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BAR-INVARIANT BASES OF THE QUANTUM CLUSTER ALGEBRA OF TYPE $A_2^{(2)}$

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Cordially dedicated to Prof. Vlastimil Dlab on the occasion of his 80th birthday

Abstract. We construct bar-invariant $\mathbb{Z}[q^{\pm 1/2}]$-bases of the quantum cluster algebra of the valued quiver $A_2^{(2)}$, one of which coincides with the quantum analogue of the basis of the corresponding cluster algebra discussed in P. Sherman, A. Zelevinsky: Positivity and canonical bases in rank 2 cluster algebras of finite and affine types, Moscow Math. J., 4, 2004, 947–974.

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1. Introduction

Cluster algebras were invented by S. Fomin and A. Zelevinsky [11], [12] in order to study total positivity in algebraic groups and canonical bases in quantum groups. The study of $\mathbb{Z}$-bases of cluster algebras has become important. There are many results involving the construction of $\mathbb{Z}$-bases of cluster algebras (for example, see [17] and [4] for cluster algebras of rank 2, [3] for finite type, [10] for type $\tilde{A}$, [5] for $\tilde{A}_2^{(1)}$, [6] for affine type and [13] for acyclic quivers). As the quantum analogue of cluster algebras, quantum cluster algebras were defined by A. Berenstein and A. Zelevinsky in [1]. A quantum cluster algebra is generated by the so-called (*quantum*) *cluster variables* inside an ambient skew-field $\mathscr{F}$. Under the specialization $q = 1$, quantum cluster algebras degenerate to cluster algebras.

Recently, D. Rupel [16] defined a quantum analogue of the Caldero-Chapoton formula [2] and conjectured that cluster variables could be expressed in terms of the refined Caldero-Chapoton formula, and then proved the conjecture for those in almost acyclic clusters. This conjecture has been proved for acyclic equally valued quivers in [15]. Naturally, one may hope to construct $\mathbb{Z}[q^{\pm 1/2}]$-bases of quantum cluster algebras. For simply-laced finite and affine quivers, the bases have been constructed in [7] and [8].

In this paper, we deal with the quantum cluster algebra of the simplest non-simply-laced valued quiver $A_2^{(2)}$ and construct various bar-invariant $\mathbb{Z}[q^{\pm 1/2}]$-bases by applying the quantum analogue of the Caldero-Chapoton formula defined in [16]. Under the specialization $q = 1$, one of these $\mathbb{Z}[q^{\pm 1/2}]$-bases is exactly the canonical basis of the cluster algebra of the valued quiver $A_2^{(2)}$ discussed in [17]. Moreover, the elements $\{s_n \colon n \in \mathbb{N}\}$ in the basis $\mathscr{S}$ (see Definition 3.4) possess representation-theoretic interpretations.

2. Preliminaries

2.1. Quantum cluster algebras

In what follows, we will give a short review on quantum cluster algebras, for details one can refer to [1]. Let L be a lattice of rank m and $\Lambda \colon L \times L \to \mathbb{Z}$ a skew-symmetric bilinear form. Let q be a formal variable and let us consider the ring of integer Laurent polynomials $\mathbb{Z}[q^{\pm 1/2}]$. The *based quantum torus* associated with a pair (L, Λ) is a $\mathbb{Z}[q^{\pm 1/2}]$-algebra $\mathscr{T}$ with a distinguished $\mathbb{Z}[q^{\pm 1/2}]$-basis $\{X^e \colon e \in L\}$ and the multiplication given by

$$X^e X^f = q^{\frac{1}{2}\Lambda(e,f)} X^{e+f}.$$

Obviously $\mathscr{T}$ is associative and the basis elements satisfy the relations

$$X^e X^f = q^{\Lambda(e,f)} X^f X^e, \quad X^0 = 1, \quad (X^e)^{-1} = X^{-e}.$$

It is well known that $\mathscr{T}$ is an Ore domain, i.e., it is contained in its skew-field of fractions $\mathscr{F}$.

A *toric frame* in $\mathscr{F}$ is a mapping $M \colon \mathbb{Z}^m \to \mathscr{F} \setminus \{0\}$ of the form

$$M(\mathbf{c}) = \varphi(X^{\eta(\mathbf{c})}) =: X^{\mathbf{c}}$$

where $\mathbf{c} \in \mathbb{Z}^{\mathbf{m}}$, φ is an automorphism of $\mathscr{F}$ and $\eta \colon \mathbb{Z}^m \to L$ is an isomorphism of lattices. By the definition, the elements $M(\mathbf{c})$ form a $\mathbb{Z}[q^{\pm 1/2}]$-basis of the based quantum torus $\mathscr{T}_M := \varphi(\mathscr{T})$ and satisfy the relations

$$\begin{aligned} M(\mathbf{c})M(\mathbf{d}) &= q^{\frac{1}{2}\Lambda_M(\mathbf{c},\mathbf{d})}M(\mathbf{c}+\mathbf{d}), \\ M(\mathbf{c})M(\mathbf{d}) &= q^{\Lambda_M(\mathbf{c},\mathbf{d})}M(\mathbf{d})M(\mathbf{c}), \\ M(\mathbf{0}) &= 1, \\ M(\mathbf{c})^{-1} &= M(-\mathbf{c}), \end{aligned}$$

where the skew-symmetric bilinear form Λ_M on $\mathbb{Z}^m$ is obtained by transferring the form Λ from L via the lattice isomorphism η. Note that Λ_M can also be identified with a skew-symmetric $m \times m$ matrix given by $\lambda_{ij} = \Lambda_M(e_i, e_j)$ where $\{e_1, \ldots, e_m\}$ is the standard basis of $\mathbb{Z}^m$.

Given a toric frame M, write $X_i = M(e_i)$; then

$$\mathscr{T}_M = \mathbb{Z}[q^{\pm 1/2}]\langle X_1^{\pm 1}, \ldots, X_m^{\pm 1} \colon\ X_iX_j = q^{\lambda_{ij}}X_jX_i\rangle.$$

Let A be an $m \times m$ skew-symmetric matrix and $\tilde{B}$ an $m \times n$ matrix with $n \leqslant m$. The pair $(A, \tilde{B})$ is called *compatible* if $\tilde{B}^{\mathrm{tr}}A = (D \mid 0)$ is an $n \times m$ matrix with $D = \operatorname{diag}(d_1, \ldots, d_n)$ where $d_i \in \mathbb{N}$ for $1 \leqslant i \leqslant n$. For a toric frame M, we call the pair $(M, \tilde{B})$ a *quantum seed* if the pair $(\Lambda_M, \tilde{B})$ is compatible. Define the $m \times m$ matrix $E = (e_{ij})_{m \times m}$ as follows:

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k; \\ -1 & \text{if } i = j = k; \\ \max(0, -b_{ik}) & \text{if } i \neq j = k. \end{cases}$$

For $n, k \in \mathbb{Z}$, $k \geqslant 0$, denote $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = (q^n - q^{-n}) \ldots (q^{n-r+1} - q^{-n+r-1})/(q^r - q^{-r}) \ldots (q - q^{-1})$. Let $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ with $c_k \geqslant 0$. We can define the toric frame $M' \colon\ \mathbb{Z}^m \to \mathscr{F} \setminus \{0\}$ as

$$M'(\mathbf{c}) = \sum_{p=0}^{c_k} \begin{bmatrix} c_k \\ p \end{bmatrix}_{q^{d_k/2}} M(E\mathbf{c} + p\mathbf{b}^k), \quad M'(-\mathbf{c}) = M'(\mathbf{c})^{-1} \tag{2.1}$$

where the vector $\mathbf{b}^k \in \mathbb{Z}^m$ is the k-th column of $\tilde{B}$.

Let $\tilde{B}' = \mu_k(\tilde{B})$ be the mutation of $\tilde{B}$ at k (see [11] for details). Then the quantum seed $(M', \tilde{B}')$ is called the mutation of $(M, \tilde{B})$ in the direction k. Two quantum seeds are mutation-equivalent if each can be obtained from the other by a sequence of mutations. Let $\mathscr{C} = \{M'(e_i) \colon\ 1 \leqslant i \leqslant n,\ (M', \tilde{B}') \text{ is mutation-equivalent to } (M, \tilde{B})\}$. The elements of $\mathscr{C}$ are called *cluster variables*. Let $\mathscr{P} = \{M(e_i) \colon\ n+1 \leqslant i \leqslant m\}$; the elements in $\mathscr{P}$ are called *coefficients*. The *quantum cluster algebra* $\mathscr{A}_q(\Lambda_M, \tilde{B})$ is the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of $\mathscr{F}$ generated by the elements in $\mathscr{C} \cup \mathscr{P}$. We can associate with $(M, \tilde{B})$ the $\mathbb{Z}$-linear *bar-involution* on $\mathscr{T}_M$ as follows:

$$\overline{q^{r/2} M(\mathbf{c})} = q^{-r/2} M(\mathbf{c}), \quad \text{where } r \in \mathbb{Z},\ \mathbf{c} \in \mathbb{Z}^n.$$

Then we can see that $\overline{XY} = \overline{Y}\,\overline{X}$ for all $X, Y \in \mathscr{A}_q(\Lambda_M, \tilde{B})$ and the elements in $\mathscr{C} \cup \mathscr{P}$ are *bar-invariant.*

2.2. The valued quiver $A_2^{(2)}$

We can associate a valued quiver (see [16, Section 2] for more details) with a given compatible pair (A, B). Now we set $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$. Thus we have $B^{\mathrm{tr}} A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ denoted by D. The valued quiver Q associated with this pair is of type $A_2^{(2)}$:

$$1 \xrightarrow{(4,1)} 2\ .$$

Let $\mathfrak{S}$ be a reduced $\mathbb{F}_q$-species of type Q, see [9] for details. The category $\mathrm{rep}(\mathfrak{S})$ of finite dimensional representations of $\mathfrak{S}$ over $\mathbb{F}_q$ is equivalent to the category of finite dimensional modules over a finite-dimensional hereditary $\mathbb{F}_q$-algebra Δ, where Δ is the tensor algebra of $\mathfrak{S}$. In the rest of the paper, we will not distinguish the representation of the valued quiver and the module of the corresponding algebra. It is well known (see [9]) that indecomposable Δ-modules are divided into three families up to isomorphism: the indecomposable regular modules with dimension vector $(nd_p, 2nd_p)$ for $p \in \mathbb{P}^1_k$ of degree d_p and $n \in \mathbb{N}$ (in particular, denote by $R_p(n)$ the indecomposable regular module with dimension vector $(n, 2n)$ for $d_p = 1$), the preprojective modules, and the preinjective modules. Define

$$R = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}, \quad R' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is well known that the Euler form on $\mathrm{rep}(\mathfrak{S})$ is given by

$$\langle V, N \rangle = \underline{m}(I - R) D \underline{n}^{\mathrm{tr}}$$

where $\underline{m}$ and $\underline{n}$ are the dimension vectors of V and N, respectively. Now, let $\mathscr{T} = \mathbb{Z}[q^{\pm 1/2}]\langle X_1^{\pm 1}, X_2^{\pm 1} \colon X_1 X_2 = q X_2 X_1 \rangle$ and let $\mathscr{F}$ be the skew field of fractions of $\mathscr{T}$. Thus the quantum cluster algebra of the valued quiver $A_2^{(2)}$ denoted by $\mathscr{A}_q(1,4)$ in the sequel is the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of $\mathscr{F}$ generated by the cluster variables X_k, $k \in \mathbb{Z}$, defined recursively by

$$X_{m-1} X_{m+1} = \begin{cases} q^{1/2} X_m + 1 & \text{if } m \text{ is odd;} \\ q^2 X_m^4 + 1 & \text{if } m \text{ is even.} \end{cases}$$

The quantum Laurent phenomenon [1] implies that each X_k belongs to the subring $\mathscr{T}$ of $\mathscr{F}$. Let V be a representation of the valued quiver $A_2^{(2)}$ with dimension vector $\underline{\dim}\, V = (v_1, v_2)$. For $\mathbf{e} = (e_1, e_2) \in \mathbb{Z}^2_{\geqslant 0}$, denote by $\mathrm{Gr}_{\mathbf{e}}(V)$ the set of all subrepresentations U of V with $\underline{\dim}\, U = \mathbf{e}$. In [16], the author defined the element X_V of the quantum torus $\mathscr{T}$ by

$$X_V = \sum_{\mathbf{e}} q^{-\frac{1}{2}d_{\mathbf{e}}^V} \mathrm{Gr}_{\mathbf{e}}(V)|X^{(-v_1+v_2-e_2, 4e_1-v_2)} \tag{2.2}$$

where $d_{\mathbf{e}}^V = 4e_1(v_1 - e_1) - (4e_1 - e_2)(v_2 - e_2)$. This formula is called the quantum analogue of the Caldero-Chapoton formula [2].

Let $C = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ be the Cartan matrix and Φ the associated root system with simple roots $\{\alpha_1, \alpha_2\}$. Then all negative real roots of Φ can be labeled by $m \in \mathbb{Z} \setminus \{1, 2\}$ as follows:

$$\alpha_{m-1} + \alpha_{m+1} = \begin{cases} \alpha_m & \text{if } m \text{ is odd,} \\ 4\alpha_m & \text{if } m \text{ is even,} \end{cases}$$

where we set $\alpha_0 = -\alpha_2$, $\alpha_3 = -\alpha_1$.

Recall the following result from [16]:

Theorem 2.1 ([16]). *For any $m \in \mathbb{Z} \setminus \{1, 2\}$, let $V(m)$ be the unique indecomposable valued representation of $A_2^{(2)}$ with dimension vector $-\alpha_m$. Then the m-th cluster variable X_m of $\mathscr{A}_q(1, 4)$ is equal to $X_{V(m)}$.*

3. Bases of the quantum cluster algebra $\mathscr{A}_q(1, 4)$

In this section, we will construct various bar-invariant $\mathbb{Z}[q^{\pm 1/2}]$-bases of the quantum cluster algebra $\mathscr{A}_q(1, 4)$. Under the specialization $q = 1$, these bases are just the $\mathbb{Z}$-bases of the cluster algebra of the valued quiver $A_2^{(2)}$.

Definition 3.1. For any (r_1, r_2) and $(s_1, s_2) \in \mathbb{Z}^2$, we write $(r_1, r_2) \preceq (s_1, s_2)$ if $r_i \leqslant s_i$ for $1 \leqslant i \leqslant 2$. Moreover, if there exists i such that $r_i < s_i$, then we write $(r_1, r_2) \prec (s_1, s_2)$.

Lemma 3.2. *The Laurent expansion in $X_{V(m)}$ has a minimal non-zero term X^{α_m}.*

Proof. It is obvious that the module $V(m)$ with dimension vector (v_1, v_2) has a submodule with dimension vector $(0, v_2)$. Thus by the definition of the q-deformation of the Caldero-Chapoton formula and the partial order in Definition 3.1, we obtain that the expansion in $X_{V(m)}$ has a minimal non-zero term X^{α_m}. $\square$

Lemma 3.3. *Let $R_p(1)$ be an indecomposable regular module of degree 1. Then*

$$X_{R_p(1)} = X^{(-1,-2)} + X^{(-1,2)} + X^{(1,-2)} + (q^{1/2}+q^{-1/2})X^{(0,-2)}.$$

Proof. Note that $R_p(1)$ contains four submodules with dimension vectors $(0,0)$, $(0,1)$, $(0,2)$ and $(1,2)$. Therefore the lemma immediately follows from the q-deformation of the Caldero-Chapoton formula. $\square$

By Lemma 3.3, the expression of $X_{R_p(1)}$ is independent of the choice of $p \in \mathbb{P}^1_k$ of degree 1. So we set

$$X_\delta := X_{R_p(1)}.$$

Definition 3.4 (Chebyshev polynomials).

(1) The n-th Chebyshev polynomial of the first kind is the polynomial $F_n(x) \in \mathbb{Z}[x]$ defined recursively by

$$\begin{cases} F_0(x) = 1, \quad F_1(x) = x, \quad F_2(x) = x^2 - 2, \\ F_{n+1}(x) = F_n(x)F_1(x) - F_{n-1}(x) \quad \text{for } n \geqslant 2. \end{cases}$$

(2) The n-th Chebyshev polynomial of the second kind is the polynomial $S_n(x) \in \mathbb{Z}[x]$ defined recursively by

$$\begin{cases} S_0(x) = 1, \quad S_1(x) = x, \quad S_2(x) = x^2 - 1, \\ S_{n+1}(x) = S_n(x)S_1(x) - S_{n-1}(x) \quad \text{for } n \geqslant 2. \end{cases}$$

It is obvious that $F_n(x) = S_n(x) - S_{n-2}(x)$. We denote $z = X_\delta$, $z_n = F_n(z)$, $s_n = S_n(z)$ for $n \geqslant 0$ and $z_n = s_n = 0$ for $n < 0$. Set

$$\begin{aligned} \mathscr{B}' &= \{X_m^a X_{m+1}^b \colon m \in \mathbb{Z},\ (a,b) \in \mathbb{Z}^2_{\geqslant 0}\} \cup \{z_n \colon n \in \mathbb{N}\}, \\ \mathscr{S}' &= \{X_m^a X_{m+1}^b \colon m \in \mathbb{Z},\ (a,b) \in \mathbb{Z}^2_{\geqslant 0}\} \cup \{s_n \colon n \in \mathbb{N}\}, \\ \mathscr{G}' &= \{X_m^a X_{m+1}^b \colon m \in \mathbb{Z},\ (a,b) \in \mathbb{Z}^2_{\geqslant 0}\} \cup \{z^n \colon n \in \mathbb{N}\}. \end{aligned}$$

Remark 3.5. It is easy to check that $X^{(r,2r)}X^{(s,2s)} = X^{(r+s,2r+2s)}$ for any $r, s \in \mathbb{Z}$, and thus the expansions of z_n, s_n and z^n have a minimal non-zero term $X^{-(n,2n)}$ according to the partial order in Definition 3.1.

We have the following immediate result.

Lemma 3.6. $X_\delta = qX_0^2X_3 - q^2(qX_1 + q^{-1/2} + q^{1/2})X_2^2$.

Proof. By $X_0X_2 = q^{1/2}X_1 + 1$, we have $X_0 = X^{(1,-1)} + X^{(0,-1)}$. By $X_1X_3 = q^2X_2^4 + 1$, we have $X_3 = X^{(-1,4)} + X^{(-1,0)}$. Then we can prove the lemma by direct computation. $\square$

The following lemma is straightforward but important.

Lemma 3.7. $\overline{X_\delta} = X_\delta$.

Proof. Note that

$$\begin{aligned}\overline{X_\delta} &= q^{-1}\overline{X_0^2X_3} - q^{-2}\overline{(qX_1 + q^{-1/2} + q^{1/2})X_2^2}\\ &= q^{-1}X_3X_0^2 - q^{-2}X_2^2(q^{-1}X_1 + q^{-1/2} + q^{1/2}) = X_\delta.\end{aligned}$$

$\square$

Remark 3.8. By Lemma 3.7, we can verify that $\overline{z_n} = z_n$, $\overline{s_n} = s_n$.

For any $\underline{d} \in \mathbb{Z}^2$, define $\underline{d}^+ = (d_1^+, d_2^+)$ such that $d_i^+ = d_i$ if $d_i > 0$ and $d_i^+ = 0$ if $d_i \leqslant 0$ for any $1 \leqslant i \leqslant 2$. Dually, we set $\underline{d}^- = \underline{d}^+ - \underline{d}$.

The proposition below is a special case of [1, Theorem 7.3].

Proposition 3.9 ([1])**.** *Let Q be the valued quiver $A_2^{(2)}$. Then the set*

$$\{X_1^{d_1^-}X_2^{d_2^-}X_{S_1}^{d_1^+}X_{S_2}^{d_2^+}\colon (d_1, d_2) \in \mathbb{Z}^2\}$$

is a $\mathbb{Z}[q^{\pm 1/2}]$-basis of $\mathscr{A}_q(1,4)$.

Proof. It is easy to check that the sets $\{X_1, X_{S_2}\}$ and $\{X_2, X_{S_1}\}$ are clusters obtained by the mutation in the direction 2 and 1, respectively, from the cluster $\{X_1, X_2\}$. Therefore the proposition immediately follows from [1, Theorem 7.3]. $\square$

The following result is an immediate consequence of the above proposition.

Corollary 3.10. *The sets $\mathscr{B}'$, $\mathscr{S}'$ and $\mathscr{G}'$ are $\mathbb{Z}[q^{\pm 1/2}]$-bases of the quantum cluster algebra $\mathscr{A}_q(1,4)$.*

Proof. Note that if $\mathscr{B}'$ is a $\mathbb{Z}[q^{\pm 1/2}]$-basis of the quantum cluster algebra $\mathscr{A}_q(1,4)$, then $\mathscr{S}'$ and $\mathscr{G}'$ are naturally $\mathbb{Z}[q^{\pm 1/2}]$-bases of $\mathscr{A}_q(1,4)$ because there exist unipotent transformations between $\{z_n\colon n \in \mathbb{N}\}$, $\{s_n\colon n \in \mathbb{N}\}$ and $\{z^n\colon n \in \mathbb{N}\}$. In what follows, we will only focus on the set $\mathscr{B}'$.

By Lemma 3.6, we obtain that X_δ is in $\mathscr{A}_q(1,4)$. Thus $\{z_n\colon n\in\mathbb{N}\}$ is contained in $\mathscr{A}_q(1,4)$. Note that for any $\underline{v}=(v_1,v_2)\in\mathbb{Z}^2$, there exists only one object X_V in $\mathscr{B}'$ such that $\underline{\dim}\,V=(v_1,v_2)\in\mathbb{Z}^2$. Then by Proposition 3.9 we have

$$X_V=b_{\underline{v}}X_1^{v_1^-}X_2^{v_2^-}X_{S_1}^{v_1^+}X_{S_2}^{v_2^+}+\sum_{\underline{v}\succ\underline{l}}b_{\underline{l}}X_1^{l_1^-}X_2^{l_2^-}X_{S_1}^{l_1^+}X_{S_2}^{l_2^+}$$

where $b_{\underline{v}},b_{\underline{l}}\in\mathbb{Z}[q^{\pm1/2}]$. Then by Lemma 3.2, Remark 3.5, we know that $b_{\underline{m}}$ must be a nonzero monomial in $q^{\pm1/2}$. Thus we obtain that $\mathscr{B}'$ is a $\mathbb{Z}[q^{\pm1/2}]$-basis of $\mathscr{A}_q(1,4)$. $\square$

Set

$$\begin{aligned}
\mathscr{B}&=\{q^{-\frac{1}{2}ab}X_m^aX_{m+1}^b\colon\ m\in\mathbb{Z},\ (a,b)\in\mathbb{Z}^2_{\geqslant0}\}\cup\{z_n\colon\ n\in\mathbb{N}\},\\
\mathscr{S}&=\{q^{-\frac{1}{2}ab}X_m^aX_{m+1}^b\colon\ m\in\mathbb{Z},\ (a,b)\in\mathbb{Z}^2_{\geqslant0}\}\cup\{s_n\colon\ n\in\mathbb{N}\},\\
\mathscr{G}&=\{q^{-\frac{1}{2}ab}X_m^aX_{m+1}^b\colon\ m\in\mathbb{Z},\ (a,b)\in\mathbb{Z}^2_{\geqslant0}\}\cup\{z^n\colon\ n\in\mathbb{N}\}.
\end{aligned}$$

Then we can obtain the following main result of the paper.

Theorem 3.11. *The sets $\mathscr{B}$, $\mathscr{S}$ and $\mathscr{G}$ are bar-invariant $\mathbb{Z}[q^{\pm1/2}]$-bases of the quantum cluster algebra $\mathscr{A}_q(1,4)$.*

Proof. By Lemma 3.7 and Remark 3.8 and the fact that every element in the set $\{q^{-\frac{1}{2}ab}X_m^aX_{m+1}^b\colon\ m\in\mathbb{Z},\ (a,b)\in\mathbb{Z}^2_{\geqslant0}\}$ is bar-invariant, the theorem follows immediately. $\square$

4. Some multiplication formulas

In this section, we prove some multiplication formulas and then give representation-theoretic interpretations of the elements $\{s_n\colon\ n\in\mathbb{N}\}$ in the basis $\mathscr{S}$.

First, we define a ring homomorphism of the quantum cluster algebra $\mathscr{A}_q(1,4)$:

$$\sigma_2\colon\ \mathscr{A}_q(1,4)\longrightarrow\mathscr{A}_q(1,4)$$

by sending X_m to X_{m+2} and $q^{\pm1/2}$ to $q^{\pm1/2}$. It is obviously an automorphism which preserves the defining relations.

We have the following result.

Lemma 4.1. $\sigma_2(X_\delta) = X_\delta$.

Proof. By direct computation, we have

$$\begin{aligned} X_3 &= X^{(-1,4)} + X^{(-1,0)}, \\ X_4 &= X^{(-1,3)} + X^{(0,-1)} + (q+q^{-1})X^{(-1,-1)}. \end{aligned}$$

Thus we obtain the identity

$$qX_4^2 + q^{-1}X_2^2 = X_3X_\delta.$$

By Lemma 3.6, we have:

$$X_\delta = qX_0^2X_3 - q^2(qX_1 + q^{-1/2} + q^{1/2})X_2^2.$$

Therefore, we have

$$\begin{aligned} \sigma_2(X_\delta) &= \sigma_2(qX_0^2X_3 - q^2(qX_1 + q^{-1/2} + q^{1/2})X_2^2) \\ &= qX_2^2X_5 - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \\ &= q^3X_2^2X_3^{-1}X_4^4 + qX_2^2X_3^{-1} - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \\ &= qX_3^{-1}X_2^2X_4^4 + q^{-1}X_3^{-1}X_2^2 - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \\ &= qX_3^{-1}X_2(q^{1/2}X_3 + 1)X_4^3 + q^{-1}X_3^{-1}X_2^2 - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \\ &= q^{3/2}X_3^{-1}X_2X_3X_4^3 + qX_3^{-1}(q^{1/2}X_3 + 1)X_4^2 \\ &\quad + q^{-1}X_3^{-1}X_2^2 - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \\ &= q^{5/2}X_2X_4^3 + q^{3/2}X_4^2 + qX_3^{-1}X_4^2 + q^{-1}X_3^{-1}X_2^2 \\ &\quad - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \\ &= q^{5/2}(q^{1/2}X_3 + 1)X_4^2 + q^{3/2}X_4^2 + X_\delta - q^2(qX_3 + q^{-1/2} + q^{1/2})X_4^2 \\ &= X_\delta. \end{aligned}$$

□

Proposition 4.2. *We have*

(1) *for* $m > n \geqslant 1$,

$$\begin{aligned} z_nz_m &= z_{m+n} + z_{m-n}, \\ z_nz_n &= z_{2n} + 2, \end{aligned}$$

(2) *for any* $n \in \mathbb{Z}$,

$$\begin{aligned} X_{2n}X_\delta &= q^{-1/2}X_{2n-2} + q^{1/2}X_{2n+2}, \\ X_{2n+1}X_\delta &= q^{-1}X_{2n}^2 + qX_{2n+2}^2. \end{aligned}$$

Proof. (1) It follows from the definition of Chebyshev polynomials.

(2) By Lemma 4.1, we only need to prove the equations

$$X_2X_\delta = q^{-1/2}X_0 + q^{1/2}X_4,$$
$$X_1X_\delta = q^{-1}X_0^2 + qX_2^2.$$

By the defining relations, we have

$$X_0 = X^{(1,-1)} + X^{(0,-1)}, \quad X_4 = X^{(-1,3)} + X^{(0,-1)} + X^{(-1,-1)}.$$

Then we can prove the above equations by Lemma 3.3 and direct computation. □

Note that for any Δ-module V, the quantum analogue of the Caldero-Chapton map of the valued quiver $Q = A_2^{(2)}$ defined in [16] can be rewritten as

$$X_V = \sum_{\underline{e}} |\operatorname{Gr}_{\underline{e}} V| q^{-1/2\langle \underline{e}, \underline{v}-\underline{e}\rangle} X^{-\underline{e}B^{\rm tr} - \underline{v}(I-R')}.$$

Lemma 4.3. *For any dimension vector $\underline{m}, \underline{e}, \underline{f} \in \mathbb{Z}^n_{\geqslant 0}$, we have*

(1) $\Lambda(\underline{m}(I-R'), \underline{e}B^{\rm tr}) = -\langle \underline{e}, \underline{m}\rangle$;

(2) $\Lambda(\underline{e}B^{\rm tr}, \underline{f}B^{\rm tr}) = \langle \underline{f}, \underline{e}\rangle - \langle \underline{e}, \underline{f}\rangle$.

Proof. It is easy to check that

$$\begin{aligned}\Lambda(\underline{m}(I-R'), \underline{e}B^{\rm tr}) &= \underline{m}(I-R')\Lambda B\underline{e}^{\rm tr} = -\underline{m}(I-R')D^{\rm tr}\underline{e}^{\rm tr}\\ &= -\underline{e}D(I-R')^{\rm tr}\underline{m}^{\rm tr} = -\underline{e}(I-R)D\underline{m}^{\rm tr} = -\langle \underline{e}, \underline{m}\rangle\end{aligned}$$

and

$$\begin{aligned}\Lambda(\underline{e}B^{\rm tr}, \underline{f}B^{\rm tr}) &= \underline{e}B^{\rm tr}\Lambda\tilde{B}\underline{f}^{\rm tr} = -\underline{e}B^{\rm tr}D\underline{f}^{\rm tr} = \underline{e}(R-R')D\underline{f}^{\rm tr}\\ &= \underline{e}((I-R')-(I-R))D\underline{f}^{\rm tr} = \underline{e}(I-R')D\underline{f}^{\rm tr} - \underline{e}(I-R)D\underline{f}^{\rm tr}\\ &= \langle \underline{f}, \underline{e}\rangle - \langle \underline{e}, \underline{f}\rangle.\end{aligned}$$

□

Corollary 4.4. *For any dimension vector $\underline{m}, \underline{l}, \underline{e}, \underline{f} \in \mathbb{Z}^n_{\geqslant 0}$, we have*

$$\begin{aligned}&\Lambda(\underline{m}(I-R') + \underline{e}B^{\rm tr}, \underline{l}(I-R') + \underline{f}B^{\rm tr})\\ &\qquad = \Lambda(\underline{m}(I-R'), \underline{l}(I-R')) + \langle \underline{f}, \underline{e}\rangle - \langle \underline{e}, \underline{f}\rangle + \langle \underline{e}, \underline{l}\rangle - \langle \underline{f}, \underline{m}\rangle.\end{aligned}$$

For Δ-modules V, T and N, we denote by F_{TN}^{V} the number of submodules U of V such that U is isomorphic to N and V/U is isomorphic to T. The following proposition gives representation-theoretic interpretations of elements s_n, $n \in \mathbb{N}$ in the basis $\mathscr{S}$. By abuse of language, we still denote by V the dimension vector of the Δ-module V in the bilinear forms involved.

Proposition 4.5. *For any $n \in \mathbb{N}$, we have*

$$X_{R_p(n)}X_{R_p(1)} = X_{R_p(n+1)} + X_{R_p(n-1)}.$$

Proof. We have the exact sequences

$$0 \longrightarrow R_p(1) \longrightarrow R_p(n+1) \longrightarrow R_p(n) \longrightarrow 0$$

and

$$0 \longrightarrow R_p(1) \xrightarrow{\varepsilon} \tau R_p(n) = R_p(n) \xrightarrow{p} R_p(n-1) \longrightarrow 0\,.$$

The term on the left-hand side is

$$\begin{aligned} &X_{R_p(n)}X_{R_p(1)} \\ &\quad = \sum_{\underline{d}} |\operatorname{Gr}_{\underline{d}} R_p(n)| q^{-1/2\langle \underline{d}, n\delta-\underline{d}\rangle} X^{-\underline{d}B^{\mathrm{tr}}-n\delta(I-R')} \\ &\quad\quad \times \sum_{\underline{b}} |\operatorname{Gr}_{\underline{b}} R_p(1)| q^{-1/2\langle \underline{b}, \delta-\underline{b}\rangle} X^{-\underline{b}B^{\mathrm{tr}}-\delta(I-R')} \\ &\quad = \sum_{\underline{b},\underline{d}} |\operatorname{Gr}_{\underline{d}} R_p(n)||\operatorname{Gr}_{\underline{b}} R_p(1)| q^{-1/2\langle \underline{d}, n\delta-\underline{d}\rangle - 1/2\langle \underline{b}, \delta-\underline{b}\rangle} \\ &\quad\quad \times q^{1/2\Lambda(-\underline{d}B^{\mathrm{tr}}-n\delta(I-R'), -\underline{b}B^{\mathrm{tr}}-\delta(I-R'))} X^{-(\underline{b}+\underline{d})B^{\mathrm{tr}}-(n+1)\delta(I-R')}. \end{aligned}$$

Then by Corollary 4.4, the above equation is equal to

$$\begin{aligned} &\sum_{\underline{b},\underline{d}} |\operatorname{Gr}_{\underline{d}} R_p(n)||\operatorname{Gr}_{\underline{b}} R_p(1)| q^{-1/2\langle \underline{d}+\underline{b}, (n+1)\delta-\underline{b}-\underline{d}\rangle} q^{\langle \underline{d}, \delta-\underline{b}\rangle} X^{-(\underline{b}+\underline{d})B^{\mathrm{tr}}-(n+1)\delta(I-R')} \\ &\quad = \sum_{N,Q} F_{PQ}^{R_p(n)} F_{TN}^{R_p(1)} q^{-1/2\langle \underline{d}+\underline{b}, (n+1)\delta-\underline{b}-\underline{d}\rangle} q^{\langle Q,T\rangle} X^{-(\underline{b}+\underline{d})B^{\mathrm{tr}}-(n+1)\delta(I-R')}. \end{aligned}$$

The first term on the right-hand side is

$$\tau_1 := X_{R_p(n+1)} = \sum_{H} F_{GH}^{R_p(n+1)} q^{-1/2\langle \underline{h}, \underline{g}\rangle} X^{-\underline{h}B^{\mathrm{tr}}-(n+1)\delta(I-R')}.$$

According to [14, Lemma 14], we have

$$\tau_1 = \sum_{N,Q} q^{\langle Q,T\rangle} \frac{q - q^{\dim_k \mathrm{Ext}^1(Q,T)}}{q-1} F_{PQ}^{R_p(n)} F_{TN}^{R_p(1)} q^{-1/2\langle N+Q,(n+1)\delta-N-Q\rangle} \times X^{-(\underline{b}+\underline{d})B^{\mathrm{tr}}-(n+1)\delta(I-R')}.$$

Now we consider the term

$$\tau_2 := \sum_{Y} q^{\langle n\delta-W,\delta\rangle} F_{WY}^{R_p(n-1)} q^{-1/2\langle Y+\delta,n\delta-Y\rangle} X^{-\underline{y}B^{\mathrm{tr}}-(n-1)\delta(I-R')}.$$

Any submodule Y of $R_p(n-1)$ induces the submodule $Q = p^{-1}(Y)$ and $N = 0$ of $R_p(n)$ and $R_p(1)$ respectively as the following commutative diagram shows:

$$\begin{tikzcd} 0 \arrow[r] & R_p(1) \arrow[r] & p^{-1}(Y) \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] & \\ & & R_p(n) \arrow[r, "p"] & R_p(n-1) \arrow[r, "0"] & R_p(1) \arrow[r, "\varepsilon"] & R_p(n) \arrow[r] & 0 \end{tikzcd}$$

Thus $\underline{y} = \underline{b} + \underline{d} - \delta$ and

$$\begin{aligned} &-\underline{y}B^{\mathrm{tr}} - (n-1)\delta(I-R') \\ &\quad = -(\underline{b}+\underline{d}-\delta)B^{\mathrm{tr}} - (n-1)\delta(I-R') \\ &\quad = -(\underline{b}+\underline{d})B^{\mathrm{tr}} + \delta B^{\mathrm{tr}} - (n-1)\delta(I-R') \\ &\quad = -(\underline{b}+\underline{d})B^{\mathrm{tr}} + \delta(R'-R) - (n-1)\delta(I-R') \\ &\quad = -(\underline{b}+\underline{d})B^{\mathrm{tr}} - \delta(I-R') + \delta(I-R) - (n-1)\delta(I-R') \\ &\quad = -(\underline{b}+\underline{d})B^{\mathrm{tr}} - 2\delta(I-R') - (n-1)\delta(I-R') \\ &\quad = -(\underline{b}+\underline{d})B^{\mathrm{tr}} - (n+1)\delta(I-R'). \end{aligned}$$

Then by [14, Lemma 16], we have

$$\tau_2 = \sum_{N,Q} q^{\langle Q,T\rangle} \frac{q^{\dim_k \mathrm{Ext}^1(Q,T)} - 1}{q-1} F_{PQ}^{R_p(n)} F_{TN}^{R_p(1)} q^{-1/2\langle \underline{b}+\underline{d},(n+1)\delta-\underline{b}-\underline{d}\rangle} \times X^{-(\underline{b}+\underline{d})B^{\mathrm{tr}}-(n+1)\delta(I-R')}.$$

Therefore

$$\begin{aligned} \tau_1 + \tau_2 &= \sum_{N,Q} F_{PQ}^{R_p(n)} F_{TN}^{R_p(1)} q^{-1/2\langle \underline{d}+\underline{b},(n+1)\delta-\underline{b}-\underline{d}\rangle} q^{\langle Q,T\rangle} X^{-(\underline{b}+\underline{d})B^{\mathrm{tr}}-(n+1)\delta(I-R')} \\ &= X_{R_p(n)} X_{R_p(1)}. \end{aligned}$$

Note that the second term on the right-hand side of the desired equation is

$$\tau_3 := X_{R_p(n-1)} = \sum_Y F_{WY}^{R_p(n-1)} q^{-1/2\langle Y,(n-1)\delta - Y\rangle} X^{-\underline{y}B^{\mathrm{tr}}-(n+1)\delta(I-R')}.$$

So it remains to prove $\tau_2 = \tau_3$, i.e., the equation

$$\langle n\delta - W, \delta\rangle - \frac{1}{2}\langle Y+\delta, n\delta - Y\rangle = -\frac{1}{2}\langle Y, (n-1)\delta - Y\rangle.$$

Note that

$$\begin{aligned}
\langle n\delta - W, \delta\rangle &- \frac{1}{2}\langle Y+\delta, n\delta - Y\rangle \\
&= \langle \delta + Y, \delta\rangle - \frac{1}{2}\langle Y, n\delta - Y\rangle - \frac{1}{2}\langle \delta, n\delta - Y\rangle \\
&= \langle Y, \delta\rangle - \frac{1}{2}\langle Y, (n-1)\delta - Y\rangle - \frac{1}{2}\langle Y, \delta\rangle + \frac{1}{2}\langle \delta, Y\rangle \\
&= -\frac{1}{2}\langle Y, (n-1)\delta - Y\rangle.
\end{aligned}$$

Here we use the fact thet

$$\langle \delta, -\rangle = -\langle -, \tau\delta\rangle = -\langle -, \delta\rangle.$$

□

By Lemma 3.3 and Proposition 4.5, we know that the expression of $X_{R_p(n)}$ is independent of the choice of $p \in \mathbb{P}^1_k$ with degree 1. Hence, we set

$$X_{n\delta} := X_{R_p(n)}.$$

The following corollary gives representation-theoretic interpretations of the elements $\{s_n \colon n \in \mathbb{N}\}$ in the basis $\mathscr{S}$.

Corollary 4.6. $X_{n\delta} = s_n$ *for every* $n \in \mathbb{N}$.

P r o o f. It follows from Proposition 4.5 and the definition of s_n. □

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