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DERIVATIONS WITH ENGEL CONDITIONS
IN PRIME AND SEMIPRIME RINGS

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Abstract. Let R be a prime ring, I a nonzero ideal of R , d a derivation of R and m, n fixed positive integers. (i) If $(d[x, y])^m = [x, y]^n$ for all $x, y \in I$, then R is commutative. (ii) If $\text{Char } R \neq 2$ and $[d(x), d(y)]_m = [x, y]^n$ for all $x, y \in I$, then R is commutative. Moreover, we also examine the case when R is a semiprime ring.

Keywords: prime and semiprime rings, ideal, derivation, GPIs

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1. INTRODUCTION

In all that follows, unless stated otherwise, R will be an associative ring, $Z(R)$ the center of R , Q its Martindale quotient ring and U its Utumi quotient ring. The center of U , denoted by C , is called the extended centroid of R (we refer the reader to [1] for these objects).

For each $x, y \in R$ and each $n \geq 0$, define $[x, y]_n$ inductively by $[x, y]_0 = x$, $[x, y]_1 = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k \geq 2$. The ring R is said to satisfy an Engel condition if there exists a positive integer n such that $[x, y]_n = 0$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. An additive mapping $d: R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For some fixed $a \in R$, the mapping $I_a: R \rightarrow R$ given by $I_a(x) = [a, x]$ for all $x \in R$ is a derivation which is called an inner derivation. If R is a ring and $S \subseteq R$, a mapping $f: R \rightarrow R$ is called strong commutativity-preserving (scp) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$.

During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific

types of derivations (see [2], where further references can be found). The Engel type identity with derivation appeared in the well-known paper of Posner [17] who proved that a prime ring admitting a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, must be commutative. Since then several authors have studied this kind of Engel type identities with derivations acting on one-sided, two-sided and Lie ideals of prime and semiprime rings (see [8] for a partial bibliography).

In the year 1992, Daif and Bell [7, Theorem 3] showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d[x, y] = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. If R is a prime ring, this implies that R is commutative. It is natural to ask what we can say about the commutativity of R in case $(d[x, y])^m = [x, y]_n$ for all $x, y \in I$. In this paper we investigate this identity and obtain the commutativity of R . In 1994, Bell and Daif [3] initiated the study of strong commutativity-preserving maps (for more information we refer to [13] and references therein) and proved that a nonzero right ideal U of a semiprime ring is central if R admits a derivation which is scp on U . Here we will examine what happens in case R is a prime ring and $[d(x), d(y)]_m = [x, y]_n$ for all $x, y \in I$, with I a nonzero ideal of R and m, n fixed positive integers. In fact, we can also prove that R is commutative under the assumption $\text{Char } R \neq 2$.

2. THE CASE: R A PRIME RING

Theorem 2.1. *Let R be a prime ring, I a nonzero ideal of R , and m, n fixed positive integers. If R admits a derivation d such that $(d[x, y])^m = [x, y]_n$ for all $x, y \in I$, then R is commutative.*

Proof. If $d = 0$, then $[x, y]_n = 0 = [I_x(y), y]_{n-1}$ for all $x, y \in I$. By Lanski [6, Theorem 1] either R is commutative or $I_x = 0$, i.e., $I \subseteq Z(R)$ in which case R is also commutative by Mayne [15, Lemma 3].

Now we assume that $d \neq 0$ and $(d[x, y])^m = [x, y]_n$ which can be rewritten as $[d(x), y] + [x, d(y)]^m = [x, y]_n$ for all $x, y \in I$. Following Kharchenko [12], we divide the proof into two cases:

Case 1. If d is Q -outer, then I satisfies the polynomial identity $([s, y] + [x, t])^m = [x, y]_n$ for all $x, y, s, t \in I$. In particular, for $x = 0$, I satisfies the blended component $[s, y]^m = 0$ for all $s, y \in I$, and R is commutative by Herstein [10, Theorem 2].

Case 2. Let now d be Q -inner induced by an element $q \in Q$, that is $d(x) = [q, x]$ for all $x \in R$. It follows that $([[q, x], y] + [x, [q, y]])^m = [x, y]_n$ for all $x, y \in I$. By Chuang [4, Theorem 2], I and Q satisfy the same generalized polynomial identities (GPIs), hence we have $([[q, x], y] + [x, [q, y]])^m = [x, y]_n$ for all $x, y \in Q$. In case the center C of Q is infinite, we have $([[q, x], y] + [x, [q, y]])^m = [x, y]_n$

for all $x, y \in Q \otimes_C \overline{C}$, where \overline{C} is the algebraic closure of C . Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [9, Theorems 2.5 & 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC = R$) which is either finite or algebraically closed and $([[q, x], y] + [x, [q, y]])^m = [x, y]_n$ for all $x, y \in R$. By Martindale [16, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H with C as the associated division ring. Hence by Jacobson's theorem [11, p. 75], R is isomorphic to a dense ring of linear transformations of some vector space V over C and H consists of the finite rank linear transformations in R .

Assume that $\dim_C V \geq 3$.

First of all, we want to show that v and qv are linearly C -dependent for all $v \in V$. Since if $qv = 0$ then v, qv is C -dependent, suppose that $qv \neq 0$. If v and qv are C -independent, since $\dim_C V \geq 3$, there exists $w \in V$ such that v, qv, w are also C -independent. By the density of R , there exist $x, y \in R$ such that: $xv = 0, xqv = w, xw = v; yv = 0, yqv = 0, yw = v$. This implies that $v = ([[q, x], y] + [x, [q, y]])^m v = [x, y]_n v = 0$, which is a contradiction. So we conclude that v and qv are linearly C -dependent for all $v \in V$.

Our next goal is to show that there exists $b \in C$ such that $qv = bv$ for all $v \in V$. In fact, choose $v, w \in V$ linearly independent. Since $\dim_C V \geq 3$, there exists $u \in V$ such that u, v, w are linearly independent, and so $b_u, b_v, b_w \in C$ such that $qu = b_u u, qv = b_v v, qw = b_w w$, that is $q(u + v + w) = b_u u + b_v v + b_w w$. Moreover, $q(u + v + w) = b_{u+v+w}(u + v + w)$ for a suitable $b_{u+v+w} \in C$. Then $0 = (b_{u+v+w} - b_u)u + (b_{u+v+w} - b_v)v + (b_{u+v+w} - b_w)w$ and because u, v, w are linearly independent, $b_u = b_v = b_w = b_{u+v+w}$, that is, b does not depend on the choice of v . Hence now we have $qv = vb$ for all $v \in V$.

Now for $r \in R, v \in V$ we have $(rq)v = r(qv) = r(vb) = (rv)b = q(rv)$, that is $[q, R]V = 0$. Since V is a left faithful irreducible R -module, hence $[q, R] = 0$, i.e., $q \in Z(R)$ and so $d = 0$, a contradiction.

Suppose now that $\dim_C V \leq 2$.

In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [6, Lemma 2], it follows that there exists a suitable field F such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F , and moreover, $M_k(F)$ satisfies the same GPI as R .

Assume $k \geq 3$, then by the same argument as above we get a contradiction.

Obviously if $k = 1$, then R is commutative.

Thus we may assume that $k = 2$, i.e., $R \subseteq M_2(F)$, where $M_2(F)$ satisfies $([[q, x], y] + [x, [q, y]])^m = [x, y]_n$.

Denote by e_{ij} the usual unit matrix with 1 in (i, j) -entry and zero elsewhere. Let $[x, y] = [e_{21}, e_{11}] = e_{21}$. In this case we have $(qe_{21} - e_{21}q)^m = e_{21}$. Right multiplying by e_{21} , we get $(-1)^m(e_{21}q)^m e_{21} = (qe_{21} - e_{21}q)^m e_{21} = e_{21}e_{21} = 0$.

Set $q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. By calculation we find that $(-1)^m \begin{pmatrix} 0 & 0 \\ q_{12}^m & 0 \end{pmatrix} = 0$, which implies that $q_{12} = 0$. Similarly we can see that $q_{21} = 0$. Therefore q is diagonal in $M_2(F)$. Let $f \in \text{Aut}(M_2(F))$. Since $([[f(q), f(x)], f(y)] + [f(x), [f(q), f(y)]])^m = [f(x), f(y)]_n$ so $f(q)$ must be a diagonal matrix in $M_2(F)$. In particular, let $f(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$, then $f(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $M_2(F)$, which leads to $d = 0$, a contradiction. This completes the proof of the theorem. \square

Theorem 2.2. *Let R be a prime ring with $\text{Char } R \neq 2$, I a nonzero ideal of R , and m, n fixed positive integers. If R admits a derivation d such that $[d(x), d(y)]_m = [x, y]^n$ for all $x, y \in I$, then R is commutative.*

Proof. If $d = 0$, then $[x, y]^n$ for all $x, y \in I$, and hence R is commutative by Herstein [10, Theorem 2]. Hence, onward we will assume that $d \neq 0$ and $[d(x), d(y)]_m = [x, y]^n$ for all $x, y \in I$. If d is not Q -inner then by Kharchenko [12] we have from the assumption that $[s, t]_m = [x, y]^n$ for all $x, y, s, t \in I$. In particular, for $s = 0$ we have $[x, y]^n = 0$ for all $x, y \in I$, and R is commutative by Herstein [10, Theorem 2]. If d is a Q -inner derivation, say $d(x) = [q, x]$ for all $x \in R$ and $q \in Q$, then we have $[[q, x], [q, y]]_m = [x, y]^n$ for all $x, y \in I$. As in the proof Theorem 2.1, we see that $[[q, x], [q, y]]_m = [x, y]^n$ for all $x, y \in R$, where R is a primitive ring with C as the associated division ring. If V is finite-dimensional over C then the density of R implies that $R \cong M_k(C)$, where $k = \dim_C V$.

We assume that $\dim_C V \geq 2$, otherwise we are done. We claim that v and qv are linearly C -dependent for all $v \in V$. Suppose that v and qv are linearly C -independent for some $v \in V$. If $q^2v \notin \text{Span}_C\{v, qv\}$ then v, qv, q^2v are linearly C -independent. By the density of R there exist $x, y \in R$ such that $xv = v, xqv = 0, xq^2v = 0; yv = 0, yqv = v, yq^2v = 3qv$. Then $qv = [[q, x], [q, y]]_m v = [x, y]^n v = 0$, a contradiction. If $q^2v \in \text{Span}_C\{v, qv\}$, then $q^2v = \alpha v + \beta qv$ for some $\alpha, \beta \in C$. Since v and qv are linearly C -independent, by the density of R there exist $x, y \in R$ such that $xv = v, xqv = 0; yv = 0, yqv = v$. Then $(-1)^m(2^m qv - \gamma v) = [[q, x], [q, y]]_m v = [x, y]^n v = 0$ for some $\gamma \in C$, which implies that $2^m qv = \gamma v$. The assumption of $\text{Char } R \neq 2$ ensures that $\gamma \neq 0$ and hence v and qv are linearly C -dependent, a contradiction. So for each $v \in V$, $qv = v\alpha_v$ for some $\alpha_v \in C$. By a standard argument, it is easy to see that α_v is independent of the choice of $v \in V$. Thus we can write $qv = v\alpha$ for all $v \in V$ and a fixed $\alpha \in C$. Reasoning as in the proof of Theorem 2.1, we conclude that $d = 0$, again a contradiction. \square

The following example demonstrates that R to be prime is essential in the hypothesis.

Example 2.1. Let S be any ring, $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in S \right\}$ and let $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S \right\}$ be a nonzero ideal of R . We define a map $d: R \rightarrow R$ by $d(x) = e_{11}x - xe_{11}$. Then it is easy to see that d is a derivation. It is straightforward to check that d satisfies the properties

- (i) $(d[x, y])^m = [x, y]_n$
- (ii) $[d(x), d(y)]_m = [x, y]_n$ for all $x, y \in I$. However, R is not commutative.

3. THE CASE: R A SEMIPRIME RING

Theorem 3.1. *Let R be a semiprime ring and m, n fixed positive integers. If R admits a derivation d such that $(d[x, y])^m = [x, y]_n$ for all $x, y \in R$, then R is commutative.*

Proof. By Beidar [1] any derivation of a semiprime ring R can be defined on the whole U , the Utumi quotient ring of R . In view of Lee [14], R and U satisfy the same differential identities, hence $(d[x, y])^m = [x, y]_n$ for all $x, y \in U$.

Let B be the complete Boolean algebra of idempotents in C and let M be any maximal ideal of B . Due to Chuang [5, p. 42] U is an orthogonal complete B -algebra and MU is a prime ideal of U , which is d -invariant. Denote $\overline{U} = U/MU$ and let \overline{d} be the derivation induced by d on \overline{U} , i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. Therefore \overline{d} has in \overline{U} the same property as d on U . In particular, \overline{U} is prime and so, by Theorem 2.1, \overline{U} is commutative. This implies that, for any maximal ideal M of B , $[U, U] \subseteq MU$ and hence $[U, U] \subseteq \bigcap_M MU = 0$, where MU runs over all prime ideals of U . In particular, $[R, R] = 0$ and so R is commutative. \square

Using arguments similar to those used in the proof of the above theorem, we can prove

Theorem 3.2. *Let R be a semiprime ring with $\text{Char } R \neq 2$ and m, n fixed positive integers. If R admits a derivation d such that $[d(x), d(y)]_m = [x, y]_n$ for all $x, y \in R$, then R is commutative.*

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