibre bundle with the structure group  $G$  and  $G$ -invariant Lagrangians, the original system of the Euler-Lagrange equations can be simplified.

The idea of reduction of the Euler-Lagrange equations originally comes from mechanics, where  $G$ -invariant Lagrangian defined on the tangent bundle of a Lie group  $G$  is considered. The equations of new kind for the reduced Lagrangian are called the

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Euler-Poincaré equations (see, e.g., [14]). The extension of such mechanism to field theory was done in [3], [4], [5], and is called the Euler-Poincaré reduction. In this case, we consider a Lagrangian defined on the first jet prolongation  $J^1P$  of a principal fibre bundle  $P$ , invariant under the natural action induced by the structure group  $G$  on  $J^1P$ . The reduced variational problem, in the sense of Euler-Poincaré reduction, is defined on the bundle of connections  $C(P)$ , which can be identified with  $(J^1P)/G$ . It was shown that locally there is an equivalence between the solutions of the Euler-Lagrange equations for  $G$ -invariant Lagrangian on the one side, and the solutions of the Euler-Poincaré equations for reduced Lagrangian plus the condition of flat connection on the other side.

In this paper, we consider the same type of invariant variational problems defined on frame bundles  $FX$  over an  $n$ -dimensional manifold  $X$ . Our aim is to generalize the reduction problem to higher order cases, i.e., to variational problems on  $J^rFX$  defined by  $\text{Gl}_n(\mathbb{R})$ -invariant Lagrangians.

In Sections 2 and 3 we recall the basic notions from the variational theory on fibred manifolds and frame bundles, respectively. Section 4 deals with the correspondence between the sections of  $J^1FX$  and  $C^1X$ , where  $C^1X$  is the bundle of linear connections of  $X$ . Using this correspondence it is shown that the  $\text{Gl}_n(\mathbb{R})$ -invariant variational problem on  $J^{r+1}FX$  can be alternatively studied as the reduced problem on  $J^rC^1X$ , which gives us the order reduction of the variational equations for extremals. Again, the necessity of additional compatibility conditions for connections, in the form of zero curvature, occurs. Having a solution of the reduced equations, the compatibility conditions ensure the existence of the associated extremal frame field, because they correspond to the conditions of complete integrability of the arising system of first order equations. In Section 5 we give explicit expressions of the reduced, first order equations for connections, which replace the second order Euler-Lagrange equations for the associated frame field.

For reduction of the solution of the Euler-Lagrange equations defined by higher order invariant Lagrangians it is also possible to use the corresponding bundles of higher order connections  $C^rX = J^rFX/\text{Gl}_n(\mathbb{R})$  (see [1]).

## 2. PRELIMINARIES

$Y$  is a *fibred manifold* with an oriented *base manifold*  $X$  and a *projection*  $\pi$ . We denote  $n = \dim X$ ,  $m = \dim Y - n$ .  $J^rY$  is the *r-jet prolongation* of  $Y$ . The *r-jet* of a section  $\gamma$  of  $Y$  at a point  $x \in X$  is denoted by  $J_x^r\gamma$ ; and  $x \mapsto J^r\gamma(x) = J_x^r\gamma$  is the *r-jet prolongation* of  $\gamma$ . If  $J_x^r\gamma \in J^rY$ , the *canonical jet projections*  $\pi^{r,0}: J^rY \rightarrow Y$  (the target projection),  $\pi^r: J^rY \rightarrow X$  (the source projection), are defined by  $\pi^{r,0}(J_x^r\gamma) = \gamma(x)$ , and  $\pi^r(J_x^r\gamma) = x$ , respectively.

Any *fibred chart*  $(V, \psi)$ ,  $\psi = (x^i, y^\sigma)$ , on  $Y$ , where  $1 \leq i \leq n$ ,  $1 \leq \sigma \leq m$ , induces the *associated charts* on  $X$  and on  $J^r Y$ ,  $(U, \varphi)$ ,  $\varphi = (x^i)$ , and  $(V^r, \psi^r)$ ,  $\psi^r = (x^i, y^\sigma, y_{j_1}^\sigma, y_{j_1 j_2}^\sigma, \dots, y_{j_1 j_2 \dots j_r}^\sigma)$ , respectively; here  $V^r = (\pi^{r,0})^{-1}(V)$ , and  $U = \pi(V)$ . For functions on  $V^r$ , we define the *formal derivative operator* (with respect to the fibred chart  $(V, \psi)$ ) by

$$d_p = \frac{\partial}{\partial x^p} + y_p^\sigma \frac{\partial}{\partial y^\sigma} + y_{i_1 p}^\sigma \frac{\partial}{\partial y_{i_1}^\sigma} + \dots + y_{i_1 i_2 \dots i_r p}^\sigma \frac{\partial}{\partial y_{i_1 i_2 \dots i_r}^\sigma}.$$

A *Lagrangian* (of order  $r$ ) for  $Y$  is any  $\pi^r$ -horizontal  $n$ -form on some  $W^r \subset J^r Y$ . If, in a fibred chart  $(V, \psi)$ ,  $\psi = (x^i, y^\sigma)$ , we denote  $\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$ , then a Lagrangian, defined on  $V^r = (\pi^{r,0})^{-1}(V)$ , has in this chart an expression

$$\lambda = \mathcal{L}\omega_0,$$

where  $\mathcal{L}: V^r \rightarrow \mathbb{R}$  is a function (the *Lagrange function* associated with  $\lambda$  and  $(V, \psi)$ ).

The *Euler-Lagrange form*  $E_\lambda$ , associated with a Lagrangian  $\lambda$  of order  $r$ , is a  $\pi^{2r,0}$ -horizontal  $(n+1)$ -form, in fibred chart defined by

$$E_\lambda = E_\sigma(\mathcal{L})\omega^\sigma \wedge \omega_0,$$

where

$$E_\sigma(\mathcal{L}) = \sum_{k=0}^r (-1)^k d_{i_1} d_{i_2} \dots d_{i_k} \frac{\partial \mathcal{L}}{\partial y_{i_1 i_2 \dots i_k}^\sigma}$$

are the *Euler-Lagrange expressions* associated with  $\lambda$ .

Let  $\Omega$  be a piece of  $X$  (a compact,  $n$ -dimensional submanifold of  $X$  with boundary  $\partial\Omega$ ), let  $\Gamma_{\Omega, W}(\pi)$  be the set of smooth sections  $\gamma$  over  $\Omega$  such that  $\gamma(\Omega) \subset W$ . If we have a Lagrangian  $\lambda$  of order  $r$  defined on  $W^r$ , the *variational function* associated with  $\lambda$  over  $\Omega$  is the mapping  $\Gamma_{\Omega, W}(\pi) \ni \gamma \rightarrow \lambda_\Omega(\gamma) \in \mathbb{R}$  defined by

$$\lambda_\Omega(\gamma) = \int_\Omega J^r \gamma^* \lambda.$$

Let  $\xi$  be a  $\pi$ -projectable vector field on  $Y$ , and let  $\partial_{J^r \xi} \lambda$  denote the Lie derivative of  $\lambda$  by the  $r$ -jet prolongation  $J^r \xi$  of  $\xi$ . The number

$$(\partial_{J^r \xi} \lambda)_\Omega(\gamma) = \int_\Omega J^r \gamma^* \partial_{J^r \xi} \lambda$$

is the *variation* of the variational function  $\lambda_\Omega$  at  $\gamma$ , *induced* by the vector field  $\xi$ . We say that a section  $\gamma \in \Gamma_{\Omega, W}(\pi)$  is *stable* with respect to a variation  $\xi$  of  $\gamma$ , if

$(\partial_{J^r \xi} \lambda)_\Omega(\gamma) = 0$ . If  $\gamma$  is stable with respect to all  $\xi$  with support contained in  $\pi^{-1}(\Omega)$ , we say that  $\gamma$  is an *extremal* of  $\lambda_\Omega$ . A section  $\gamma$  which is an extremal of every  $\lambda_\Omega$  is called an *extremal* of  $\lambda$ . A section  $\gamma$  is an extremal of a Lagrangian  $\lambda$  of order  $r$  if and only if, for every fibred chart on  $Y$ ,  $\gamma$  satisfies the system of partial differential equations

$$E_\sigma(\mathcal{L}) \circ J^{2r} \gamma = 0.$$

For more detailed exposition of the variational theory on fibred spaces, including local variational principles, see [2], [11].

### 3. FRAME BUNDLES AND THEIR PROLONGATIONS

In this section we recall the basic notions from the theory of frame bundles; the reader can also consult [1], [9], [13].

Henceforth, instead of  $Y$ , let us consider the *frame bundle*  $\mu: FX \rightarrow X$  over an  $n$ -dimensional manifold  $X$ , which has the structure of a principal fiber bundle with the structure group  $\text{Gl}_n(\mathbb{R})$ . A *frame* at a point  $x \in X$  is a pair  $\Xi = (x, \zeta)$ , where  $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$  is an ordered basis of the tangent space  $T_x X$ . For every chart  $(U, \varphi)$ ,  $\varphi = (x^i)$ , on  $X$ , the *associated chart*  $(V, \psi)$ ,  $\psi = (x^i, x_j^i)$ , on  $FX$ , is defined by  $V = \mu^{-1}(U)$ , and

$$x^i(\Xi) = x^i(\mu(\Xi)), \quad \zeta_j = x_j^i(\Xi) \left( \frac{\partial}{\partial x^i} \right)_x,$$

where  $\Xi \in V$ . We denote by  $y_k^j$  the *inverse matrix* of  $x_j^i$ .

Let us denote by  $J^r FX$  an *r-jet prolongation* of  $FX$ . With any chart  $(U, \varphi)$ ,  $\varphi = (x^i)$ , on  $X$ , we also associate a chart  $(V^r, \psi^r)$  on  $J^r FX$ , where  $V^r = (\mu^r)^{-1}(U)$  and  $\psi^r = (x^i, x_j^i, x_{j,k_1}^i, x_{j,k_1 k_2}^i, \dots, x_{j,k_1 k_2 \dots k_r}^i)$ .

The *right action*  $(\Xi, A) \mapsto \Xi \cdot A \equiv R_A(\Xi)$  of  $\text{Gl}_n(\mathbb{R})$  on  $FX$  is expressed, in a chart  $(U, \varphi)$ ,  $\varphi = (x^i)$ , on  $X$ , by the equations

$$\bar{x}^i = x^i \circ R_A = x^i, \quad \bar{x}_j^i = x_j^i \circ R_A = x_k^i a_j^k,$$

where  $a_j^k$  are the canonical coordinates on  $\text{Gl}_n(\mathbb{R})$ . This action can be canonically prolonged to the action of  $\text{Gl}_n(\mathbb{R})$  on  $J^r FX$ . If  $A \in \text{Gl}_n(\mathbb{R})$ ,  $J_x^r \gamma \in J^r FX$ , we define

$$(J_x^r \gamma, A) \mapsto J_x^r \gamma \cdot A \equiv J^r R_A(J_x^r \gamma) = J_x^r (R_A \circ \gamma).$$

In the associated charts on  $FX$  and  $J^r FX$ , we have

$$\begin{aligned} \bar{x}^i &= x^i \circ J^r R_A = x^i, \\ \bar{x}_{j,k_1 k_2 \dots k_m}^i &= x_{j,k_1 k_2 \dots k_m}^i \circ J^r R_A = x_{l,k_1 k_2 \dots k_m}^i a_j^l, \quad 0 \leq m \leq r. \end{aligned}$$

A form  $\eta$  on  $J^r FX$  is *invariant* with respect to the action of  $\text{Gl}_n(\mathbb{R})$  (or,  $\text{Gl}_n(\mathbb{R})$ -*invariant*), if

$$(J^r R_A)^* \eta = \eta$$

holds for all  $A \in \text{Gl}_n(\mathbb{R})$ . In particular, a function  $f$  on  $J^r FX$  is  $\text{Gl}_n(\mathbb{R})$ -invariant if  $(J^r R_A)^* f = f \circ J^r R_A = f$  for all  $A \in \text{Gl}_n(\mathbb{R})$ .

The functions

$$(3.1) \quad \Gamma_{k_1 k_2 \dots k_m p}^i = -y_p^l x_{l, k_1 k_2 \dots k_m}^i,$$

defined for  $1 \leq m \leq r$ , are examples of  $\text{Gl}_n(\mathbb{R})$ -invariant functions on  $J^r FX$ . Note that the functions  $\Gamma_{k_1 k_2 \dots k_m p}^i$  are *symmetric* in  $k_1, k_2, \dots, k_m$ .

Let  $(V^r, \psi^r)$  be any fibred chart on  $J^r FX$ . Setting  $\Psi^r = (x^i, x_p^i, \Gamma_{k_1 p}^i, \Gamma_{k_1 k_2 p}^i, \dots, \Gamma_{k_1 k_2 \dots k_r p}^i)$ , we obtain the associated chart  $(V^r, \Psi^r)$  on  $J^r FX$ , which is said to be  $\text{Gl}_n(\mathbb{R})$ -*adapted*.

For  $r \geq 1$ , let us denote

$$C^r X = J^r FX / \text{Gl}_n(\mathbb{R}).$$

The quotient projection  $\mu^{(r)}: J^r FX \rightarrow C^r X$  is in  $\text{Gl}_n(\mathbb{R})$ -adapted coordinates expressed as

$$\mu^{(r)}: (x^i, x_j^i, \Gamma_{k_1 j}^i, \Gamma_{k_1 k_2 j}^i, \dots, \Gamma_{k_1 k_2 \dots k_r j}^i) \rightarrow (x^i, \Gamma_{k_1 j}^i, \Gamma_{k_1 k_2 j}^i, \dots, \Gamma_{k_1 k_2 \dots k_r j}^i).$$

Let  $\{V_\alpha^r, \Psi_\alpha^r\}_{\alpha \in \mathcal{A}}$  be a covering of  $J^r FX$  by  $\text{Gl}_n(\mathbb{R})$ -adapted charts. For each  $\alpha \in \mathcal{A}$ , we set  $W_\alpha^r = \mu^{(r)}(V_\alpha^r)$ , and define the system of coordinate functions  $\chi_\alpha^r$  by  $\chi_\alpha^r \circ \mu^{(r)} = \Psi_\alpha^r$ . Then  $\{W_\alpha^r, \chi_\alpha^r\}_{\alpha \in \mathcal{A}}$  is a covering of  $C^r X$  consisting of the fibred charts and  $C^r X$  is the fibred manifold over  $X$  with the projection  $\kappa^r: C^r X \rightarrow X$ , in the corresponding coordinates defined by  $\kappa^r: (x^i, \Gamma_{k_1 j}^i, \Gamma_{k_1 k_2 j}^i, \dots, \Gamma_{k_1 k_2 \dots k_r j}^i) \rightarrow (x^i)$ .

**Lemma 3.1.**  *$J^r FX$  has the structure of a right principal  $\text{Gl}_n(\mathbb{R})$ -bundle over  $C^r X$  such that the diagram*

$$\begin{array}{ccc} J^r FX & \xrightarrow{\mu^{(r)}} & C^r X \\ \downarrow \mu^{r,0} & & \downarrow \kappa^r \\ FX & \xrightarrow{\mu} & X \end{array}$$

*commutes.*

For  $r = 1$ , the space  $C^1 X = J^1 FX / \text{Gl}_n(\mathbb{R})$  can be identified with the bundle of linear connections (García Pérez [7], Krupka [12]). In the case  $r > 1$ , the space  $C^r X$  is said to be the bundle of an  $r$ th order connections (Kolář [10]).

**Lemma 3.2.**

- (a) A function on  $J^r FX$  is  $\text{Gl}_n(\mathbb{R})$ -invariant if and only if it depends, locally, on the coordinates  $(x^i, \Gamma_{k_1 p}^i, \Gamma_{k_1 k_2 p}^i, \dots, \Gamma_{k_1 k_2 \dots k_r p}^i)$  on  $C^r X$  only.
- (b) A  $k$ -form  $\eta$  on  $J^r FX$  is  $\text{Gl}_n(\mathbb{R})$ -invariant if and only if in any chart on  $X$ ,

$$\eta = \Delta_0 + y_{s_1}^{q_1} dx_{q_1}^{p_1} \wedge \Delta_{p_1}^{s_1} + y_{s_1}^{q_1} y_{s_2}^{q_2} dx_{q_1}^{p_1} \wedge dx_{q_2}^{p_2} \wedge \Delta_{p_1 p_2}^{s_1 s_2} \\ + \dots + y_{s_1}^{q_1} y_{s_2}^{q_2} \dots y_{s_k}^{q_k} dx_{q_1}^{p_1} \wedge dx_{q_2}^{p_2} \dots \wedge dx_{q_k}^{p_k} \wedge \Delta_{p_1 p_2 \dots p_k}^{s_1 s_2 \dots s_k},$$

where  $\Delta_0, \Delta_{p_1}^{s_1}, \Delta_{p_1 p_2}^{s_1 s_2}, \dots, \Delta_{p_1 p_2 \dots p_k}^{s_1 s_2 \dots s_k}$  are arbitrary forms defined on  $C^r X$ .

- (c) Let  $f$  be a  $\text{Gl}_n(\mathbb{R})$ -invariant function on  $J^r FX$ . Then the  $p$ th formal derivative  $d_p f$  is again a  $\text{Gl}_n(\mathbb{R})$ -invariant function on  $J^{r+1} FX$ , and in the corresponding adapted chart it can be expressed as

$$(3.2) \quad d_p f = \frac{\partial f}{\partial x^p} + \sum_{s=1}^r \sum_{k_1 \leq k_2 \leq \dots \leq k_s} \frac{\partial f}{\partial \Gamma_{k_1 k_2 \dots k_s j}^i} (\Gamma_{k_1 k_2 \dots k_s p j}^i + \Gamma_{k_1 k_2 \dots k_s q}^i \Gamma_{p j}^q).$$

Note that the operator  $d_p$  introduced in (3.2), restricted to the invariant functions on  $J^r FX$ , corresponds to the operator of the formal derivative of any function on  $C^r X$ .

The functions (3.1) can be rewritten in another form. By (3.2),

$$\Gamma_{k l j}^i = d_l \Gamma_{k j}^i - \Gamma_{k s}^i \Gamma_{l j}^s = d_k \Gamma_{l j}^i - \Gamma_{l s}^i \Gamma_{k j}^s,$$

where the functions  $\Gamma_{k j}^i$  and their formal derivatives  $d_l \Gamma_{k j}^i$  satisfy the compatibility condition

$$(3.3) \quad d_l \Gamma_{k j}^i - d_k \Gamma_{l j}^i + \Gamma_{l s}^i \Gamma_{k j}^s - \Gamma_{k s}^i \Gamma_{l j}^s = 0.$$

Analogously,

$$\Gamma_{k_1 k_2 k_3 j}^i = d_{k_3} \Gamma_{k_1 k_2 j}^i - \Gamma_{k_1 k_2 s}^i \Gamma_{k_3 j}^s \\ = d_{k_2} \Gamma_{k_1 k_3 j}^i - \Gamma_{k_1 k_3 s}^i \Gamma_{k_2 j}^s = d_{k_1} \Gamma_{k_2 k_3 j}^i - \Gamma_{k_2 k_3 s}^i \Gamma_{k_1 j}^s \\ \vdots \\ \Gamma_{k_1 k_2 \dots k_r j}^i = d_{k_r} \Gamma_{k_1 k_2 \dots k_{r-1} j}^i - \Gamma_{k_1 k_2 \dots k_{r-1} s}^i \Gamma_{k_r j}^s \\ = d_{k_{r-1}} \Gamma_{k_1 k_2 \dots k_{r-2} k_r j}^i - \Gamma_{k_1 k_2 \dots k_{r-2} k_r s}^i \Gamma_{k_{r-1} j}^s \\ = \dots \\ = d_{k_1} \Gamma_{k_2 k_3 \dots k_r j}^i - \Gamma_{k_2 k_3 \dots k_r s}^i \Gamma_{k_1 j}^s,$$

and we have the corresponding compatibility conditions

$$\begin{aligned}
d_{k_1} \Gamma_{k_2 k_3 p}^i - \Gamma_{k_2 k_3 s}^i \Gamma_{k_1 p}^s &= d_{k_2} \Gamma_{k_1 k_3 p}^i - \Gamma_{k_1 k_3 s}^i \Gamma_{k_2 p}^s \\
&= d_{k_3} \Gamma_{k_1 k_2 p}^i - \Gamma_{k_1 k_2 s}^i \Gamma_{k_3 p}^s \\
&\vdots \\
d_{k_1} \Gamma_{k_2 k_3 \dots k_r p}^i - \Gamma_{k_2 k_3 \dots k_r s}^i \Gamma_{k_1 p}^s &= d_{k_2} \Gamma_{k_1 k_3 \dots k_r p}^i - \Gamma_{k_1 k_3 \dots k_r s}^i \Gamma_{k_2 p}^s \\
&= \dots \\
&= d_{k_{r-1}} \Gamma_{k_1 k_2 \dots k_{r-2} k_r p}^i - \Gamma_{k_1 k_2 \dots k_{r-2} k_r s}^i \Gamma_{k_{r-1} p}^s \\
&= d_{k_r} \Gamma_{k_1 k_2 \dots k_{r-1} p}^i - \Gamma_{k_1 k_2 \dots k_{r-1} s}^i \Gamma_{k_r p}^s.
\end{aligned}$$

Let  $\lambda$  be a Lagrangian on  $J^r FX$ . Recall that the coordinate expression of  $\lambda$  in any  $\text{Gl}_n(\mathbb{R})$ -adapted chart is  $\lambda = \mathcal{L}\omega_0$ , where  $\mathcal{L}: V^r \rightarrow \mathbb{R}$  is the associated Lagrange function.

**Lemma 3.3.** *The Euler-Lagrange form  $E_\lambda$ , associated with the  $\text{Gl}_n(\mathbb{R})$ -invariant Lagrangian  $\lambda$  on  $J^r FX$ , is  $\text{Gl}_n(\mathbb{R})$ -invariant.*

**Corollary 3.1.**

- (a) *A Lagrangian  $\lambda$  on  $J^r FX$  is  $\text{Gl}_n(\mathbb{R})$ -invariant if and only if in any  $\text{Gl}_n(\mathbb{R})$ -adapted chart, the associated Lagrange function  $\mathcal{L}$  is  $\text{Gl}_n(\mathbb{R})$ -invariant.*
- (b) *The coordinate expression of the Euler-Lagrange form  $E_\lambda$ , associated with  $\text{Gl}_n(\mathbb{R})$ -invariant Lagrangian  $\lambda$  on  $J^r FX$ , is*

$$E_\lambda = F_i^l(\mathcal{L}) y_t^j dx_j^i \wedge \omega_0,$$

where  $F_i^l(\mathcal{L})$  are  $\text{Gl}_n(\mathbb{R})$ -invariant functions.

**Corollary 3.2.** *A Lagrangian  $\lambda$  on  $J^r FX$  is  $\text{Gl}_n(\mathbb{R})$ -invariant if and only if there exists a Lagrangian  $\tilde{\lambda}$  on  $C^r X$  such that  $\lambda = (\mu^{(r)})^* \tilde{\lambda}$ .*

Finally, we prove an auxiliary assertion.

**Lemma 3.4.** *In a  $\text{Gl}_n(\mathbb{R})$ -adapted chart  $(V^r, \Psi^r)$ ,  $\Psi^r = (x^i, x_j^i, \Gamma_{p_1 j}^i, \Gamma_{p_1 p_2 j}^i, \dots, \Gamma_{p_1 p_2 \dots p_r j}^i)$ , on  $J^r FX$ ,  $r > 2$ ,*

$$\Gamma_{p_1 p_2 \dots p_m k l j}^i - \Gamma_{p_1 p_2 \dots p_m l k j}^i = -\Gamma_{p_1 p_2 \dots p_m q}^i (\Gamma_{k l j}^q - \Gamma_{l k j}^q)$$

holds for all  $1 \leq m \leq r - 2$ .



Proof. According to (3.2), for adapted coordinates  $\Gamma_{p_1 p_2 \dots p_m j}^i$ ,  $m \geq 1$  we have

$$(3.4) \quad \Gamma_{p_1 p_2 \dots p_m k j}^i = d_k \Gamma_{p_1 p_2 \dots p_m j}^i - \Gamma_{p_1 p_2 \dots p_m q}^i \Gamma_{k j}^q$$

and

$$(3.5) \quad d_l \Gamma_{p_1 p_2 \dots p_m k j}^i = d_l d_k \Gamma_{p_1 p_2 \dots p_m j}^i - d_l \Gamma_{p_1 p_2 \dots p_m q}^i \Gamma_{k j}^q - \Gamma_{p_1 p_2 \dots p_m q}^i d_l \Gamma_{k j}^q.$$

Then, using (3.4) and (3.5), we obtain

$$\begin{aligned} & \Gamma_{p_1 p_2 \dots p_m k l j}^i - \Gamma_{p_1 p_2 \dots p_m l k j}^i \\ &= d_l \Gamma_{p_1 p_2 \dots p_m k j}^i - \Gamma_{p_1 p_2 \dots p_m k t}^i \Gamma_{l j}^t - d_k \Gamma_{p_1 p_2 \dots p_m l j}^i + \Gamma_{p_1 p_2 \dots p_m l t}^i \Gamma_{k j}^t \\ &= d_l d_k \Gamma_{p_1 p_2 \dots p_m j}^i - d_l \Gamma_{p_1 p_2 \dots p_m q}^i \Gamma_{k j}^q - \Gamma_{p_1 p_2 \dots p_m q}^i d_l \Gamma_{k j}^q \\ &\quad - d_k \Gamma_{p_1 p_2 \dots p_m t}^i \Gamma_{l j}^t + \Gamma_{p_1 p_2 \dots p_m q}^i \Gamma_{k t}^q \Gamma_{l j}^t \\ &\quad - d_k d_l \Gamma_{p_1 p_2 \dots p_m j}^i + d_k \Gamma_{p_1 p_2 \dots p_m q}^i \Gamma_{l j}^q + \Gamma_{p_1 p_2 \dots p_m q}^i d_k \Gamma_{l j}^q \\ &\quad + d_l \Gamma_{p_1 p_2 \dots p_m t}^i \Gamma_{k j}^t - \Gamma_{p_1 p_2 \dots p_m q}^i \Gamma_{l t}^q \Gamma_{k j}^t \\ &= -\Gamma_{p_1 p_2 \dots p_m q}^i (d_l \Gamma_{k j}^q - d_k \Gamma_{l j}^q + \Gamma_{l t}^q \Gamma_{k j}^t - \Gamma_{k t}^q \Gamma_{l j}^t) \\ &= -\Gamma_{p_1 p_2 \dots p_m q}^i (\Gamma_{k l j}^q - \Gamma_{l k j}^q). \end{aligned}$$

□

#### 4. VARIATIONAL PROBLEMS DEFINED BY HIGHER ORDER INVARIANT LAGRANGIANS

First we discuss relations between linear connections on  $X$  and sections of the fibred manifold  $J^1FX$  (see [15]).

**Lemma 4.1.** *Equations*

$$(4.1) \quad \mu^{1,0} \circ w = \text{id}_{\mu^{-1}(U)}, \quad \mu^{(1)} \circ w = \Gamma \circ \mu$$

define a bijective correspondence between the connections  $\Gamma$ , defined on open sets  $U \subset X$ , and the  $\text{Gl}_n(\mathbb{R})$ -equivariant sections  $w$  of  $J^1FX$ , defined on open,  $\mu$ -saturated sets  $V = \mu^{-1}(U) \subset FX$ .

If  $w: V \rightarrow J^1FX$  is the  $\text{Gl}_n(\mathbb{R})$ -equivariant section, then the connection  $\Gamma$ , defined by (4.1), is said to be *associated* with  $w$ . The diagram

$$\begin{array}{ccc} J^1FX & \xrightarrow{\mu^{(1)}} & C^1X \\ \uparrow w & \swarrow \delta & \uparrow \Gamma \\ V & \xrightarrow{\mu} & U \\ & \xleftarrow{\gamma} & \end{array}$$

also induces the natural correspondence between the sections of  $J^1FX$  over  $X$ , and the sections of  $C^1X$ : if  $\delta$  is a section of  $J^1FX$ , then  $\Gamma = \mu^{(1)} \circ \delta$  is a section of  $C^1X$  over the same domain of definition. Conversely, given  $\Gamma$ , one can construct, for every chart  $(U, \varphi)$ ,  $\varphi = (x^i)$ , on  $X$ , a section  $\delta: U \rightarrow J^1FX$ . We choose *any* frame field  $\gamma: U \rightarrow FX$ , and then define  $\delta: U \rightarrow J^1FX$  by setting  $x_j^i \circ \delta = x_j^i \circ \gamma$ ,  $x_{j,k}^i \circ \delta = -(x_j^l \circ \gamma) \cdot \Gamma_{kl}^i$ . If

$$\Gamma = \mu^{(1)} \circ \delta$$

over an open set  $U$ , we say that  $\delta$  *generates*  $\Gamma$  on  $U$ .

Consider the restriction of the correspondence  $\delta \rightarrow \mu^{(1)} \circ \delta$  to *holonomic* sections, i.e., to sections  $\delta$  of the form  $\delta = J^1\gamma$ , where  $\gamma$  is a section of  $FX$ . If

$$\Gamma = \mu^{(1)} \circ J^1\gamma$$

for some frame field  $\gamma$ , we say that  $\gamma$  *generates*  $\Gamma$  on  $U$ . If every point of  $X$  has a neighborhood  $U$  such that  $\Gamma = \mu^{(1)} \circ J^1\gamma$  for some frame field  $\gamma$ , we say that  $\Gamma$  is *locally generated* by frame fields.

**Lemma 4.2.** *The following conditions are equivalent.*

- (a)  $\Gamma$  is locally generated by frame fields.
- (b)  $\Gamma$  is flat.

**Proof.** Let  $\Gamma$  be a linear connection on  $X$ . Denote by

$$(4.2) \quad R_{ljk}^i = \frac{\partial \Gamma_{kj}^i}{\partial x^l} - \frac{\partial \Gamma_{lj}^i}{\partial x^k} + \Gamma_{ls}^i \Gamma_{kj}^s - \Gamma_{ks}^i \Gamma_{lj}^s$$

the components of the *curvature tensor* of  $\Gamma$  in a chart  $(U, \varphi)$ ,  $\varphi = (x^i)$ , on  $X$  (see [8]).

(a)  $\Rightarrow$  (b): Assume that the first condition holds. Then there exists a system of functions  $x_j^i: U \rightarrow \mathbb{R}$  such that

$$(4.3) \quad \frac{\partial x_j^i}{\partial x^k} = -x_j^l \Gamma_{kl}^i.$$

This system satisfies

$$\begin{aligned} -\frac{\partial x_j^s}{\partial x^l} \Gamma_{ks}^i - x_j^s \frac{\partial \Gamma_{ks}^i}{\partial x^l} &= \frac{\partial}{\partial x^l} (-x_j^s \Gamma_{ks}^i) = \frac{\partial^2 x_j^i}{\partial x^l \partial x^k} \\ &= \frac{\partial^2 x_j^i}{\partial x^k \partial x^l} = \frac{\partial}{\partial x^k} (-x_j^s \Gamma_{ls}^i) = -\frac{\partial x_j^s}{\partial x^k} \Gamma_{ls}^i - x_j^s \frac{\partial \Gamma_{ls}^i}{\partial x^k}, \end{aligned}$$

i.e.,

$$-x_j^t \Gamma_{lt}^s \Gamma_{ks}^i + x_j^s \frac{\partial \Gamma_{ks}^i}{\partial x^l} + x_j^t \Gamma_{kt}^s \Gamma_{ls}^i - x_j^s \frac{\partial \Gamma_{ls}^i}{\partial x^k} = 0.$$

Thus,

$$(4.4) \quad -\Gamma_{lj}^s \Gamma_{ks}^i + \frac{\partial \Gamma_{kj}^i}{\partial x^l} + \Gamma_{kj}^s \Gamma_{ls}^i - \frac{\partial \Gamma_{lj}^i}{\partial x^k} = 0,$$

or,

$$R_{lkj}^i = 0,$$

which means that  $\Gamma$  is flat.

(b)  $\Rightarrow$  (a): If  $\Gamma$  is a flat connection, it satisfies the system of equations (4.4). This system represents the conditions of complete integrability for the system (4.3), which means that  $\Gamma$  is locally generated by the frame fields (see [6]).  $\square$

Note that the conditions of zero curvature (4.4) for a connection  $\Gamma$  to be locally generated by frame fields are equivalent to the compatibility conditions (3.3) (symmetry of  $\Gamma_{klj}^i$  in the indices  $k, l$ ) for the associated section  $\gamma$ .

The correspondence between sections, as described above, allows us to consider a variational problem for a frame field  $\gamma: X \rightarrow FX$  as a problem to find the corresponding linear connection  $\Gamma: X \rightarrow C^1X$ . For higher order variational problems on frame bundles we consider the commutative diagram

$$\begin{array}{ccc} J^r J^1 FX & \xrightarrow{J^r \mu^{(1)}} & J^r C^1 X \\ \downarrow & & \downarrow \\ J^1 FX & \xrightarrow{\mu^{(1)}} & C^1 X \end{array}$$

where  $J^r \mu^{(1)}$  denotes the  $r$ -jet prolongation of the quotient projection  $\mu^{(1)}$  (over  $X$ ), and the canonical inclusions  $C^{r+1}X \hookrightarrow J^{r+1}FX$ , and  $J^{r+1}FX \hookrightarrow J^r J^1 FX$ . The composed mapping

$$(4.5) \quad \nu: C^{r+1}X \rightarrow J^r C^1 X$$

has the expression

$$(4.6) \quad \begin{aligned} x^i \circ \nu &= x^i, \\ \Gamma_{kj}^i \circ \nu &= \Gamma_{kj}^i, \\ \Gamma_{kj,l}^i \circ \nu &= d_l \Gamma_{kj}^i = \Gamma_{klj}^i + \Gamma_{ks}^i \Gamma_{lj}^s, \end{aligned}$$

$$\begin{aligned} \Gamma_{kj,l_1l_2}^i \circ \nu &= d_{l_2} d_{l_1} \Gamma_{kj}^i \\ &= \Gamma_{kl_1l_2j}^i + \Gamma_{kl_1s}^i \Gamma_{l_2j}^s + \Gamma_{kl_2s}^i \Gamma_{l_1j}^s + \Gamma_{ks}^i (\Gamma_{l_1l_2j}^s + \Gamma_{l_1q}^s \Gamma_{l_2j}^q + \Gamma_{l_2q}^s \Gamma_{l_1j}^q), \\ &\quad \dots \\ \Gamma_{kj,l_1l_2\dots l_r}^i \circ \nu &= d_{l_r} \dots d_{l_2} d_{l_1} \Gamma_{kj}^i. \end{aligned}$$

To describe the image of  $\nu$  in  $J^r C^1 X$ , we denote by  $\tau_3^1 X$  the bundle of tensors of type  $(1, 3)$  over  $X$ , with induced coordinates  $S_{lkj}^i$ . We introduce the *formal curvature tensor* on  $J^r C^1 X$  (see [12]) as a morphism  $\varrho: J^r C^1 X \rightarrow \tau_3^1 X$  expressed by

$$S_{lkj}^i \circ \varrho = R_{lkj}^i = \Gamma_{kj,l}^i - \Gamma_{lj,k}^i + \Gamma_{ls}^i \Gamma_{kj}^s - \Gamma_{ks}^i \Gamma_{lj}^s$$

(compare with (4.2)).

Consider the sequence

$$0 \longrightarrow C^{r+1} X \xrightarrow{\nu} J^r C^1 X \xrightarrow{\varrho} \tau_3^1 X \longrightarrow 0$$

of bundles over  $X$ . We have the following result.

**Lemma 4.3.** *Let  $J_x^r \Gamma \in J^r C^1 X$ . Then  $\varrho(J_x^r \Gamma) = 0$  if and only if there exists an element  $\Delta \in C^{r+1} X$  such that  $\nu(\Delta) = J_x^r \Gamma$ .*

**Proof.**  $\Rightarrow$ : Let  $J_x^r \Gamma = (x^i, \Gamma_{kj}^i, \Gamma_{kj,l}^i, \Gamma_{kj,l_1l_2}^i, \dots, \Gamma_{kj,l_1l_2\dots l_r}^i)$  be the coordinates of  $J_x^r \Gamma \in J^r C^1 X$  with respect to a fibred coordinate system  $(W^r, \chi^r)$  on  $J^r C^1 X$ , associated with the coordinate system  $(W, \chi)$ ,  $\chi = (x^i, \Gamma_{kj}^i)$  on  $C^1 X$ , where  $\Gamma$  is a section of  $C^1 X$  defined in the neighborhood of  $x \in X$ . Let  $\varrho(J_x^r \Gamma) = 0$ , i.e., the coordinates of  $J_x^r \Gamma$  satisfy  $\Gamma_{kj,l}^i - \Gamma_{lj,k}^i + \Gamma_{ls}^i \Gamma_{kj}^s - \Gamma_{ks}^i \Gamma_{lj}^s = 0$ . It means that  $\Gamma_{klj}^i = (\Gamma_{kj,l}^i - \Gamma_{ks}^i \Gamma_{lj}^s) \circ \nu = (\Gamma_{lj,k}^i - \Gamma_{ls}^i \Gamma_{kj}^s) \circ \nu = \Gamma_{lkj}^i$ , and  $\Gamma_{klj}^i$  are symmetric in  $k, l$ . Moreover, by (3.2),  $\Gamma_{klmj}^i = d_m \Gamma_{klj}^i - \Gamma_{kls}^i \Gamma_{mj}^s$ , which gives us symmetry of  $\Gamma_{klmj}^i$  in  $k, l$ . According to Lemma 3.4,  $\Gamma_{klmj}^i - \Gamma_{kmlj}^i = -\Gamma_{kq}^i (\Gamma_{lmj}^q - \Gamma_{mlj}^q) = 0$ , which gives us symmetry of  $\Gamma_{klmj}^i$  in  $l, m$ . It means that  $\Gamma_{klmj}^i$  is symmetric in the indices  $k, l, m$  and these coordinates of  $\Delta$  are well defined. Continuing in this way, suppose that the coordinates  $\Gamma_{kl_1l_2\dots l_{s-1}j}^i$  of  $\Delta$  are defined. Again, by (3.2),  $\Gamma_{kl_1l_2\dots l_{s-1}l_sj}^i = d_{l_s} \Gamma_{kl_1l_2\dots l_{s-1}j}^i - \Gamma_{kl_1l_2\dots l_{s-1}q}^i \Gamma_{l_sj}^q$ , and symmetry of  $\Gamma_{kl_1l_2\dots l_{s-1}j}^i$  in the indices  $k, l_1, l_2, \dots, l_{s-1}$  ensures symmetry of  $\Gamma_{kl_1l_2\dots l_{s-1}l_sj}^i$  in  $k, l_1, l_2, \dots, l_{s-1}$ . According to Lemma 3.4,  $\Gamma_{kl_1l_2\dots l_{s-1}l_sj}^i - \Gamma_{kl_1l_2\dots l_{s-1}l_sj}^i = -\Gamma_{kl_1l_2\dots l_{s-1}q}^i (\Gamma_{l_{s-1}l_sj}^q - \Gamma_{l_sl_{s-1}j}^q) = 0$ , which gives us symmetry of  $\Gamma_{kl_1l_2\dots l_{s-1}l_sj}^i$  in  $l_{s-1}, l_s$ . It means that  $\Gamma_{kl_1l_2\dots l_{s-1}l_sj}^i$  is symmetric in the indices  $k, l_1, l_2, \dots, l_{s-1}, l_s$  and  $\Delta = (x^i, \Gamma_{kj}^i, \Gamma_{kj,l}^i, \Gamma_{kj,l_1l_2}^i, \dots, \Gamma_{kj,l_1l_2\dots l_r}^i)$  is well defined. By (4.6), we have  $\nu(\Delta) = J_x^r \Gamma$ .

$\Leftarrow$ : Let there exist  $\Delta = J_x^{r+1} \gamma \in C^{r+1} X$  such that  $\nu(J_x^{r+1} \gamma) = J_x^r \Gamma$ . Then for the adapted coordinates  $(x^i, \Gamma_{kj}^i, \Gamma_{kj,l}^i, \Gamma_{kj,l_1l_2}^i, \dots, \Gamma_{kj,l_1l_2\dots l_r}^i)$  of  $J_x^{r+1} \gamma$ ,  $\Gamma_{klj}^i = \Gamma_{lkj}^i$  holds, which is equivalent to the relation  $\Gamma_{kj,l}^i - \Gamma_{lj,k}^i + \Gamma_{ls}^i \Gamma_{kj}^s - \Gamma_{ks}^i \Gamma_{lj}^s = 0$  for the coordinates of  $J_x^r \Gamma$  and  $\varrho(J_x^r \Gamma) = 0$ .  $\square$

Let us denote by  $\mathcal{S}$  the submanifold of  $J^r C^1 X$  determined by  $\mathcal{S} = \{\Lambda \in J^r C^1 X : \varrho(\Lambda) = 0\}$ . Lemma 4.3 shows that for  $\Lambda \in \mathcal{S}$  the inverse

$$(4.7) \quad \sigma: \mathcal{S} \rightarrow C^{r+1} X$$

of  $\nu$  (4.5) is defined. In the corresponding coordinates,

$$(4.8) \quad \begin{aligned} x^i \circ \sigma &= x^i, \\ \Gamma_{kj}^i \circ \sigma &= \Gamma_{kj}^i, \\ \Gamma_{klj}^i \circ \sigma &= \Gamma_{kj,l}^i - \Gamma_{ks}^i \Gamma_{lj}^s, \\ \Gamma_{kl_1 l_2 j}^i \circ \sigma &= \tilde{d}_{l_2}(\Gamma_{kl_1 j}^i \circ \sigma) - (\Gamma_{kl_1 s}^i \circ \sigma) \Gamma_{l_2 j}^s \\ &= \Gamma_{kj, l_1 l_2}^i - \Gamma_{ks, l_1}^i \Gamma_{l_2 j}^s - \Gamma_{ks, l_2}^i \Gamma_{l_1 j}^s - \Gamma_{ks}^i \Gamma_{l_1 j, l_2}^s + \Gamma_{kq}^i \Gamma_{l_2 s}^q \Gamma_{pj}^s, \\ &\vdots \\ \Gamma_{kl_1 l_2 \dots l_r j}^i \circ \sigma &= \tilde{d}_{l_r}(\Gamma_{kl_1 l_2 \dots l_{r-1} j}^i \circ \sigma) - (\Gamma_{kl_1 l_2 \dots l_{r-1} s}^i \circ \sigma) \Gamma_{l_r j}^s. \end{aligned}$$

The operator  $\tilde{d}_m$  of formal derivative on  $J^r C^1 X$  corresponds to the operator of formal derivative  $d_m$  on  $C^{r+1} X$ , given by Lemma 3.2. More exactly,  $\tilde{d}_m(f \circ \sigma) = d_m f \circ \sigma$  for any function  $f$  on  $C^{r+1} X$ .

Using the mapping  $\sigma$  (4.7), it is possible to rewrite the Euler-Lagrange equations of the invariant variational problem, defined on  $C^{r+1} X$  by a Lagrangian  $\lambda$ , as associated equations on  $J^r C^1 X$  for connections with compatibility conditions as constraints.

**Theorem 4.1.** *Let  $\lambda = \mathcal{L}\omega_0$  be a  $\text{Gl}_n(\mathbb{R})$ -invariant Lagrangian on  $J^q FX$  and let  $E_\lambda = F_i^l(\mathcal{L})y_l^j dx_j^i \wedge \omega_0$  be its associated Euler-Lagrange form. The system of  $n^2$  equations of order  $r + 1$  for extremals of the given variational problem,*

$$(4.9) \quad F_i^l(\mathcal{L}) \circ J^{r+1} \gamma = 0,$$

is equivalent to the system of  $n^2$   $r$ th order equations

$$(4.10) \quad G_i^l(\mathcal{L}) \circ J^r \Gamma = 0,$$

where

$$G_i^l(\mathcal{L}) = F_i^l(\mathcal{L}) \circ \sigma,$$

and  $\sigma$  is defined by (4.8), together with the system of  $\binom{n}{2} \cdot n^2$  first order equations in the form

$$(4.11) \quad (\Gamma_{kj,l}^i - \Gamma_{lj,k}^i + \Gamma_{ls}^i \Gamma_{kj}^s - \Gamma_{ks}^i \Gamma_{lj}^s) \circ J^1 \Gamma = 0.$$

Finding a solution  $\Gamma: X \rightarrow C^1 X$  of (4.10), the corresponding solution  $\gamma: X \rightarrow FX$  of (4.9) is then obtained by solving the system (4.3).

## 5. EXAMPLE

Let  $\lambda$  be a  $\text{Gl}_n(\mathbb{R})$ -invariant Lagrangian on  $J^1FX$ . The coordinate expression of  $\lambda$ , in the coordinates  $(x^i, x_j^i, \Gamma_{jk}^i)$  adapted to the action of  $\text{Gl}_n(\mathbb{R})$  on  $J^1FX$ , is given by

$$\lambda = \mathcal{L}\omega_0, \quad \mathcal{L} = \mathcal{L}(x^i, \Gamma_{jk}^i).$$

The Euler-Lagrange form  $E_\lambda$  of  $\lambda$  is expressed as

$$E_\lambda = F_i^l(\mathcal{L})y_l^j dx_j^i \wedge \omega_0,$$

where

$$F_i^l(\mathcal{L}) = -\Gamma_{qi}^p \frac{\partial \mathcal{L}}{\partial \Gamma_{ql}^p} + \Gamma_{pq}^l \frac{\partial \mathcal{L}}{\partial \Gamma_{pq}^i} + \frac{\partial^2 \mathcal{L}}{\partial x^p \partial \Gamma_{pl}^i} + (\Gamma_{mpq}^k + \Gamma_{ms}^k \Gamma_{pq}^s) \frac{\partial^2 \mathcal{L}}{\partial \Gamma_{mq}^k \partial \Gamma_{pl}^i}.$$

A section  $\gamma: X \rightarrow FX$  is an extremal of the first order variational problem defined by  $\lambda$  if and only if the system of second order partial differential equations

$$F_i^l(\mathcal{L}) \circ J^2\gamma = 0$$

is satisfied.

According to Theorem 4.1, the problem of finding the extremal  $\gamma$  can be also represented as to find the corresponding section  $\Gamma: X \rightarrow C^1X$  which is the solution of the system

$$G_i^l(\mathcal{L}) \circ J^1\Gamma = 0$$

of first order equations, where

$$G_i^l(\mathcal{L}) = -\Gamma_{qi}^p \frac{\partial \mathcal{L}}{\partial \Gamma_{ql}^p} + \Gamma_{pq}^l \frac{\partial \mathcal{L}}{\partial \Gamma_{pq}^i} + \frac{\partial^2 \mathcal{L}}{\partial x^p \partial \Gamma_{pl}^i} + \Gamma_{mq,p}^k \frac{\partial^2 \mathcal{L}}{\partial \Gamma_{mq}^k \partial \Gamma_{pl}^i},$$

satisfying the additional conditions (4.11) for  $\Gamma$ .

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