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Bar-invariant bases of the quantum cluster algebra of type $A_2^{(2)}$
BAR-IN Variant Bases of the Quantum Cluster Algebra of Type $A^{(2)}$

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(Received October 3, 2010)

Cordially dedicated to Prof. Vlastimil Dlab on the occasion of his 80th birthday

Abstract. We construct bar-invariant $\mathbb{Z}[q^{\pm 1/2}]$-bases of the quantum cluster algebra of the valued quiver $A_2^{(2)}$, one of which coincides with the quantum analogue of the basis of the corresponding cluster algebra discussed in P. Sherman, A. Zelevinsky: Positivity and canonical bases in rank 2 cluster algebras of finite and affine types, Moscow Math. J., 4, 2004, 947–974.

Keywords: quantum cluster algebra, $\mathbb{Z}[q^{\pm 1/2}]$-basis, valued quiver

MSC 2010: 16G20, 20G42, 14M17

1. Introduction

Cluster algebras were invented by S. Fomin and A. Zelevinsky [11], [12] in order to study total positivity in algebraic groups and canonical bases in quantum groups. The study of $\mathbb{Z}$-bases of cluster algebras has become important. There are many results involving the construction of $\mathbb{Z}$-bases of cluster algebras (for example, see [17] and [4] for cluster algebras of rank 2, [3] for finite type, [10] for type $\tilde{A}$, [5] for $\tilde{A}_2^{(1)}$, [6] for affine type and [13] for acyclic quivers). As the quantum analogue of cluster algebras, quantum cluster algebras were defined by A. Berenstein and A. Zelevinsky in [1]. A quantum cluster algebra is generated by the so-called (quantum) cluster variables inside an ambient skew-field $\mathcal{F}$. Under the specialization $q = 1$, quantum cluster algebras degenerate to cluster algebras.

Recently, D. Rupel [16] defined a quantum analogue of the Caldero-Chapoton formula [2] and conjectured that cluster variables could be expressed in terms of
the refined Caldero-Chapoton formula, and then proved the conjecture for those in almost acyclic clusters. This conjecture has been proved for acyclic equally valued quivers in [15]. Naturally, one may hope to construct \( \mathbb{Z}[q^{\pm 1/2}] \)-bases of quantum cluster algebras. For simply-laced finite and affine quivers, the bases have been constructed in [7] and [8].

In this paper, we deal with the quantum cluster algebra of the simplest non-simply-laced valued quiver \( A_2^{(2)} \) and construct various bar-invariant \( \mathbb{Z}[q^{\pm 1/2}] \)-bases by applying the quantum analogue of the Caldero-Chapoton formula defined in [16]. Under the specialization \( q = 1 \), one of these \( \mathbb{Z}[q^{\pm 1/2}] \)-bases is exactly the canonical basis of the cluster algebra of the valued quiver \( A_2^{(2)} \) discussed in [17]. Moreover, the elements \( \{s_n : n \in \mathbb{N}\} \) in the basis \( \mathcal{S} \) (see Definition 3.4) possess representation-theoretic interpretations.

2. Preliminaries

2.1. Quantum cluster algebras

In what follows, we will give a short review on quantum cluster algebras, for details one can refer to [1]. Let \( L \) be a lattice of rank \( m \) and \( \Lambda: L \times L \to \mathbb{Z} \) a skew-symmetric bilinear form. Let \( q \) be a formal variable and let us consider the ring of integer Laurent polynomials \( \mathbb{Z}[q^{\pm 1/2}] \). The based quantum torus associated with a pair \((L, \Lambda)\) is a \( \mathbb{Z}[q^{\pm 1/2}] \)-algebra \( \mathcal{T} \) with a distinguished \( \mathbb{Z}[q^{\pm 1/2}] \)-basis \( \{X^e : e \in L\} \) and the multiplication given by

\[
X^e X^f = q^{\frac{1}{2} \Lambda(e,f)} X^{e+f}.
\]

Obviously \( \mathcal{T} \) is associative and the basis elements satisfy the relations

\[
X^e X^f = q^{\Lambda(e,f)} X^f X^e, \quad X^0 = 1, \quad (X^e)^{-1} = X^{-e}.
\]

It is well known that \( \mathcal{T} \) is an Ore domain, i.e., it is contained in its skew-field of fractions \( \mathcal{F} \).

A toric frame in \( \mathcal{F} \) is a mapping \( M: \mathbb{Z}^m \to \mathcal{F} \setminus \{0\} \) of the form

\[
M(c) = \varphi(X^{\eta(c)}) =: X^c
\]

where \( c \in \mathbb{Z}^m \), \( \varphi \) is an automorphism of \( \mathcal{F} \) and \( \eta: \mathbb{Z}^m \to L \) is an isomorphism of lattices. By the definition, the elements \( M(c) \) form a \( \mathbb{Z}[q^{\pm 1/2}] \)-basis of the based
quantum torus $\mathcal{T}_M := \varphi(\mathcal{T})$ and satisfy the relations

\[
M(c)M(d) = q^{\Lambda_M(c,d)}M(c + d),
\]
\[
M(c)M(d) = q^{\Lambda_M(c,d)}M(d)M(c),
\]
\[
M(0) = 1,
\]
\[
M(c)^{-1} = M(-c),
\]

where the skew-symmetric bilinear form $\Lambda_M$ on $\mathbb{Z}^m$ is obtained by transferring the form $\Lambda$ from $L$ via the lattice isomorphism $\eta$. Note that $\Lambda_M$ can also be identified with a skew-symmetric $m \times m$ matrix given by $\lambda_{ij} = \Lambda_M(e_i, e_j)$ where \{0, $\ldots$, $m$\} is the standard basis of $\mathbb{Z}^m$.

Given a toric frame $M$, write $X_i = M(e_i)$; then

\[
\mathcal{T}_M = \mathbb{Z}[q^{\pm 1/2}](X_1^\pm 1, \ldots, X_m^\pm 1): X_iX_j = q^{\lambda_{ij}}X_jX_i).
\]

Let $A$ be an $m \times m$ skew-symmetric matrix and $\tilde{B}$ an $m \times n$ matrix with $n \leq m$. The pair $(A, \tilde{B})$ is called compatible if $\tilde{B}^rA = (D \mid 0)$ is an $n \times m$ matrix with $D = \text{diag}(d_1, \ldots, d_n)$ where $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. For a toric frame $M$, we call the pair $(M, \tilde{B})$ a quantum seed if the pair $(\Lambda_M, \tilde{B})$ is compatible. Define the $m \times m$ matrix $E = (e_{ij})_{m \times m}$ as follows:

\[
e_{ij} = \begin{cases} 
\delta_{ij} & \text{if } j \neq k; \\
-1 & \text{if } i = j = k; \\
\max(0, -b_{ik}) & \text{if } i \neq j = k.
\end{cases}
\]

For $n, k \in \mathbb{Z}$, $k \geq 0$, denote $[n]_q = (q^n - q^{-n}) \ldots (q^{n-r+1} - q^{-n+r-1})/(q^r - q^{-r}) \ldots (q - q^{-1})$. Let $c = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ with $c_k \geq 0$. We can define the toric frame $M': \mathbb{Z}^m \to \mathcal{T} \setminus \{0\}$ as

\[
(2.1) \quad M'(c) = \sum_{p=0}^{c_k} \left[ \begin{array}{c} c_k \\ p \end{array} \right] q^{p_k/2} M(Ec + pb_k), \quad M'(-c) = M'(c)^{-1}
\]

where the vector $b_k \in \mathbb{Z}^m$ is the $k$-th column of $\tilde{B}$.

Let $\tilde{B}' = \mu_k(\tilde{B})$ be the mutation of $\tilde{B}$ at $k$ (see [11] for details). Then the quantum seed $(M', \tilde{B}')$ is called the mutation of $(M, \tilde{B})$ in the direction $k$. Two quantum seeds are mutation-equivalent if each can be obtained from the other by a sequence of mutations. Let $\mathcal{C} = \{M'(e_i): 1 \leq i \leq n, \ (M', \tilde{B}')$ is mutation-equivalent to $(M, \tilde{B})\}$. The elements of $\mathcal{C}$ are called cluster variables. Let $\mathcal{P} = \{M(e_i): n + 1 \leq i \leq m\}$; the elements in $\mathcal{P}$ are called coefficients. The quantum cluster algebra $\mathcal{A}_q(\Lambda_M, \tilde{B})$ is
the \( \mathbb{Z}[q^{\pm 1/2}] \)-subalgebra of \( \mathcal{F} \) generated by the elements in \( \mathcal{C} \cup \mathcal{P} \). We can associate with \((M, \tilde{B})\) the \( \mathbb{Z} \)-linear bar-involution on \( \mathcal{T}_M \) as follows:

\[
q^r/2 M(c) = q^{r/2} M(c), \quad \text{where } r \in \mathbb{Z}, \ c \in \mathbb{Z}^n.
\]

Then we can see that \( XY = YX \) for all \( X, Y \in \mathcal{A}_q(\Lambda M, \tilde{B}) \) and the elements in \( \mathcal{C} \cup \mathcal{P} \) are bar-invariant.

### 2.2. The valued quiver \( A^{(2)}_2 \)

We can associate a valued quiver (see [16, Section 2] for more details) with a given compatible pair \((A, B)\). Now we set \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \). Thus we have \( B^{tr} A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \) denoted by \( D \). The valued quiver \( Q \) associated with this pair is of type \( A^{(2)}_2 \):

\[
1 \rightarrow (4,1) \rightarrow 2.
\]

Let \( \mathcal{S} \) be a reduced \( \mathbb{F}_q \)-species of type \( Q \), see [9] for details. The category \( \text{rep}(\mathcal{S}) \) of finite dimensional representations of \( \mathcal{S} \) over \( \mathbb{F}_q \) is equivalent to the category of finite dimensional modules over a finite-dimensional hereditary \( \mathbb{F}_q \)-algebra \( \Delta \), where \( \Delta \) is the tensor algebra of \( \mathcal{S} \). In the rest of the paper, we will not distinguish the representation of the valued quiver and the module of the corresponding algebra.

It is well known (see [9]) that indecomposable \( \Delta \)-modules are divided into three families up to isomorphism: the indecomposable regular modules with dimension vector \((nd_p, 2nd_p)\) for \( p \in \mathbb{P}^1_k \) of degree \( d_p \) and \( n \in \mathbb{N} \) (in particular, denote by \( R_p(n) \) the indecomposable regular module with dimension vector \((n, 2n)\) for \( d_p = 1 \)), the preprojective modules, and the preinjective modules. Define

\[
R = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}, \quad R' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

It is well known that the Euler form on \( \text{rep}(\mathcal{S}) \) is given by

\[
\langle V, N \rangle = m(I - R)D_{\mathbb{F}_q}^{tr}
\]

where \( m \) and \( n \) are the dimension vectors of \( V \) and \( N \), respectively. Now, let \( \mathcal{T} = \mathbb{Z}[q^{\pm 1/2}][X_1^{\pm 1}, X_2^{\pm 1}] \) and \( \mathcal{T} \) be the skew field of fractions of \( \mathcal{T} \). Thus the quantum cluster algebra of the valued quiver \( A^{(2)}_2 \) denoted by \( \mathcal{A}_q(1, 4) \) in the sequel is the \( \mathbb{Z}[q^{\pm 1/2}] \)-subalgebra of \( \mathcal{F} \) generated by the cluster variables \( X_k \), \( k \in \mathbb{Z} \), defined recursively by

\[
X_{m-1}X_{m+1} = \begin{cases} q^{1/2}X_m + 1 & \text{if } m \text{ is odd;} \\ q^2X_m^4 + 1 & \text{if } m \text{ is even.} \end{cases}
\]
The quantum Laurent phenomenon [1] implies that each $X_k$ belongs to the subring $\mathcal{T}$ of $\mathcal{F}$. Let $V$ be a representation of the valued quiver $A_2^{(2)}$ with dimension vector $\dim V = (v_1, v_2)$. For $e = (e_1, e_2) \in \mathbb{Z}_{\geq 0}^2$, denote by $\text{Gr}_e(V)$ the set of all subrepresentations $U$ of $V$ with $\dim U = e$. In [16], the author defined the element $X_V$ of the quantum torus $\mathcal{T}$ by

$$X_V = \sum_e q^{-\frac{1}{2}d_e^V} \text{Gr}_e(V)|X^{(-v_1+v_2-e_2, 4e_1-v_2)}$$

where $d_e^V = 4e_1(v_1 - e_1) - (4e_1 - e_2)(v_2 - e_2)$. This formula is called the quantum analogue of the Caldero-Chapoton formula [2].

Let $C = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ be the Cartan matrix and $\Phi$ the associated root system with simple roots $\{\alpha_1, \alpha_2\}$. Then all negative real roots of $\Phi$ can be labeled by $m \in \mathbb{Z} \setminus \{1, 2\}$ as follows:

$$\alpha_{m-1} + \alpha_{m+1} = \begin{cases} \alpha_m & \text{if } m \text{ is odd}, \\ 4\alpha_m & \text{if } m \text{ is even}, \end{cases}$$

where we set $\alpha_0 = -\alpha_2$, $\alpha_3 = -\alpha_1$.

Recall the following result from [16]:

**Theorem 2.1 ([16]).** For any $m \in \mathbb{Z} \setminus \{1, 2\}$, let $V(m)$ be the unique indecomposable valued representation of $A_2^{(2)}$ with dimension vector $-\alpha_m$. Then the $m$-th cluster variable $X_m$ of $\mathcal{A}_q(1, 4)$ is equal to $X_{V(m)}$.

3. BASES OF THE QUANTUM CLUSTER ALGEBRA $\mathcal{A}_q(1, 4)$

In this section, we will construct various bar-invariant $\mathbb{Z}[q^{\pm 1/2}]$-bases of the quantum cluster algebra $\mathcal{A}_q(1, 4)$. Under the specialization $q = 1$, these bases are just the $\mathbb{Z}$-bases of the cluster algebra of the valued quiver $A_2^{(2)}$.

**Definition 3.1.** For any $(r_1, r_2)$ and $(s_1, s_2) \in \mathbb{Z}^2$, we write $(r_1, r_2) \preceq (s_1, s_2)$ if $r_i \leq s_i$ for $1 \leq i \leq 2$. Moreover, if there exists $i$ such that $r_i < s_i$, then we write $(r_1, r_2) \prec (s_1, s_2)$.

**Lemma 3.2.** The Laurent expansion in $X_{V(m)}$ has a minimal non-zero term $X^{\alpha_m}$.

**Proof.** It is obvious that the module $V(m)$ with dimension vector $(v_1, v_2)$ has a submodule with dimension vector $(0, v_2)$. Thus by the definition of the $q$-deformation of the Caldero-Chapoton formula and the partial order in Definition 3.1, we obtain that the expansion in $X_{V(m)}$ has a minimal non-zero term $X^{\alpha_m}$. \qed
Lemma 3.3. Let $R_p(1)$ be an indecomposable regular module of degree 1. Then

$$X_{R_p(1)} = X^{(-1, -2)} + X^{(-1, 2)} + X^{(1, -2)} + (q^{1/2} + q^{-1/2})X^{(0, -2)}.$$ 

Proof. Note that $R_p(1)$ contains four submodules with dimension vectors $(0, 0)$, $(0, 1)$, $(0, 2)$ and $(1, 2)$. Therefore the lemma immediately follows from the $q$-deformation of the Caldero-Chapoton formula. □

By Lemma 3.3, the expression of $X_{R_p(1)}$ is independent of the choice of $p \in \mathbb{P}^1_k$ of degree 1. So we set

$$X_\delta := X_{R_p(1)}.$$

Definition 3.4 (Chebyshev polynomials).

1. The $n$-th Chebyshev polynomial of the first kind is the polynomial $F_n(x) \in \mathbb{Z}[x]$ defined recursively by

$$\begin{cases} F_0(x) = 1, & F_1(x) = x, & F_2(x) = x^2 - 2, \\ F_{n+1}(x) = F_n(x)F_1(x) - F_{n-1}(x) \quad \text{for } n \geq 2. \end{cases}$$

2. The $n$-th Chebyshev polynomial of the second kind is the polynomial $S_n(x) \in \mathbb{Z}[x]$ defined recursively by

$$\begin{cases} S_0(x) = 1, & S_1(x) = x, & S_2(x) = x^2 - 1, \\ S_{n+1}(x) = S_n(x)S_1(x) - S_{n-1}(x) \quad \text{for } n \geq 2. \end{cases}$$

It is obvious that $F_n(x) = S_n(x) - S_{n-2}(x)$. We denote $z = X_\delta$, $z_n = F_n(z)$, $s_n = S_n(z)$ for $n \geq 0$ and $z_n = s_n = 0$ for $n < 0$. Set

$$\mathcal{B}' = \{X_m^aX_{m+1}^b: m \in \mathbb{Z}, (a, b) \in \mathbb{Z}_{\geq 0}^2\} \cup \{z_n: n \in \mathbb{N}\},$$

$$\mathcal{G}' = \{X_m^aX_{m+1}^b: m \in \mathbb{Z}, (a, b) \in \mathbb{Z}_{\geq 0}^2\} \cup \{s_n: n \in \mathbb{N}\},$$

$$\mathcal{B}' = \{X_m^aX_{m+1}^b: m \in \mathbb{Z}, (a, b) \in \mathbb{Z}_{\geq 0}^2\} \cup \{z^n: n \in \mathbb{N}\}.$$ 

Remark 3.5. It is easy to check that $X^{(r, 2r)}X^{(s, 2s)} = X^{(r+s, 2r+2s)}$ for any $r, s \in \mathbb{Z}$, and thus the expansions of $z_n$, $s_n$ and $z^n$ have a minimal non-zero term $X^{-(n, 2n)}$ according to the partial order in Definition 3.1.

We have the following immediate result.

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Lemma 3.6. $X_{\delta} = qX_0^2X_3 - q^2(qX_1 + q^{-1/2} + q^{1/2})X_2^2$.

Proof. By $X_0X_2 = q^{1/2}X_1 + 1$, we have $X_0 = X^{(1, -1)} + X^{(0, -1)}$. By $X_1X_3 = q^2X_4 + 1$, we have $X_3 = X^{(-1, 4)} + X^{(-1, 0)}$. Then we can prove the lemma by direct computation. \qed

The following lemma is straightforward but important.

Lemma 3.7. $\overline{X_{\delta}} = X_{\delta}$.

Proof. Note that $X_{\delta} = q^{-1}X_0^2X_3 - q^{-2}(qX_1 + q^{-1/2} + q^{1/2})X_2^2 = q^{-1}X_3X_0^2 - q^{-2}X_2^2(q^{-1}X_1 + q^{-1/2} + q^{1/2}) = \overline{X_{\delta}}$. \qed

Remark 3.8. By Lemma 3.7, we can verify that $\overline{z_n} = z_n$, $\overline{s_n} = s_n$.

For any $\underline{d} \in \mathbb{Z}^2$, define $\underline{d}^+ = (d_1^+, d_2^+)$ such that $d_i^+ = d_i$ if $d_i > 0$ and $d_i^+ = 0$ if $d_i \leq 0$ for any $1 \leq i \leq 2$. Dually, we set $\underline{d}^- = \underline{d}^+ - \underline{d}$.

The proposition below is a special case of [1, Theorem 7.3].

Proposition 3.9 ([1]). Let $Q$ be the valued quiver $A_2^{(2)}$. Then the set

$$\{X_{d_1^+}X_{d_2^+}X_{S_1}X_{S_2} : (d_1, d_2) \in \mathbb{Z}^2\}$$

is a $\mathbb{Z}[q^{\pm 1/2}]$-basis of $\mathcal{A}_q(1, 4)$.

Proof. It is easy to check that the sets $\{X_1, X_{S_2}\}$ and $\{X_2, X_{S_1}\}$ are clusters obtained by the mutation in the direction 2 and 1, respectively, from the cluster $\{X_1, X_2\}$. Therefore the proposition immediately follows from [1, Theorem 7.3]. \qed

The following result is an immediate consequence of the above proposition.

Corollary 3.10. The sets $\mathcal{B}'$, $\mathcal{I}'$ and $\mathcal{G}'$ are $\mathbb{Z}[q^{\pm 1/2}]$-bases of the quantum cluster algebra $\mathcal{A}_q(1, 4)$.

Proof. Note that if $\mathcal{B}'$ is a $\mathbb{Z}[q^{\pm 1/2}]$-basis of the quantum cluster algebra $\mathcal{A}_q(1, 4)$, then $\mathcal{I}'$ and $\mathcal{G}'$ are naturally $\mathbb{Z}[q^{\pm 1/2}]$-bases of $\mathcal{A}_q(1, 4)$ because there exist unipotent transformations between $\{z_n : n \in \mathbb{N}\}$, $\{s_n : n \in \mathbb{N}\}$ and $\{z^n : n \in \mathbb{N}\}$. In what follows, we will only focus on the set $\mathcal{B}'$. 1083
By Lemma 3.6, we obtain that $X_{\delta}$ is in $\mathcal{A}_q(1,4)$. Thus $\{z_n: n \in \mathbb{N}\}$ is contained in $\mathcal{A}_q(1,4)$. Note that for any $v = (v_1, v_2) \in \mathbb{Z}^2$, there exists only one object $X_V$ in $\mathcal{B}'$ such that $\dim V = (v_1, v_2) \in \mathbb{Z}^2$. Then by Proposition 3.9 we have

$$X_V = b_v X_1^{1^+} X_2^{1^+} X_{S_1}^{-1^+} X_{S_2}^{-1^+} + \sum_{\omega \neq 1} b_\omega X_1^{1^+} X_2^{1^+} X_{S_1}^{-1^+} X_{S_2}^{-1^+}$$

where $b_v, b_\omega \in \mathbb{Z}[q^{\pm 1/2}]$. Then by Lemma 3.2, Remark 3.5, we know that $b_m$ must be a nonzero monomial in $q^{\pm 1/2}$. Thus we obtain that $\mathcal{B}'$ is a $\mathbb{Z}[q^{\pm 1/2}]$-basis of $\mathcal{A}_q(1,4)$.

Then we can obtain the following main result of the paper.

**Theorem 3.11.** The sets $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{G}$ are bar-invariant $\mathbb{Z}[q^{\pm 1/2}]$-bases of the quantum cluster algebra $\mathcal{A}_q(1,4)$.

**Proof.** By Lemma 3.7 and Remark 3.8 and the fact that every element in the set $\{q^{-\frac{1}{2}} ab X_m X_{m+1}^b: m \in \mathbb{Z}, (a, b) \in \mathbb{Z}^2_{\geq 0}\}$ is bar-invariant, the theorem follows immediately.

□

4. Some multiplication formulas

In this section, we prove some multiplication formulas and then give representation-theoretic interpretations of the elements $\{s_n: n \in \mathbb{N}\}$ in the basis $\mathcal{S}$.

First, we define a ring homomorphism of the quantum cluster algebra $\mathcal{A}_q(1,4)$:

$$\sigma_2: \mathcal{A}_q(1,4) \rightarrow \mathcal{A}_q(1,4)$$

by sending $X_m$ to $X_{m+2}$ and $q^{\pm 1/2}$ to $q^{\pm 1/2}$. It is obviously an automorphism which preserves the defining relations.

We have the following result.

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**Lemma 4.1.** \( \sigma_2(X_\delta) = X_\delta. \)

**Proof.** By direct computation, we have

\[
X_3 = X^{(-1,4)} + X^{(-1,0)}, \\
X_4 = X^{(-1,3)} + X^{(0,-1)} + (q + q^{-1})X^{(-1,-1)}.
\]

Thus we obtain the identity

\[ qX_4^2 + q^{-1}X_2^2 = X_3X_\delta. \]

By Lemma 3.6, we have:

\[ X_\delta = qX_3^2X_3 - q^2(qX_1 + q^{-1/2} + q^{1/2})X_2^2. \]

Therefore, we have

\[
\sigma_2(X_\delta) = \sigma_2(qX_3^2X_3 - q^2(qX_1 + q^{-1/2} + q^{1/2})X_2^2)
\]

\[ = qX_3^2X_5 - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \]

\[ = q^3X_3^{-1}X_4^2 + qX_3^{-1}X_3^{-1} - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \]

\[ = qX_3^{-1}X_2X_3X_4^3 + q^{-1}X_3^{-1}X_2^2 - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \]

\[ = X_4^3 + q^{-1}X_3^{-1}X_2^2 - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \]

\[ = X_\delta. \]

\[ \square \]

**Proposition 4.2.** We have

1. for \( m > n \geq 1, \)

\[ z_nz_m = z_{m+n} + z_{m-n}, \]

\[ z_nz_n = z_{2n} + 2, \]

2. for any \( n \in \mathbb{Z}, \)

\[ X_{2n}X_\delta = q^{-1/2}X_{2n-2} + q^{1/2}X_{2n+2}, \]

\[ X_{2n+1}X_\delta = q^{-1}X_{2n}^2 + qX_{2n+2}^2. \]

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Proof. (1) It follows from the definition of Chebyshev polynomials.
(2) By Lemma 4.1, we only need to prove the equations

\[ X_2X_\delta = q^{-1/2}X_0 + q^{1/2}X_4, \]
\[ X_1X_\delta = q^{-1}X_0^2 + qX_2^2. \]

By the defining relations, we have

\[ X_0 = X^{(1,-1)} + X^{(0,-1)}, \quad X_4 = X^{(-1,3)} + X^{(0,-1)} + X^{(-1,-1)}. \]

Then we can prove the above equations by Lemma 3.3 and direct computation. □

Note that for any \( \Delta \)-module \( V \), the quantum analogue of the Caldero-Chapton
map of the valued quiver \( Q = A^{(2)}_2 \) defined in [16] can be rewritten as

\[ X_V = \sum_\mathcal{L} |\text{Gr}_\mathcal{L} V|q^{-1/2}(e, e) X_{-e} B^{\text{tr}} - (I - R'). \]

**Lemma 4.3.** For any dimension vector \( m, e, f \in \mathbb{Z}^n_{\geq 0} \), we have

1. \( \Lambda(m(I - R'), e B^{\text{tr}}) = -\langle e, m \rangle \);
2. \( \Lambda(e B^{\text{tr}}, f B^{\text{tr}}) = \langle f, e \rangle - \langle e, f \rangle \).

**Proof.** It is easy to check that

\[ \Lambda(m(I - R'), e B^{\text{tr}}) = m(I - R')\Lambda B^{\text{tr}} = -m(I - R')D^{\text{tr}} e^{\text{tr}} \]
\[ = -e D(I - R')^{\text{tr}}m^{\text{tr}} = -e(I - R)D^{\text{tr}} e^{\text{tr}} = -\langle e, m \rangle \]

and

\[ \Lambda(e B^{\text{tr}}, f B^{\text{tr}}) = e B^{\text{tr}} \Lambda f^{\text{tr}} = -e B^{\text{tr}} Df^{\text{tr}} = e(R - R')Df^{\text{tr}} \]
\[ = e((I - R') - (I - R))Df^{\text{tr}} = e(I - R')Df^{\text{tr}} - e(I - R)Df^{\text{tr}} \]
\[ = \langle f, e \rangle - \langle e, f \rangle. \]

□

**Corollary 4.4.** For any dimension vector \( m, l, e, f \in \mathbb{Z}^n_{\geq 0} \), we have

\[ \Lambda(m(I - R') + e B^{\text{tr}}, l(I - R') + f B^{\text{tr}}) \]
\[ = \Lambda(m(I - R'), l(I - R')) + \langle f, e \rangle - \langle e, f \rangle + \langle e, l \rangle - \langle f, m \rangle. \]
For $\Delta$-modules $V$, $T$ and $N$, we denote by $F_{T,N}^{V}$ the number of submodules $U$ of $V$ such that $U$ is isomorphic to $N$ and $V/U$ is isomorphic to $T$. The following proposition gives representation-theoretic interpretations of elements $s_n$, $n \in \mathbb{N}$ in the basis $\mathcal{S}$. By abuse of language, we still denote by $V$ the dimension vector of the $\Delta$-module $V$ in the bilinear forms involved.

**Proposition 4.5.** For any $n \in \mathbb{N}$, we have

$$X_{R_p(n)}X_{R_p(1)} = X_{R_p(n+1)} + X_{R_p(n-1)}.$$ 

**Proof.** We have the exact sequences

$$0 \rightarrow R_p(1) \rightarrow R_p(n+1) \rightarrow R_p(n) \rightarrow 0$$

and

$$0 \rightarrow R_p(1) \epsilon \tau R_p(n) = R_p(n) \rightarrow R_p(n-1) \rightarrow 0.$$

The term on the left-hand side is

$$X_{R_p(n)}X_{R_p(1)} = \sum_{d} |Gr_d R_p(n)|q^{-1/2/(d;n\delta-n\delta(I-R'))}X^{-dB^{tr}-(n\delta(I-R'))}$$

$$\times \sum_{b} |Gr_b R_p(1)|q^{-1/2/(b;\delta-n\delta(I-R'))}X^{-bB^{tr}-(\delta(I-R'))}$$

$$= \sum_{b,d} |Gr_d R_p(n)||Gr_b R_p(1)|q^{-1/2/(d;n\delta-n\delta(I-R'))}X^{-dB^{tr}-(n\delta(I-R'))}X^{-bB^{tr}-(n+1)\delta(I-R')}.$$ 

Then by Corollary 4.4, the above equation is equal to

$$\sum_{b,d} |Gr_d R_p(n)||Gr_b R_p(1)|q^{-1/2/(d;n\delta-n\delta(I-R'))}X^{-dB^{tr}-(n+1)\delta(I-R')}.$$ 

The first term on the right-hand side is

$$\tau_1 := X_{R_p(n+1)} = \sum_{H} F_{G,H}^{R_p(n+1)} q^{-1/2/(b;\delta-n\delta(I-R'))}X^{-bB^{tr}-(n+1)\delta(I-R')}.$$ 

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According to [14, Lemma 16], we have

$$\tau_1 = \sum_{N,Q} q^{(Q,T)} q - q^{\dim_k \Ext^1(Q,T)} F_{FQ}^{R_p(n)} F_{TN}^{R_p(1)} q - 1/2(N+Q,(n+1)\delta - N-Q) X^{-(\delta + d)B^{tr} - (n+1)\delta(I-R')}.$$ 

Now we consider the term

$$\tau_2 := \sum_{Y} q^{(n\delta - W,\delta)} F_{WY}^{R_p(n-1)} q^{-1/2(Y+\delta,n\delta - Y')} X^{-(\delta + d)B^{tr} - (n-1)\delta(I-R')}.$$ 

Any submodule $Y$ of $R_p(n-1)$ induces the submodule $Q = p^{-1}(Y)$ and $N = 0$ of $R_p(n)$ and $R_p(1)$ respectively as the following commutative diagram shows:

$$
\begin{array}{c}
0 \rightarrow R_p(1) \rightarrow p^{-1}(Y) \rightarrow Y \rightarrow 0 \\
\downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
R_p(n) \rightarrow R_p(n-1) \rightarrow 0 \rightarrow R_p(1) \rightarrow 0
\end{array}
$$

Thus $y = b + d - \delta$ and

$$-yB^{tr} - (n-1)\delta(I-R')$$

$$= -(b + d - \delta)B^{tr} - (n-1)\delta(I-R')$$

$$= -(b + d)B^{tr} + \delta B^{tr} - (n-1)\delta(I-R')$$

$$= -(b + d)B^{tr} + \delta(R' - R) - (n-1)\delta(I-R')$$

$$= -(b + d)B^{tr} - \delta(I-R') + \delta(I-R) - (n-1)\delta(I-R')$$

$$= -(b + d)B^{tr} - 2\delta(I-R') - (n-1)\delta(I-R')$$

$$= -(b + d)B^{tr} - (n+1)\delta(I-R').$$

Then by [14, Lemma 16], we have

$$\tau_2 = \sum_{N,Q} q^{(Q,T)} q^{\dim_k \Ext^1(Q,T)} F_{FQ}^{R_p(n)} F_{TN}^{R_p(1)} q^{-1/2(N+Q,(n+1)\delta - N-Q)} X^{-(\delta + d)B^{tr} - (n+1)\delta(I-R')}.$$ 

Therefore

$$\tau_1 + \tau_2 = \sum_{N,Q} F_{FQ}^{R_p(n)} F_{TN}^{R_p(1)} q^{-1/2(N+Q,(n+1)\delta - N-Q)} X^{-(\delta + d)B^{tr} - (n+1)\delta(I-R')} X_{R_p(n)} X_{R_p(1)}.$$ 

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Note that the second term on the right-hand side of the desired equation is
\[ \tau_3 := X_{R_p(n-1)} = \sum_Y F^{R_p(n-1)}_{WY} q^{-1/2\langle Y,(n-1)\delta - Y \rangle} X_{-y^{R'}+(n+1)\delta(I-R')} . \]

So it remains to prove \( \tau_2 = \tau_3 \), i.e., the equation
\[ \langle n\delta - W, \delta \rangle - \frac{1}{2} \langle Y + \delta, n\delta - Y \rangle = -\frac{1}{2} \langle Y, (n-1)\delta - Y \rangle. \]

Note that
\[
\begin{align*}
\langle n\delta - W, \delta \rangle & - \frac{1}{2} \langle Y + \delta, n\delta - Y \rangle \\
& = \langle \delta + Y, \delta \rangle - \frac{1}{2} \langle Y, n\delta - Y \rangle - \frac{1}{2} \langle \delta, n\delta - Y \rangle \\
& = \langle Y, \delta \rangle - \frac{1}{2} \langle Y, (n-1)\delta - Y \rangle - \frac{1}{2} \langle Y, \delta \rangle + \frac{1}{2} \langle \delta, Y \rangle \\
& = -\frac{1}{2} \langle Y, (n-1)\delta - Y \rangle.
\end{align*}
\]

Here we use the fact that
\[ \langle \delta, - \rangle = -\langle -, \tau \delta \rangle = -\langle -, \delta \rangle. \]

By Lemma 3.3 and Proposition 4.5, we know that the expression of \( X_{R_p(n)} \) is independent of the choice of \( p \in \mathbb{P}^1_k \) with degree 1. Hence, we set
\[ X_{n\delta} := X_{R_p(n)}. \]

The following corollary gives representation-theoretic interpretations of the elements \( \{s_n : n \in \mathbb{N}\} \) in the basis \( \mathcal{S} \).

**Corollary 4.6.** \( X_{n\delta} = s_n \) for every \( n \in \mathbb{N} \).

**Proof.** It follows from Proposition 4.5 and the definition of \( s_n \). \( \square \)

**Acknowledgement.** The authors would like to thank Professor Jie Xiao and Dr. Fan Xu for many helpful discussions and comments.
References


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