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SMOOTHED PROLONGATION MULTIGRID WITH RAPID COARSENING AND MASSIVE SMOOTHING*

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Abstract. We prove that within the frame of smoothed prolongations, rapid coarsening between first two levels can be compensated by massive prolongation smoothing and pre- and post-smoothing derived from the prolongator smoother.

Keywords: smoothed prolongations, rapid coarsening, massive smoothing

MSC 2010: 35J25, 65M55

1. Introduction

In multigrid context, assuming no regularity it is impossible to improve convergence by adding more smoothing steps and therefore rapid coarsening cannot be compensated by massive smoothing. In this paper we show that this is possible if, alongside with adding more smoothing steps, also prolongator smoothing is performed. We prove that within the frame of smoothed prolongations ([5], [4], [6]), rapid coarsening between first two levels can be compensated by massive prolongation smoothing and pre- and post-smoothing derived from the prolongator smoother while preserving coarse-level matrices reasonably sparse. We prove an abstract convergence result and apply it to the simplest regular multigrid for the twodimensional model problem with H^1_0-equivalent form on a unit square. We prove a uniform

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convergence result for the case that finest level prolongator smoother, pre- and post-smoothers are based on transformed Chebyshev polynomials in matrix A of degree about $\frac{1}{2}h_2/h_1$, where h_1 and h_2 are resolutions on first and second level. By uniform convergence result we mean rate of convergence independent of h_1, h_2/h_1 and dependent linearly on number of levels as in [1]. In [3], we presented such result for variational two-level method. Here, we extend it to multigrid V-cycle with rapid coarsening between first two levels. Our theory is based on interpretation of a complex key result in [3] and is simpler and more general. Restriction to V-cycle is only for the sake of brevity and extension to W-cycle is trivial. Paper presupposes basic expertise in multigrid.

2. Algorithm

We are interested in solving the system of linear algebraic equations

$$Ax = f, \tag{2.1}$$

where A is an $n \times n$ symmetric positive definite matrix and $f \in \mathbb{R}^n$. Set $n_1 = n$. We consider a standard variational multigrid V-cycle with prolongators

$$I_{l+1}^{l} \colon \mathbb{R}^{n_{l+1}} \to \mathbb{R}^{n_l}, \quad l = 1, \ldots, L-1, \; n_{l+1} < n_l, \tag{2.2}$$

that is, the multigrid algorithm with coarse level matrices

$$A_l = (I_l^1)^\top A I_l^1, \quad I_l^1 = I_2^1 \ldots I_l^{l-1} \quad \text{for } l > 2 \text{ and } I_1^1 = I,$$

and restrictions given by the transpose of prolongators. With abstract convergence theory [1] in mind we define coarse spaces U_l, associated norms $\|\cdot\|_l$ and inner products $(\cdot,\cdot)_l$ by

$$U_l = \operatorname{Rng}(I_l^1), \quad (\cdot,\cdot)_l \colon I_l^1 x, I_l^1 y \mapsto \sum_{i=1}^{n_l} x_i y_i \quad \text{and} \quad \|\cdot\|_l = (\cdot,\cdot)_l^{1/2}. \tag{2.3}$$

In this paper we are interested in rapid coarsening between the first two levels compensated by massive prolongator smoothing and massive pre- and post-smoothing derived from the prolongator smoother. More specifically, we will investigate the multigrid V-cycle with the prolongator I_2^1 in the form

$$I_2^1 = S\tilde{I}_2^1, \quad \tilde{I}_2^1 \colon \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}, \quad n_2 \ll n_1,$$

where the *prolongator smoother* S is the tranformed Chebyshev polynomial in A such that

$$\varrho(S^2 A) \ll \varrho(A). \tag{2.4}$$

In what follows we assume that S is a polynomial in A satisfying

$$\varrho(S) \leqslant 1. \tag{2.5}$$

With (2.4) in mind a specific choice of the operator S will be given in Section 4. On finest level we choose pre-smoother with error propagation operator S and post-smoother with error propagation operator

$$S' = I - \frac{\omega}{\varrho(A_S)} A_S, \quad A_S = S^2 A, \ \omega \in (0,1). \tag{2.6}$$

Note that resulting iteration can be implemented by formula

$$x \leftarrow \Big(I - \frac{\omega}{\varrho(A_S)} A_S\Big)x + \frac{\omega}{\varrho(A_S)} S^2 f.$$

On other levels we choose both pre- and post-smoothers to be simple Jacobi iterations with error propagation operator

$$I - \frac{\omega}{\varrho(A_l)} A_l. \tag{2.7}$$

The choice of smoothers will be justified in next section. In Section 4 we propose operator S in the form $S = (I - \alpha_1 A) \dots (I - \alpha_d A)$. Finest level pre-smoother can be then implemented as sequence of Jacobi sweeps $x \leftarrow (I - \alpha_i A)x + \alpha_i f$, $i = 1, \dots, d$. Note that smoothers (2.7) satisfy smoothing condition of [1].

3. Analysis

The error propagation operator of our algorithm is

$$E = S'(I - I_2^1 V_2 (I_2^1)^\top A)S = S'(I - S\tilde{I}_2^1 V_2 (S\tilde{I}_2^1)^\top A)S, \tag{3.1}$$

where V_2 is a single iteration of V-cycle on level 2 started from the zero approximation. By direct calculation,

$$E = S'S(I - \tilde{I}_2^1 V_2 (\tilde{I}_2^1)^\top A_S) = SS'(I - \tilde{I}_2^1 V_2 (\tilde{I}_2^1)^\top A_S). \tag{3.2}$$

Let $u \in U_1$. It is well known that since both the pre- and post-smoothers on all coarse levels are simple Jacobi smoothers, the coarse-grid correction part of the error propagation operator of the variational multigrid V-cycle is A-symmetric and satisfies

$$\|I - I_2^1 V_2 (I_2^1)^\top A\|_A \leqslant 1,$$

where $\|\cdot\|_A = (A\cdot,\cdot)_1^{1/2}$. Moreover, $\|S'\|_A \leqslant 1$ and therefore by (3.1),

$$\|Eu\|_A \leqslant \|S'\|_A \|I - I_2^1 V_2 (I_2^1)^\top A\|_A \|Su\|_A \leqslant \|Su\|_A. \tag{3.3}$$

Further, setting

$$R = I - \tilde{I}_2^1 V_2 (\tilde{I}_2^1)^\top A_S$$

and using (3.2) and (2.5), we get

$$\begin{aligned} \frac{\|Eu\|_A}{\|u\|_A} = \frac{\|SS'Ru\|_A}{\|u\|_{A_S}} \cdot \frac{\|u\|_{A_S}}{\|u\|_A} &= \frac{\|S'Ru\|_{A_S}}{\|u\|_{A_S}} \cdot \frac{\|Su\|_A}{\|u\|_A} \\ &\leqslant \frac{\|S'(I - \tilde{I}_2^1 V_2 (\tilde{I}_2^1)^\top A_S)u\|_{A_S}}{\|u\|_{A_S}}, \end{aligned} \tag{3.4}$$

where $\|\cdot\|_{A_S} = (A_S\cdot,\cdot)_1^{1/2}$. Let $t \in (0,1)$ be a given parameter. We set

$$W = \Big\{ u \in U \colon \frac{\|Su\|_A}{\|u\|_A} \geqslant t \Big\}.$$

Note that

$$t\|u\|_A \leqslant \|u\|_{A_S} = \|Su\|_A \leqslant \|u\|_A \quad \text{for all } u \in W. \tag{3.5}$$

In view of (3.3) and (3.4) it becomes clear that to establish a convergence estimate for our method it is sufficient to estimate the term on the right-hand side of (3.4) for all $u \in W$. We will do so using the equivalence (3.5) as a key argument. Note that the term $S'(I - \tilde{I}_2^1 V_2 (\tilde{I}_2^1)^\top A_S)$ on the right-hand side of (3.4) is an error propagation operator of the multigrid V-cycle for solving the problem with the smoothed matrix A_s, prolongators $\tilde{I}_2^1, I_3^2, \ldots, I_L^{L-1}$, Jacobi post-smoother S' on the finest level and simple Jacobi pre and post smoothers (2.7) on the other levels.

Theorem 3.1 ([1]). *Let E be the error propagation operator of the variational multigrid V-cycle algorithm for solving problem* (2.1) *with prolongators* (2.2) *and simple Jacobi pre- and/or post-smoothers* (2.7) *on all levels $l = 1, \ldots, L-1$. Let $V \subset U_1$ be a subset. We assume that there are constants $C_1, C_2 > 0$ and linear mappings $Q_l \colon U_1 \to U_l$, $l = 2, \ldots, L$, $Q_1 = I$ such that for every $u \in V$*

$$\|(Q_l - Q_{l+1})u\|_l \leqslant \frac{C_1}{\sqrt{\varrho(A_l)}} \|u\|_A, \quad l = 1, \ldots, L-1, \tag{3.6}$$

and

$$\|Q_l u\|_A \leqslant C_2 \|u\|_A, \quad l = 1, \ldots, L. \tag{3.7}$$

Then for every $u \in V$

$$\|Eu\|_A \leqslant 1 - \frac{1}{CL} \|u\|_A, \quad \text{where } C = \Bigl(1 + C_2^{1/2} + \Bigl(\frac{C_1}{\omega}\Bigr)^{1/2}\Bigr)^2.$$

P r o o f. Proof follows by the original proof of [1]. $\square$

We apply the above theorem, choosing $V = W$, to estimate the right-hand side of (3.4). Recall that the term

$$S'(I - \tilde{I}_2^1 V_2 (\tilde{I}_2^1)^\top A_S)$$

is the variational multigrid V-cycle for solving the problem with the matrix A_S, prolongators $\tilde{I}_2^1, I_3^2, \ldots, I_L^{L-1}$ and Jacobi post-smoother (2.6) on the finest level and Jacobi pre- and post-smoothers (2.7) on the other levels. The coarse level matrices are given by

$$\tilde{A}_l = (\tilde{I}_l^1)^\top A_S \tilde{I}_l^1, \quad l = 1, \ldots, L,$$

where $\tilde{I}_l^1 = \tilde{I}_2^1 I_3^2 \ldots I_l^{l-1}$ for $l > 2$, $\tilde{I}_1^1 = I$. Note that $\tilde{A}_1 = A_S$ and $\tilde{A}_l = A_l$ for $l > 1$. We define coarse spaces and associated norms by

$$\tilde{U}_l = \mathrm{Rng}(\tilde{I}_l^1), \quad \|\cdot\|_{\tilde{U}_l} \colon \tilde{I}_l^1 x \mapsto \left(\sum_{i=1}^{n_l} x_i^2\right)^{1/2}, \quad x \in \mathbb{R}^{n_l}.$$

Note that $\tilde{U}_1 = U_1 = \mathbb{R}^{n_1}$.

Theorem 3.2. *Assume there are constants $C_1, C_2 > 0$ and linear mappings $\tilde{Q}_l\colon \tilde{U}_1 \to \tilde{U}_l$, $l = 2, \ldots, L$, $\tilde{Q}_1 = I$ such that for every $u \in \tilde{U}_1$*

$$\|(\tilde{Q}_l - \tilde{Q}_{l+1})u\|_{\tilde{U}_l} \leqslant \frac{C_1}{\sqrt{\varrho(\tilde{A}_l)}} \|u\|_A, \quad l = 1, \ldots, L-1, \tag{3.8}$$

and

$$\|\tilde{Q}_l\|_A \leqslant C_2, \quad l = 1, \ldots, L. \tag{3.9}$$

Then the error propagation operator E of our method satisfies

$$\|Eu\|_A \leqslant q\|u\|_A \quad \forall\, u \in \tilde{U}_1 \tag{3.10}$$

with a positive constant $q < 1$ dependent only on C_1, C_2 and a parameter ω in (2.6) *and* (2.7).

Proof. Assume $u \notin W$. Then (3.10) follows by (3.3) with $q = t$. Let $u \in W$. We use Theorem 3.1 choosing $V = W$ to estimate the right-hand side of (3.4). We need to verify

$$\|(\tilde{Q}_l - \tilde{Q}_{l+1})u\|_{\tilde{U}_l} \leqslant \frac{C_a}{\sqrt{\varrho(\tilde{A}_l)}} \|u\|_{A_S}, \quad l = 1, \ldots, L-1 \tag{3.11}$$

and

$$\|\tilde{Q}_l u\|_{A_S} \leqslant C_s \|u\|_{A_s}, \quad l = 1, \ldots, L \tag{3.12}$$

for every $u \in W$. Using the equivalence (3.5), (3.11) follows from (3.8) with $C_a = C_1/t$. Further, $\|\tilde{Q}_l u\|_{A_s} = \|S\tilde{Q}_l u\|_A \leqslant \|\tilde{Q}_l u\|_A$ and $\|u\|_{A_S} \geqslant t\|u\|_A$ from (3.5). Therefore,

$$\frac{\|\tilde{Q}_l u\|_{A_S}}{\|u\|_{A_S}} \leqslant \frac{1}{t} \cdot \frac{\|\tilde{Q} u\|_A}{\|u\|_A}$$

and (3.12) follows from (3.9) with $C_s = C_2/t$. Hence, by Theorem 3.1 we have the estimate (3.10) with

$$q = 1 - \frac{1}{\left(1 + \left(\frac{C_2}{t}\right)^{1/2} + \left(\frac{C_1}{\omega t}\right)^{1/2}\right)^2 L}, \quad u \in W$$

and for $u \notin W$ we have (3.10) with $q = t$, where t is a paremeter we choose. Minimizing

$$\min_{t \in (0,1)} \max\{t, q(C_1, C_2, \omega, t)\}$$

eliminates the dependence of the estimate q on t. □

When proving convergence of the multigrid in case of rapid coarsening between levels 1 and 2 by means of Theorem 3.1, the critical issue is to verify the approximation condition (3.6) for $l=1$. If $\varrho(S^2A) \ll \varrho(A)$, the condition (3.8) is much easier to satisfy than (3.6).

4. Prolongator smoother

Lemma 4.1 ([2]). *For any $\varrho > 0$ and the integer $d > 0$ there is a unique polynomial p of degree d such that*

$$\max\{p^2(\lambda)\lambda \colon\ 0 \leqslant \lambda \leqslant \varrho\}$$

is minimal under the constraint $p(0)=1$. The polynomial p is given by

$$p(\lambda) = \Bigl(1-\frac{\lambda}{r_1}\Bigr)\ldots\Bigl(1-\frac{\lambda}{r_d}\Bigr), \quad r_k = \frac{\varrho}{2}\Bigl(1-\cos\frac{2k\pi}{2d+1}\Bigr), \tag{4.1}$$

$k=1,\ldots,d$. The polynomial p satisfies

$$\max_{0\leqslant\lambda\leqslant\varrho} p^2(\lambda)\lambda = \frac{\varrho}{(2d+1)^2} \tag{4.2}$$

and

$$\max_{0\leqslant\lambda\leqslant\varrho} |p(\lambda)| = 1. \tag{4.3}$$

We choose S to be the polynomial (4.1) in A with $\varrho = \varrho(A)$. For simplicity we assume that the spectral bound $\varrho(A)$ is available. To use the upper bound $\varrho = \bar{\varrho}$ satisfying

$$\varrho(A) \leqslant \bar{\varrho} \leqslant C\varrho(A)$$

represents a minimal technical problem.

Using the spectral mapping theorem and (4.3), we get (2.5). By the spectral mapping theorem and (4.2) we get

$$\varrho(A_S) \leqslant \frac{1}{(2\deg(S)+1)^2}\,\varrho(A). \tag{4.4}$$

5. Application to regular multigrid for problem with H_0^1-equivalent form

The purpose of this section is to demonstrate the abstract result on the simplest possible nontrivial case. A more natural application to smoothed aggregation spaces will be published elsewhere.

Let Ω be unit square. We consider the second order elliptic problem

$$\text{find } u \in H_0^1(\Omega) \text{ such that } a(u,v) = (f,v)_{L^2(\Omega)} \quad \forall\, v \in H_0^1(\Omega),$$

where $f \in L^2(\Omega)$ and the bilinear form $a(\cdot,\cdot)$ satisfies

$$c\|u\|^2_{H^1(\Omega)} \leqslant a(u,u) \leqslant C\|u\|^2_{H^1(\Omega)} \quad \forall\, u \in H_0^1(\Omega).$$

We assume a system of nested regular triangulations τ_{h_l} with meshsizes h_l, where

$$h_2 = N h_1, \quad h_{l+1} = 2h_l, \quad l = 2,\dots,L,$$

and N, L are integers $N > 2$, $L \geqslant 2$. Let n_l be the number of interior vertices of τ_{h_l}. On each level we consider the standard P1 finite element basis $\{\varphi_i^l\}_{i=1}^{n_l}$ and denote the corresponding finite element space by V_{h_l}. We assume the standard scaling $\|\varphi_i^l\|_{L^\infty(\Omega)} = 1$. It is well known that the corresponding stiffness matrices satisfy

$$\varrho\{a(\varphi_l^l,\varphi_j^l),\ i,j = 1,\dots,n_l\} \leqslant C. \tag{5.1}$$

Here and in what follows c, C are positive constants independent of the meshsize on any level and the number of levels. We assume that the prolongators $\tilde{I}_2^1, I_3^2, \dots, I_L^{L-1}$ are constructed so that

$$\Pi_h \tilde{I}_l^1 e_i = \varphi_i^l, \quad i = 1,\dots,n_l, \quad \Pi_h \colon\, x \in \mathbb{R}^{n_1} \mapsto \sum_{i=1}^{n_l} x_i \varphi_i^1.$$

Here, e_i denotes the ith canonical basis vector. Note that

$$V_{h_l} = \{\Pi_h u,\ u \in \tilde{U}_l\} \quad \text{and} \quad (\tilde{I}_l^1)^\top A(\tilde{I}_l^1) = \{a(\varphi_l^l,\varphi_j^l),\ i,j = 1,\dots,n_l\}.$$

In finite element stiffness matrices, $a_{ij} \neq 0$ if the vertex j belongs to an element adjacent to the vertex i. Ussage of the prolongator smoother of degree 1 causes that the fill-in of all coarse level matrices A_l increases; an entry a_{ij} of A_l, $l = 2,\dots,L$ becomes nonzero if the vertex j of τ_{h_l} belongs to two layers of elements adjacent to the vertex i. We choose the prolongator smoother S, of the largest degree such that coarse-level matrices A_l, $l > 1$, have such pattern. It is routine to verify that such degree is the nearest integer that is smaller or equal to $\frac{1}{2}h_2/h_1$. Then (4.4) and (5.1) give

$$\varrho(\tilde{A}_1) = \varrho(A_s) \leqslant C\Big(\frac{h_1}{h_2}\Big)^2. \tag{5.2}$$

In what follows we will verify assumptions (3.8) and (3.9). We choose $\tilde{Q}_l$, $l = 2, \ldots, L$ so that $\Pi_h \tilde{Q}_l$ is the $L^2(\Omega)$-orthogonal projection onto V_{h_l}. The following are well-known properties of finite element functions:

$$ch_l\|u\|_{\tilde{U}_l} \leqslant \|\Pi_h u\|_{L^2(\Omega)} \leqslant Ch_l\|u\|_{\tilde{U}_l}, \quad u \in \tilde{U}_l, \tag{5.3}$$

$$\|\Pi_h(I - \tilde{Q}_l)u\|_{L^2(\Omega)} \leqslant Ch_l|\Pi_h u|_{H^1(\Omega)}, \quad u \in \tilde{U}_1, \tag{5.4}$$

$$|\Pi_h \tilde{Q}_l u|_{H^1(\Omega)} \leqslant C|\Pi_h u|_{H^1(\Omega)}, \quad u \in \tilde{U}_1. \tag{5.5}$$

Clearly,

$$c|\Pi_h u|_{H^1(\Omega)} \leqslant a(\Pi_h u, \Pi_h u) = \|u\|_A^2 \leqslant C|\Pi_h u|_{H^1(\Omega)}, \quad u \in \tilde{U}_1 \tag{5.6}$$

and (3.9) follows from (5.5).

Using (5.4) and the well-known properties of the orthogonal projections we get

$$\|\Pi_h(\tilde{Q}_l - \tilde{Q}_{l+1})u\|_{L^2(\Omega)} \leqslant \|\Pi_h(I - \tilde{Q}_{l+1})u\|_{L^2(\Omega)} \leqslant Ch_{l+1}|\Pi_h u|_{H^1(\Omega)}.$$

Hence by (5.3) and (5.6),

$$\|(\tilde{Q}_l - \tilde{Q}_{l+1})u\|_{\tilde{U}_l} \leqslant C\frac{h_{l+1}}{h_l}\,\|u\|_A. \tag{5.7}$$

This together with (5.2) gives (3.8) for $l = 1$. For $l > 1$ we first estimate

$$\varrho(\tilde{A}_l) = \sup \frac{(\tilde{I}_l^1 x)^\top S^2 A(\tilde{I}_l^1 x)}{x^\top x} \leqslant \varrho((I_l^1)^\top A I_l^1).$$

Since $(I_l^1)^\top A I_l^1$ is a stiffness matrix corresponding to the basis $\{\varphi_i^l\}$, the last inequality together with (5.1) gives

$$\varrho(\tilde{A}_l) \leqslant \varrho((I_l^1)^\top A I_l^1) \leqslant C \quad \text{for } l = 2, \ldots, L.$$

This estimate, $h_{l+1} = 2h_l$ and (5.7) give (3.8) for $l > 2$.

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