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EXISTENCE OF WEAK SOLUTIONS TO DOUBLY DEGENERATE DIFFUSION EQUATIONS*

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Abstract. We prove existence of weak solutions to doubly degenerate diffusion equations

$$\dot{u} = \Delta_p u^{m-1} + f \quad (m, p \geqslant 2)$$

by Faedo-Galerkin approximation for general domains and general nonlinearities. More precisely, we discuss the equation in an abstract setting, which allows to choose function spaces corresponding to bounded or unbounded domains $\Omega \subset \mathbb{R}^n$ with Dirichlet or Neumann boundary conditions. The function f can be an inhomogeneity or a nonlinearity involving terms of the form $f(u)$ or $\operatorname{div}(F(u))$. In the appendix, an introduction to weak differentiability of functions with values in a Banach space appropriate for doubly nonlinear evolution equations is given.

Keywords: p-Laplacian, doubly nonlinear evolution equation, weak solution

MSC 2010: 35K92, 35D30, 37L65

Introduction

We study the quasilinear parabolic equation

$$\dot{u} = \Delta_p u^{m-1} + f$$

on $(0,T)\times\Omega$ for a domain $\Omega \subset \mathbb{R}^n$, where $u^{m-1} := |u|^{m-2}u$ denotes the signed power, $\Delta_p u := \operatorname{div}((\nabla u)^{p-1})$ is the p-Laplacian and f is a nonlinearity possibly depending on t, x, u^{m-1}, and ∇u^{m-1}.

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(i) The case $p = 2$, $m = 2$, is the ordinary semilinear diffusion equation with nonlinearity f.

(ii) The case $p = 2$, $m \neq 2$, is the porous media equation, it is degenerate at $u = 0$ for $2 < m < \infty$ and singular at $u = 0$ for $1 < m < 2$.

(iii) The case $p \neq 2$, $m = 2$, is the p-diffusion equation, it is degenerate at $\nabla u = 0$ for $2 < p < \infty$ and singular at $\nabla u = 0$ for $1 < p < 2$.

(iv) The case $p \neq 2$, $m \neq 2$, is the doubly nonlinear diffusion equation, singularity and degeneracy at $u = 0$ and $\nabla u = 0$, respectively, occur in arbitrary combinations.

(v) A doubly nonlinear diffusion is called slow (normal, fast), if $(m-1)(p-1) > 1$ $(= 1, < 1)$, or equivalently $mp > m'p'$ $(= m'p', < m'p')$.

Our main aim is to study existence of weak solutions to doubly nonlinear diffusion equations for parameters $m, p \geqslant 2$, where the equation is doubly degenerate, but we also comment on general slow and normal diffusions.

The existence of weak solutions to doubly degenerate diffusion equations has been proved by miscellaneous methods for different types of nonlinearities and domains, see e.g. [6], [8], [14]. However, if fully-discretized or time-discretized Galerkin methods like Rothe's method are used to prove the existence of solutions, then the estimates often strongly depend on assumptions about the domain or the type of the nonlinearity. Thus it is not so easy to say, how the results obtained by such a method can be transfered to other types of domains or nonlinearities. Further, if the implicit Euler scheme and the theorem of Crandall-Liggett are used to prove the existence of solutions (see e.g. [3]), then it is not so clear whether the constructed solution is also a weak solution.

In this paper we give an elementary proof of the existence of weak solutions by a Faedo-Galerkin method, which is valid for a large class of domains and nonlinearities. Particularly, it is easy to see where additional assumptions allow to prove stronger results.

In the first section the appropriate notion of weak solutions to doubly nonlinear diffusion equations with inhomogeneities is developed. The existence of weak solutions is proved in the second section by a Faedo-Galerkin method. To the best of our knowledge, this has not been done before in such a general situation. Finally, in the third section, the existence of weak solutions in presence of nonlinearities is discussed. In the appendix, the basic theory of Banach space valued functions on intervals adapted to doubly nonlinear evolution equations is presented.

1. Weak solutions

A modern treatment of a partial differential equation like

$$\dot{u} = \Delta_p u^{m-1} + f$$

requires to realize the partial differential equation as an equation in a Banach space, where unique solvability can be guaranteed by functional analytic methods, and solutions of this Banach space equation are called weak solutions.

To obtain the appropriate notion of a weak solution for the particular case of doubly nonlinear diffusion equations, let us first reformulate the equation. The function $\Phi\colon L^{m'} \to \mathbb{R}$ defined by

$$\Phi\colon u \mapsto \frac{1}{m'}\|u\|_{m'}^{m'}$$

has the derivative $\varphi\colon L^{m'} \to L^m$, $\varphi(u) = u^{m'-1}$, and φ satisfies $\varphi(u^{m-1}) = u$. Thus substituting u^{m-1} in the original equation $\dot{u} = \Delta_p u^{m-1} + f$ leads to the reformulated equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u) = \Delta_p u + f. \tag{1.1}$$

R e m a r k. If u is a solution of the reformulated equation, then u^{m-1} is a solution of the original equation, and if the original nonlinearity has the form $f(u^{m-1}, \nabla u^{m-1})$, then the nonlinearity in the reformulated equation is $f(u, \nabla u)$.

Note that in the reformulation the double nonlinearity in space has been removed at the price of single nonlinearities in space and time. From now on we work with the reformulated equation.

To establish the appropriate functional analytic setup for weak solutions of the reformulated equation, let X_p be a reflexive Banach space of functions on Ω with a norm (equivalent to) $\|\nabla u\|_p$, or at least a space of distributions on Ω such that $X_p \cap L^{m'}$ is a reflexive Banach space with a norm (equivalent to) $\|\nabla u\|_p + \|u\|_{m'}$. Further, X_p is assumed to be such that the (nonlinear) map $\varphi\colon X_p \cap L^{m'} \to L^m$ is compact, at least when restricted to bounded subdomains Ω' of Ω. For the following examples of such spaces X_p, compactness of φ is guaranteed by Lemma B.1:

(i) For arbitrary $\Omega \subset \mathbb{R}^n$ (possibly unbounded) let X_p be the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\nabla u\|_p$.

The choice of this space corresponds to Dirichlet boundary conditions on $\partial\Omega$. If Ω is bounded in at least one direction, then X_p is identical with $W_0^{1,p}(\Omega)$ by Poincaré's inequality $\|u\|_p \leqslant C\|\nabla u\|_p$ for $u \in C_c^\infty(\Omega)$. For arbitrary $\Omega \subset \mathbb{R}^n$

and $p < n$, the space X_p is by Sobolev's inequality identical with the space of functions $u \in L^{p^*}(\Omega)$ having a distributional derivative $\nabla u \in L^p(\Omega, \mathbb{R}^n)$.

(ii) For arbitrary $\Omega \subset \mathbb{R}^n$ (possibly unbounded) satisfying the cone condition (e.g. for Ω with C^1-boundary) let X_p be the space of distributions on Ω with distributional derivative $\nabla u \in L^p(\Omega, \mathbb{R}^n)$.

The choice of this space corresponds to Neumann boundary conditions on $\partial\Omega$, and $X_p \cap L^{m'}$ is the reflexive Banach space of functions $u \in L^{m'}(\Omega)$ with distributional derivative $\nabla u \in L^p(\Omega, \mathbb{R}^n)$.

The p-Laplacian can be considered as an operator $\Delta_p \colon L^p(0,T;X_p) \to L^{p'}(0,T;X_p')$, $\langle \Delta_p u, v\rangle := \int_\Omega (\nabla u)^{p-1} \nabla v \,\mathrm{d}x$, due to

$$|\langle \Delta_p u, v\rangle| = \left| -\int_\Omega (\nabla u)^{p-1} \nabla v \,\mathrm{d}x \right| \leqslant \| |\nabla u|^{p-1} \|_{p'} \|\nabla v\|_p = \|\nabla u\|_p^{p-1} \|\nabla v\|_p$$

for $u, v \in X_p$, and

$$\int_0^T \|\Delta_p u\|_{X_p'}^{p'} \,\mathrm{d}t \leqslant \int_0^T (\|\nabla u\|_p^{p-1})^{p'} \,\mathrm{d}t = \int_0^T \|\nabla u\|_p^p \,\mathrm{d}t$$

for $u \in L^p(0,T;X_p)$. This also shows that Δ_p maps bounded sets in $L^p(0,T;X_p)$ to bounded sets in $L^{p'}(0,T;X_p')$.

Moreover, the map φ induces a map from $L^\infty(0,T;L^{m'})$ to $L^\infty(0,T;L^m)$ because of $\|\varphi(u)\|_m^m = \|u\|_{m'}^{m'}$, and bounded sets in $L^\infty(0,T;L^{m'})$ are mapped to bounded sets in $L^\infty(0,T;L^m)$ by φ.

Now consider $u \in L^p(0,T;X_p) \cap L^\infty(0,T;L^{m'})$. Then on the one hand $\varphi(u) \in L^\infty(0,T;L^m)$, and on the other hand—at least for $f \in L^{p'}(0,T;X_p')$ not depending on u—the right-hand side of the equation $(\mathrm{d}/\mathrm{d}t)\varphi(u) = \Delta_p u + f$ lies in $L^{p'}(0,T;X_p')$. Hence, the left-hand side $(\mathrm{d}/\mathrm{d}t)\varphi(u)$ should also lie in $L^{p'}(0,T;X_p')$.

Recall from appendix A that by definition a function $\varphi(u) \in L^\infty(0,T;L^m)$ is weakly differentiable with weak derivative $(\mathrm{d}/\mathrm{d}t)\varphi(u) \in L^{p'}(0,T;X_p')$ and initial value $\varphi(u(0)) \in L^m$, if

$$\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\varphi(u), v - v(T) \right\rangle = -\int_0^T \langle \varphi(u) - \varphi(u(0)), \dot{v} \rangle$$

holds for all $v \in L^p(0,T;X_p)$ with weak derivative $\dot{v} \in L^1(0,T;L^{m'})$ and final value $v(T) \in X_p$.

Therefore, we are led to the following definition of a weak solution:

Definition 1.1. Let $f \in L^{p'}(0,T;X_p')$ be an inhomogeneity. A function u is called a weak solution of $(\mathrm{d}/\mathrm{d}t)\varphi(u) = \Delta_p u + f$ with the initial value $u(0) \in L^{m'}$, if

- $u \in L^p(0,T;X_p) \cap L^\infty(0,T;L^{m'})$ is such that the function $\varphi(u) \in L^\infty(0,T;L^m)$ has a weak derivative $(\mathrm{d}/\mathrm{d}t)\varphi(u) \in L^{p'}(0,T;X_p')$ and initial value $\varphi(u(0)) \in L^m$,
- the equation $(\mathrm{d}/\mathrm{d}t)\varphi(u) = \Delta_p u + f$ is valid in the sense that the integral equation

$$-\int_0^T \langle \varphi(u) - \varphi(u(0)), \dot{v} \rangle = -\int_0^T \langle (\nabla u)^{p-1}, \nabla v \rangle + \int_0^T \langle f, v \rangle \tag{1.2}$$

holds for all $v \in L^p(0,T;X_p)$ with weak derivative $\dot{v} \in L^1(0,T;L^{m'})$ and final value $v(T) = 0$.

Note that by this definition we have realized equation (1.1) as an equation in a Banach space. Further, the Banach space $L^p(0,T;X_p) \cap L^\infty(0,T;L^{m'})$ is the appropriate one adapted to the problem, not only by the former argumentation, but also because the fundamental a priori estimate

$$\frac{1}{m}\|u(t)\|_{m'}^{m'} + \frac{1}{p'}\int_0^t \|\nabla u\|_p^p \leqslant \frac{1}{m}\|u(0)\|_{m'}^{m'} + \frac{1}{p'}\int_0^T \|f\|_{X_p'}^{p'}$$

proved in Section 2.2 provides bounds with respect to the norm in the chosen Banach space. These bounds and a compactness criterion allow to establish existence of weak solutions by a Faedo-Galerkin method.

Theorem 1.2. *Let $p, m \geqslant 2$, let X_p be a function space as above, let $f \in L^p(0,T;X_p')$ be an inhomogeneity and let $u(0) \in L^{m'}$. Then there exists a weak solution u in $L^p(0,T;X_p) \cap L^\infty(0,T;L^{m'})$ of the equation $(\mathrm{d}/\mathrm{d}t)\varphi(u) = \Delta_p u + f$ with the initial value $u(0)$.*

The proof of this theorem is given in Section 2.

Note that for various choices of X_p—as indicated above—we in particular obtain the existence of weak solutions for $\Omega = \mathbb{R}^n$, for bounded domains $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary, and for bounded C^1-domains $\Omega \subset \mathbb{R}^n$ with Neumann boundary.

Further, the existence can also be established for nonlinearities f depending on u. The corresponding results can be found in Section 3.

Finally, let us make some remarks about regularity and uniqueness of weak solutions. In [7], [5], [12] it is shown that weak solutions of doubly nonlinear diffusion equations with initial values $u(0) \in L^{m'}$ are instantly regularized to functions $u(t) \in L^\infty$ for $t > 0$. Using boundedness of $u(t)$, in the next step Hölder continuity of weak solutions in space can be verified (see e.g. [10]).

The uniqueness of weak solutions and continuous dependence on the data can be guaranteed in the special case $m = 2$ by testing the difference of two equations with the difference of the corresponding two solutions, but in the general case more advanced methods have to be used (see e.g. [9]).

2. The existence of weak solutions

We prove the existence of weak solutions by a Faedo-Galerkin method, which discretizes the evolution equation in space to obtain an ODE on a finite-dimensional space. Therefore, we first project the PDE to an ODE on a finite-dimensional subspace and prove existence of solutions to these approximate problems. Thus for an increasing sequence of finite-dimensional subspaces we obtain a sequence u_k of approximate solutions. Then we establish the fundamental a priori estimate, which allows to extract a weakly convergent subsequence of u_k. Afterwards, we show that the weak limit u of this subsequence is a weak solution.

2.1. Finite-dimensional approximations. The projection of the following PDE $(\mathrm{d}/\mathrm{d}t)\varphi(u) = \Delta_p u + f$ to an ODE on a finite-dimensional subspace of $X_p \cap L^{m'}$ with a basis w_j, $1 \leqslant j \leqslant k$, is given by

$$\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\varphi(u)(t), w_j \Big\rangle = \langle \Delta_p u(t), w_j \rangle + \langle f(t), w_j \rangle_{X_p}$$

under the constraint $u(t) \in \operatorname{span}(w_1, \ldots, w_k)$. As we interpret the PDE according to (1.2) in the integral form, the projected equation also makes sense only in the integral form

$$\langle \varphi(u(t)), w_j \rangle = \langle \varphi(u(0)), w_j \rangle - \int_0^t \langle (\nabla u)^{p-1}, \nabla w_j \rangle - \langle f(s), w_j \rangle_{X_p} \,\mathrm{d}s. \tag{2.1}$$

For technical reasons we additionally require that the chosen basis w_j of $X_p \cap L^{m'}$ also lies in L^∞, and instead of the projection of $\varphi(u(0))$ to $\operatorname{span}(w_1, \ldots, w_k)$ we choose the initial values $\varphi(u(0)_k)$ of the approximate equation so that $\varphi(u(0)_k) \to \varphi(u(0))$ strongly in $L^{m'}$.

Before we prove that for a given initial value the integral equation (2.1) has locally in time a unique solution, i.e. the ODE has a solution in the sense of Carathéodory, for the reader's convenience let us show that the integral equation corresponds to an implicit ODE under the assumption that $\varphi(u)$ can be differentiated according to the chain rule. In fact, let $u(t)$ be given in coordinates y_i with respect to the basis w_i by $u(t) = \sum_i y_i(t) w_i$, then under this assumption the projected equation $\langle (\mathrm{d}/\mathrm{d}t)\varphi(u), w_j \rangle = \langle \Delta_p u, w_j \rangle + \langle f, w_j \rangle$ would be equivalent to

$$\sum_i \Big\langle \frac{1}{m-1} \Big(\sum_k y_k w_k \Big)^{m'-2} w_i, w_j \Big\rangle_{L^m} \dot{y}_i = \Big\langle \Big(\sum_i y_i \nabla w_i \Big)^{p-1}, \nabla w_j \Big\rangle_{L^p} + \langle f, w_j \rangle,$$

and thus would have the form $B(y)\dot{y} = F(y)$ with a matrix B depending on y, while the initial value $y(0)$ can be computed from the equation $\sum_i y_i(0)\langle w_i, w_j\rangle_{L^m} = \langle u(0), w_j\rangle_{L^m}$. Note that the matrix B not only depends on y, but may also be not well-defined for certain values of y (e.g. for $y = 0$ and $m > 2$ it is divided by zero) or singular (e.g. for $m < 2$ and $y = 0$). Thus contrary to the case of semilinear equations or the case $m = 2$, the projected equation shoud not be viewed as an ODE, it only makes sense as an integral equation.

For $m, p \geqslant 2$ the integral equation (2.1) can be solved with help of Banach's fixed point theorem in analogy to the existence theorem of Picard-Lindelöf.

Lemma 2.1. *For $m, p \geqslant 2$ and an initial value $u(0) \in L^{m'}$ the projected equation* (2.1) *has locally in time a unique solution.*

Proof. We show that φ has a locally Lipschitz continuous inverse and that the right-hand side $\langle F(s,u), w_j\rangle := -\langle (\nabla u)^{p-1}, \nabla w_j\rangle + \langle f(s), w_j\rangle_{X_p}$ is locally Lipschitz continuous in u, hence the integral equation can be solved locally by Picard iteration.

To prove that φ^{-1} is locally Lipschitz continuous on the finite-dimensional subspace, let us prove

$$\|u - v\| \leqslant C\|\varphi(u) - \varphi(v)\|$$

for $m \geqslant 2$ and u, v near u_0 with a constant C depending on u_0. This inequality is valid due to the elementary inequality

$$|a - b|^2 \leqslant C(a^{m'-1} - b^{m'-1})(a - b)(|a| + |b|)^{2-m'}$$

for real numbers $a, b \in \mathbb{R}$ (proved e.g. in [4]), which implies by Hölder's inequality

$$\begin{aligned} \|u - v\|_2^2 &= \int_\Omega |u - v|^2 \,\mathrm{d}x \\ &\leqslant C \int_\Omega (u^{m'-1} - v^{m'-1})(u - v)(|u| + |v|)^{2-m'} \,\mathrm{d}x \\ &\leqslant C\|\varphi(u) - \varphi(v)\|_m \|u - v\|_{m'} \operatorname*{ess\,sup}_\Omega (|u| + |v|)^{2-m'}. \end{aligned}$$

As we are on a finite dimensional subspace, all norms are equivalent, and if u, v are near u_0, then $\operatorname{ess\,sup}(|u| + |v|)$ is close to $2 \operatorname{ess\,sup} |u_0|$ (recall $w_j \in L^\infty$). This allows to conclude

$$\|u - v\| \leqslant C\|\varphi(u) - \varphi(v)\|$$

for all u, v near to u_0 with a constant C depending on u_0. Hence, φ^{-1} is locally Lipschitz continuous on the chosen finite-dimensional subspace.

Moreover, the right-hand side $\langle F(s,u), w_j\rangle := -\langle(\nabla u)^{p-1}, \nabla w_j\rangle + \langle f(s), w_j\rangle_{X_p}$ is locally Lipschitz continuous in u for $p \geqslant 2$ when restricted to a finite-dimensional subspace, because the elementary inequality

$$|a^{p-1} - b^{p-1}| \leqslant C|a-b|(|a|+|b|)^{p-2}$$

is valid for real numbers $a, b \in \mathbb{R}$ (proved e.g. in [4]). In fact, this inequality implies

$$\|F(s,u) - F(s,v)\|_{X_p'} \leqslant C\|\nabla u - \nabla v\|_p \||\nabla u| + |\nabla v|\|_p^{p-2}.$$

If u, v are near u_0, then $\||\nabla u| + |\nabla v|\|_p^{p-2}$ is close to $2^{p-2}\|\nabla u_0\|_p^{p-2}$, and thus we obtain

$$\forall\, j\colon\ |\langle F(s,u), w_j\rangle - \langle F(s,v), w_j\rangle| \leqslant C\|u-v\|$$

for u, v near u_0 with a constant C depending on u_0. Hence, the right-hand side $\langle F(s,u), w_j\rangle$ is locally Lipschitz continuous on the finite-dimensional subspace. $\square$

Thus under the conditions $m \geqslant 2$, $p \geqslant 2$, i.e. in the doubly degenerate case, and for a basis $w_j \in X_p \cap L^{m'} \cap L^\infty$ there exists locally a solution of the projected integral equation.

Let us remark that in the more general case $(m-1)(p-1) \geqslant 1$ it should be possible to verify the existence of approximate solutions via the Leray-Schauder fixed point theorem. Further, the proof of existence of solutions to the projected equation and the proof of Lemma B.1 are the only parts where the stronger assumption $m, p \geqslant 2$ is used, all other parts are valid for general parameters.

2.2. The fundamental a priori estimate. For weak solutions u of either the PDE or the ODE we obtain from the energy identity $(\mathrm{d}/\mathrm{d}t)m^{-1}\|u\|_{m'}^{m'} = \langle(\mathrm{d}/\mathrm{d}t)\varphi(u), u\rangle$ and from the equation $(\mathrm{d}/\mathrm{d}t)\varphi(u) = \Delta_p u + f$ (or its projection) the equation

$$\frac{1}{m}\|u(t)\|_{m'}^{m'} - \frac{1}{m}\|u(0)\|_{m'}^{m'} = -\int_0^t \|\nabla u\|_p^p + \int_0^t \langle f, u\rangle_{X_p}.$$

Now use $|\langle f, u\rangle_{X_p}| \leqslant \|f\|_{X_p'}\|\nabla u\|_p$, Hölder's inequality

$$\int_0^t \|f\|_{X_p'}\|\nabla u\|_p \leqslant \left(\int_0^t \|f\|_{X_p'}^{p'}\right)^{1/p'} \left(\int_0^t \|\nabla u\|_p^p\right)^{1/p}$$

and Young's inequality $ab \leqslant (p')^{-1}a^{p'} + p^{-1}b^p$ to prove

$$\frac{1}{m}\|u(t)\|_{m'}^{m'} + \frac{1}{p'}\int_0^t \|\nabla u\|_p^p \leqslant \frac{1}{m}\|u(0)\|_{m'}^{m'} + \frac{1}{p'}\int_0^t \|f\|_{X_p'}^{p'}$$

for a.e. $t \in (0,T)$, and finally take on the right-hand side the supremum in t to obtain the fundamental a priori estimate

$$\frac{1}{m}\|u(t)\|_{m'}^{m'} + \frac{1}{p'}\int_0^t \|\nabla u\|_p^p \leqslant \frac{1}{m}\|u(0)\|_{m'}^{m'} + \frac{1}{p'}\int_0^T \|f\|_{X_p'}^{p'}.$$

By this inequality, $m^{-1}\|u(t)\|_{m'}^{m'}$ and $(p')^{-1}\int_0^T \|\nabla u\|_p^p$ are bounded by a constant not depending on $t \in (0,T)$.

Applied to weak solutions u_k of the projected equation for an increasing sequence of finite-dimensional subspaces $\operatorname{span}(w_j\colon\ j \leqslant k)$ the fundamental a priori estimate has two major consequences: On the one hand, each approximate solution u_k exists on the whole time interval $(0,T)$. In fact, local existence in time has been proved by Lemma 2.1, and the only possibility for non-existence on the whole time interval $(0,T)$ is that the norm blows up at a time $t < T$. But this possibility is excluded by the fundamental a priori estimate. On the other hand, by Alaoglu's theorem boundedness of the norm allows to extract a weakly convergent subsequence of the sequence u_k. In the next section it is shown that the weak limit of this subsequence is a weak solution of the PDE.

2.3. Extraction of an appropriate subsequence. Having obtained a sequence u_k of solutions to the integral form of the projected equation

$$\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\varphi(u)(t), w_j \Big\rangle = \langle \Delta_p u(t), w_j\rangle + \langle f(t), w_j\rangle, \quad 1 \leqslant j \leqslant k,$$

let us discuss in which sense we can form the limit to obtain a weak solution of the PDE.

Because of the fundamental a priori estimate, u_k is uniformly bounded in $L^p(0,T;X_p)$ and in $L^\infty(0,T;L^{m'})$. Further, as

- φ maps bounded sets in $L^\infty(0,T;L^{m'})$ to bounded sets in $L^\infty(0,T;L^m)$,
- the restrictions f_k of the inhomogeneity are uniformly bounded in $L^{p'}(0,T;X_p')$ by $\int_0^T \|f\|_{X_p'}^{p'}$,
- Δ_p maps bounded sets in $L^p(0,T;X_p)$ to bounded sets in $L^{p'}(0,T;X_p')$,
- $(\mathrm{d}/\mathrm{d}t)\varphi(u_k)$ is uniformly bounded in $L^{p'}(0,T;X_p')$ (because $(\mathrm{d}/\mathrm{d}t)\varphi(u_k) = \Delta_p u_k + f_k$ and $\Delta_p u_k$, f_k are uniformly bounded), by repeatedly using Alaoglu's theorem there are indices k and corresponding subsequences such that the following weak limits exist:

$$\begin{aligned} u_k &\rightharpoonup u \text{ in } L^p(0,T;X_p),\\ u_k &\overset{*}{\rightharpoonup} (u)_{\mathrm{ex}} \text{ in } L^\infty(0,T;L^{m'}), \end{aligned}$$

$$
\begin{aligned}
\varphi(u_k) &\overset{*}{\rightharpoonup} (\varphi(u))_{\mathrm{ex}} \text{ in } L^\infty(0,T;L^m),\\
f_k &\rightharpoonup (f)_{\mathrm{ex}} \text{ in } L^{p'}(0,T;X_p'),\\
\Delta_p u_k &\rightharpoonup (\Delta_p u)_{\mathrm{ex}} \text{ in } L^{p'}(0,T;X_p'),\\
\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u_k) &\rightharpoonup \Big(\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u)\Big)_{\mathrm{ex}} \text{ in } L^{p'}(0,T;X_p').
\end{aligned}
$$

Here $(\cdot)_{\mathrm{ex}}$ denotes a weak limit, which we expect to coincide with $\cdot$, but till now we do not know whether the limit really is $\cdot$. Thus the main task is to show that the weak limits $(\cdot)_{\mathrm{ex}}$ coincide with their expected values $\cdot$.

- $(u)_{\mathrm{ex}} = u$:

  The equation $\int_0^T \langle v, u_k\rangle_{X_p} = \int_0^T \langle u_k, v\rangle_{L^m}$ is valid for all functions $v \in L^{p'}(0,T;X_p') \cap L^1(0,T;L^m)$. As the left-hand side converges to $\int_0^T \langle v, u\rangle_{X_p}$ and the right-hand side converges to $\int_0^T \langle (u)_{\mathrm{ex}}, v\rangle_{L^m}$, the equation $u = (u)_{\mathrm{ex}}$ is valid as an equation in the dual space of $L^{p'}(0,T;X_p') \cap L^1(0,T;L^m)$, and thus due to density $u = (u)_{\mathrm{ex}} \in L^p(0,T;X_p) \cap L^\infty(0,T;L^{m'})$.
- $(\varphi(u))_{\mathrm{ex}} = \varphi(u)$:

  By our assumptions on X_p, the map $\varphi\colon (X_p \cap L^{m'})(\Omega) \to L^m(\Omega')$ is compact for every bounded subdomain $\Omega' \subset \Omega$. Recall that this assumption is valid for the main examples by Lemma B.1.

  Thus we can use Lemma 2.2 (see below) to conclude that the sequence of the functions $\varphi(u_k)$ restricted to a bounded subdomain $\Omega' \subset \Omega$ is relatively compact in $L^1(0,T;L^m(\Omega'))$. Hence, $\varphi(u_k)\big|_{\Omega'}$ has a cluster point in $L^1(0,T;L^m(\Omega'))$. But due to weak convergence this cluster point is unique and coincides with the restriction of $(\varphi(u))_{\mathrm{ex}}$ to Ω'.

  Hence, $\varphi(u_k)\big|_{\Omega'}$ converges strongly to $(\varphi(u))_{\mathrm{ex}}\big|_{\Omega'}$ in $L^1(0,T;L^m(\Omega'))$. Further, by monotonicity of $\varphi\colon L^{m'} \to L^m$ we have

  $$\int_0^T \langle \varphi(v) - \varphi(u_k), v - u_k\rangle_{L^m(\Omega')} \geqslant 0$$

  for every $v \in L^\infty(0,T;L^{m'})$. Because $\varphi(u_k)\big|_{\Omega'}$ converges strongly in $L^1(0,T;L^m(\Omega'))$, we can form the limit in this inequality to obtain $\int_0^T \langle \varphi(v) - (\varphi(u))_{\mathrm{ex}}, v - u\rangle \geqslant 0$.

  Now replace v by $u + \lambda v$ to conclude $\int_0^T \langle \varphi(u+\lambda v) - (\varphi(u))_{\mathrm{ex}}, v\rangle \geqslant 0$ for all $\lambda > 0$ and $v \in L^\infty(0,T;L^{m'})$. Let $\lambda \searrow 0$, then by hemicontinuity the inequality $\int_0^T \langle \varphi(u) - (\varphi(u))_{\mathrm{ex}}, v\rangle \geqslant 0$ holds for all $v \in L^\infty(0,T;L^{m'})$, and therefore $(\varphi(u))_{\mathrm{ex}}\big|_{\Omega'} = \varphi(u)\big|_{\Omega'}$. But as $\Omega' \subset \Omega$ was an arbitrary bounded subdomain, this proves $(\varphi(u))_{\mathrm{ex}} = \varphi(u)$ as functions on Ω.
- $((\mathrm{d}/\mathrm{d}t)\varphi(u))_{\mathrm{ex}} = (\mathrm{d}/\mathrm{d}t)\varphi(u)$ and the initial value of $\varphi(u)$ is $\varphi(u(0))$:

As the equality $\int_0^T \langle (\mathrm{d}/\mathrm{d}t)\varphi(u_k), v - v(T)\rangle_{X_p} = -\int_0^T \langle \varphi(u_k) - \varphi(u(0)_k), \dot{v}\rangle_{L^{m'}}$ holds for all functions $v \in L^p(0,T;\mathrm{span}(w_1,\ldots,w_k))$ with $\dot{v} \in L^1(0,T;$ $\mathrm{span}(w_1,\ldots,w_k))$, by convergence of $\varphi(u_k)$ to $\varphi(u)$ and of the initial value $\varphi(u(0)_k)$ to $\varphi(u(0))$ we obtain in the limit

$$\int_0^T \left\langle \Big(\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u)\Big)_{\mathrm{ex}}, v - v(T)\right\rangle_{X_p} = -\int_0^T \langle \varphi(u) - \varphi(u(0)), \dot{v}\rangle_{L^{m'}}$$

for all $v \in L^p(0,T;X_p)$ with $\dot{v} \in L^1(0,T;L^{m'})$ and $v(t) \in \mathrm{span}(w_j)$ for a.e. t. As these functions v form a dense subspace, $\varphi(u)$ is weakly differentiable with weak the derivative $((\mathrm{d}/\mathrm{d}t)\varphi(u))_{\mathrm{ex}}$ and the initial value $\varphi(u(0))$.

- $(f)_{\mathrm{ex}} = f$:
  By construction of f_k we have $\int_0^T \langle f_k, v\rangle_{X_p} = \int_0^T \langle f, v\rangle_{X_p}$ for all $v \in L^p(0,T;X_p)$ with $v(t)$ in the finite-dimensional subspace $\mathrm{span}(w_1,\ldots,w_k)$ of X_p for a.e. t. As the left-hand side converges to $\int_0^T \langle (f)_{\mathrm{ex}}, v\rangle_{X_p}$, the equation $\int_0^T \langle (f)_{\mathrm{ex}}, v\rangle_{X_p} = \int_0^T \langle f, v\rangle_{X_p}$ holds for all $v \in L^p(0,T;X_p)$ with $v(t) \in \mathrm{span}(w_j)$ for a.e. t, and because these functions form a dense subspace, the equation $(f)_{\mathrm{ex}} = f$ is valid.
- $\varphi(u_k(T)) \rightharpoonup \varphi(u(T))$ in L^m:
  Because

$$\begin{aligned} -\int_0^T \langle \varphi(u_k) - \varphi(u_k(T)), \dot{v}\rangle_{L^{m'}} &= \int_0^T \langle \dot{\varphi(u_k)}, v - v(0)\rangle_{X_p} \\ &\to \int_0^T \langle \dot{\varphi(u)}, v - v(0)\rangle_{X_p} \end{aligned}$$

  and $\varphi(u_k) \to \varphi(u)$, by the definition of final values we obtain $\varphi(u(T)_k) \rightharpoonup \varphi(u(T))$ in L^m.
- $(\Delta_p u)_{\mathrm{ex}} = \Delta_p u$:
  As u_k solves the approximate problem, from the energy identity we have

$$\int_0^T \langle -\Delta_p u_k, u_k\rangle = \int_0^T \langle f, u_k\rangle + \frac{1}{m}\|u(0)_k\|_{m'}^{m'} - \frac{1}{m}\|u_k(T)\|_{m'}^{m'}.$$

By strong convergence of the initial values $\varphi(u(0)_k)$ to $\varphi(u(0))$ in L^m we have $\|u(0)_k\|_{m'} \to \|u(0)\|'_m$. Further, $\int_0^T \langle f, u_k\rangle \to \int_0^T \langle f, u\rangle$ follows from weak convergence $u_k \rightharpoonup u$ in $L^p(0,T;X_p)$. Finally, the values $\varphi(u_k(T))$ converge weakly to $\varphi(u(T))$ in $L^{m'}$, so by weak lower-semicontinuity of the norm in $L^{m'}$ we have $\liminf_k \|u_k(T)\|_{m'} \geqslant \|u(T)\|_{m'}$.

Thus $\limsup_k \int_0^T \langle -\Delta_p u_k, u_k\rangle \leqslant \int_0^T \langle f, u\rangle + m^{-1}\|u(0)\|_{m'} - m^{-1}\|u(T)\|_{m'}^{m'}$, and because the weak limit of $(\mathrm{d}/\mathrm{d}t)\varphi(u_k) = \Delta_p u_k + f_k$ is $(\mathrm{d}/\mathrm{d}t)\varphi(u) = (\Delta_p u)_{\mathrm{ex}} + f$, from $\langle -(\Delta_p u)_{\mathrm{ex}}, u\rangle = \langle f, u\rangle - \langle (\mathrm{d}/\mathrm{d}t)\varphi(u), u\rangle$ we get $\limsup_k \int_0^t \langle -\Delta_p u_k, u_k\rangle \leqslant \int_0^T \langle -(\Delta_p u)_{\mathrm{ex}}, u\rangle$.

Hence, as the p-Laplacian Δ_p is a monotone operator which is bounded by $\|\Delta_p u\|_{L^{p'}(0,T;X_p')} \leqslant \|u\|^{p-1}_{L^p(0,T;X_p)}$ and hemicontinuous, i.e. $\lambda \to \langle \Delta_p(u+\lambda v), v\rangle$ is continuous, the monotonicity lemma B.3 can be applied to conclude the identity $-\Delta_p u = -(\Delta_p u)_{\mathrm{ex}}$.

Finally, as all the weak limits in the equation $(\mathrm{d}/\mathrm{d}t)\varphi(u_k) = \Delta_p u_k + f_k$ converge to the expected values, the equation $(\mathrm{d}/\mathrm{d}t)\varphi(u) = \Delta_p u + f$ is valid when applied to a test function $v \in L^p(0,T;X_p)$ with weak derivative $\dot v \in L^1(0,T;L^{m'})$, i.e. u is a weak solution.

To complete the proof, merely the following lemma remains to be proved:

Lemma 2.2. *Assume that $\varphi\colon X_p \cap L^{m'} \to L^m$ is compact, and let $u_k \in L^p(0,T;X_p) \cap L^\infty(0,T;L^{m'})$ be the sequence of weak solutions of the projected equations. Then $\varphi(u_k)$ is relatively compact in $L^1(0,T;L^m)$.*

Proof. We want to apply the nonlinear compactness lemma (see Lemma A.2). As u_k is uniformly bounded in $L^p(0,T;X_p)\cap L^\infty(0,T;L^{m'}) \subset L^p(0,T;X_p\cap L^{m'})$ and the sequence u_k is mapped by φ to a bounded set in $L^\infty(0,T;L^m)$, by the nonlinear compactness lemma (see again Lemma A.2) we only have to show

$$\int_0^{T-h} \|\varphi(u_k(t+h)) - \varphi(u_k(t))\|_m \to 0 \tag{2.2}$$

uniformly in k for $h \searrow 0$.

The first step to guarantee this limit is to verify

$$\int_0^{T-h} \langle \varphi(u(t+h)) - \varphi(u(t)), u(t+h) - u(t)\rangle \leqslant hC \tag{2.3}$$

for weak solutions of (1.1) by a proof which also works for the projected equation:

Integrate equation (1.1) with respect to $t' \in [t, t+h]$ to obtain $\varphi(u(t+h)) - \varphi(u(t)) = \int_t^{t+h} \Delta_p u + f$. Define $H_1(t) := h^{-1}\int_0^h \Delta_p u(t+h')\,\mathrm{d}h'$, $H_2(t) := h^{-1}\int_0^h f(t+h')\,\mathrm{d}h'$, then the right-hand side is $h\cdot(H_1(t)+H_2(t))$, and note that both $H_i(t)$ are convolutions respectively of $\Delta_p u$ and f with the Dirac sequence $h^{-1}1_{[0,h]}$, so that in particular $\int_0^T \|H_1(t)\|^{p'}_{X_p'} \leqslant \int_0^T \|\Delta_p u(t)\|^{p'}_{X_p'}$ and $\int_0^T \|H_2(t)\|^{p'}_{X_p'} \leqslant \int_0^T \|f\|^{p'}_{X_p'}$. Now apply the equation to $u(t+h)-u(t)$ and integrate with respect to $t \in [0, T-h]$ to obtain

$$\int_0^{T-h} \langle \varphi(u(t+h)) - \varphi(u(t)), u(t+h) - u(t)\rangle\,\mathrm{d}t = h\int_0^{T-h} \langle H_1(t) + H_2(t), u(t+h) - u(t)\rangle\,\mathrm{d}t.$$

Estimate $\langle H_1(t)+H_2(t), u(t+h)-u(t)\rangle \leqslant (\|H_1(t)\|_{X_p'}+\|H_2(t)\|_{X_p'})(\|u(t+h)\|_{X_p}+\|u(t)\|_{X_p})$ and use Hölder's inequality to obtain

$$\begin{aligned}
&\int_0^{T-h} \langle H_1(t)+H_2(t), u(t+h)-u(t)\rangle \\
&\qquad \leqslant \Big(\int_0^{T-h} \|H_1(t)\|_{X_p'}^{p'}\Big)^{1/p'} \Big(\int_0^{T-h} \|u(t+h)\|_{X_p}^{p}\Big)^{1/p} \\
&\qquad\quad + \Big(\int_0^{T-h} \|H_2(t)\|_{X_p'}^{p'}\Big)^{1/p'} \Big(\int_0^{T-h} \|u(t+h)\|_{X_p}^{p}\Big)^{1/p} \\
&\qquad\quad + \Big(\int_0^{T-h} \|H_1(t)\|_{X_p'}^{p'}\Big)^{1/p'} \Big(\int_0^{T-h} \|u(t)\|_{X_p}^{p}\Big)^{1/p} \\
&\qquad\quad + \Big(\int_0^{T-h} \|H_2(t)\|_{X_p'}^{p'}\Big)^{1/p'} \Big(\int_0^{T-h} \|u(t)\|_{X_p}^{p}\Big)^{1/p} \\
&\qquad \leqslant 2\bigg(\Big(\int_0^{T} \|H_1(t)\|_{X_p'}^{p'}\Big)^{1/p'} + \Big(\int_0^{T} \|H_1(t)\|_{X_p'}^{p'}\Big)^{1/p'}\bigg)\Big(\int_0^{T} \|u(t)\|_{X_p}^{p}\Big)^{1/p}.
\end{aligned}$$

Finally, observe that by the fundamental a priori estimate the right-hand side is bounded by a constant (even if u is replaced by the approximate solutions u_k), so that inequality (2.3) is valid with a constant C not depending on h (and not depending on k, if u is replaced by u_k).

To deduce from (2.3) the limit (2.2), in the second step we prove by the arguments of [2, Lemma 1.8] for every $M>0$ the existence of a continuous function ω_M with $\omega_M(0)=0$ such that if u, v satisfy $\|u\|_{X_p}, \|v\|_{X_p} \leqslant M$, $\|u\|_{m'}, \|v\|_{m'} \leqslant M$ and $\langle \varphi(u)-\varphi(v), u-v\rangle \leqslant \delta$, then $\|\varphi(u)-\varphi(v)\|_m \leqslant \omega_M(\delta)$.

In fact, assume that $u_\delta, v_\delta \in X_p \cap L^{m'}$ satisfy the proposed estimates, but that the inequality $\|\varphi(u_\delta)-\varphi(v_\delta)\|_m \geqslant \kappa > 0$ holds. Due to boundedness of u_δ, v_δ and compactness of the map φ there are subsequences $\varphi(u_\delta) \to \varphi(u)$, $\varphi(v_\delta) \to \varphi(v)$ in L^m. By assumption

$$\langle \varphi(u_\delta)-\varphi(v_\delta), u_\delta - v_\delta\rangle \leqslant \delta,$$

and for $\delta \to 0$ we obtain $\langle \varphi(u)-\varphi(v), u-v\rangle \leqslant 0$. By monotonicity of φ this inequality implies $\langle \varphi(u)-\varphi(v), u-v\rangle = 0$. Thus $\langle \varphi(\cdot), v-u\rangle$ is constant along the line segment between u and v due to

$$0 \leqslant \langle \varphi(u+\theta(v-u))-\varphi(u), v-u\rangle \leqslant \langle \varphi(v)-\varphi(u), v-u\rangle = 0$$

for every $\theta \in [0,1]$, where we used monotonicity of φ. Therefore, the potential Φ of φ is linear along this line segment. Hence, convexity of Φ allows to conclude from the inequality

$$\begin{aligned}\Phi(v+w)-\Phi(v) &= \Phi(v+w)-\Phi(u)-\langle d\Phi(u), v-u\rangle \\ &\geqslant \langle d\Phi(u), v+w-u\rangle - \langle d\Phi(u), v-u\rangle = \langle d\Phi(u), w\rangle\end{aligned}$$

for all $w \in L^{m'}$ the equality $\varphi(u) = d\Phi(v) = \varphi(v)$ contradicting $\|\varphi(u)-\varphi(v)\|_m \geqslant \kappa > 0$.

Finally, to establish (2.2), in the third step consider the set

$$\begin{aligned}E := \{t \in (0, T-h)\colon\ &\|u_k(t+h)\|_{X_p\cap L^{m'}} \leqslant M,\ \|u_k(t)\|_{X_p\cap L^{m'}} \leqslant M, \\ &\langle \varphi(u_k(t+h)) - \varphi(u_k(t)), u_k(t+h)-u_k(t)\rangle \leqslant hM\}.\end{aligned}$$

Obviously the complement of this set has measure $|E^c| \leqslant 2C/M$ for large M with the constant $2C$ from inequality (2.3). Split the integral of $\|\varphi(u_k(t+h)) - \varphi(u_k(t))\|_m$ over $(0, T-h)$ into an integral over E and over E^c. The first part is smaller than $T\omega_M(hM)$. For the second part the inequality

$$\begin{aligned}\int_{E^c} \|\varphi(u_k(t+h)) - \varphi(u_k(t))\|_m &\leqslant \int_{E^c+h} \|\varphi(u_k(t))\|_m + \int_{E^c} \|\varphi(u_k(t))\|_m \\ &\leqslant 2|E^c|C^{1/m}\end{aligned}$$

holds due to the boundedness of $\|\varphi(u_k(t))\|_m^m = \|u_k(t)\|_{m'}^{m'}$ for a.e. t.

Thus

$$\int_0^{T-h} \|\varphi(u_k(t+h)) - \varphi(u_k(t))\|_m \leqslant \omega_M(hM) + 2C^{1/m}C/M,$$

and the left-hand side is smaller than $\varepsilon > 0$ if M is chosen such that $2C^{1/m}C/M \leqslant \varepsilon/2$, and then h is chosen such that $\omega_M(hM) \leqslant \varepsilon/2$ (which is possible by continuity of ω_M at zero). This proves that the limit (2.2) is valid uniformly in k as $h \searrow 0$. $\square$

3. Nonlinearities

If $f \in L^{p'}(0, T; X_p')$ is not an inhomogeneity, but a nonlinearity f depending on u, we have to impose conditions on f to guarantee the validity of a priori estimates. These conditions strongly depend on two facts:

- If f has monotone parts, then better a priori estimates can be proved, which allow to compensate worse non-monotone parts of f.

- If X_p can be embedded into another function space, e.g. due to the properties of the domain Ω, then the norm in this function space can be used in the estimates, and this changes the conditions on the nonlinearity.

In the case that f has monotone parts, weak solutions automatically lie in better function spaces, and hence the notion of a weak solution has to be changed. Therefore, let us first concentrate on nonlinearities without monotone parts, then the definition of a weak solution need not be changed as long as we can guarantee $f(u) \in L^{p'}(0,T;X_p')$.

3.1. Non-monotone nonlinearities. If we have no additional information about imbeddings of X_p into a space of functions, we have to work abstractly in the space X_p' to solve problems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u) = \Delta_p u + \operatorname{div}(F(u)) + f_{\text{inhom}}.$$

Note that every function in $L^{p'}(0,T;X_p')$ can be written as the divergence of a time-dependent vector field $F \in L^{p'}(0,T;L^{p'})$.

Assumption 3.1. Assume that the nonlinearity has the form $f(t,x,u) = \operatorname{div}(F(t,x,u))$ with a Carathéodory vector field F satisfying an $L^{p'}$-X_p-Lipschitz and growth condition, i.e.

(i) $u \mapsto F(t,x,u)$ is continuous for almost every (t,x),

(ii) $(t,x) \mapsto F(t,x,u)$ is measurable for every $u \in \mathbb{R}$,

(iii) the local Lipschitz condition is satisfied, i.e. for every $u_0 \in X_p \cap L^m$ there exists a neighbourhood $B_r(u_0) \subset X_p \cap L^{m'}$ and a function $L \in L^{p'}(0,T)$ such that

$$\|F(t,\cdot,u(\cdot)) - F(t,\cdot,v(\cdot))\|_{p'} \leqslant L(t)\|\nabla u - \nabla v\|_p \tag{3.1}$$

for every $u, v \in B_r(u_0)$ and a.e. $t \in (0,T)$,

(iv) the growth condition is satisfied, i.e. there exists an index $1 \leqslant q \leqslant p$ and a function $C \in L^{p/(p-q)}(0,T)$ ($\|C\|_\infty < 1$ in the case $q = p$) such that

$$\|F(t,\cdot,u(\cdot))\|_{p'} \leqslant C(t)\|\nabla u\|_p^{q-1} \tag{3.2}$$

for every $u \in X_p \cap L^{m'}$ and a.e. $t \in (0,T)$.

Under this assumption, by Hölder's inequality and Young's inequality with $\varepsilon > 0$ for $1 \leqslant q < p$ we obtain

$$\begin{aligned}\int_0^T \langle F(t,u), \nabla u\rangle \,\mathrm{d}t &\leqslant \int_0^T C(t)\|\nabla u\|_p^{q-1}\|\nabla u\|_p \,\mathrm{d}t \\ &\leqslant \Big(\int_0^T C^{p/(p-q)}\Big)^{(p-q)/p}\Big(\int_0^T \|\nabla u\|_p^p\Big)^{q/p} \\ &\leqslant \frac{p-q}{p\varepsilon^{p/(p-q)}}\Big(\int_0^T C^{p/(p-q)}\Big) + \frac{q\varepsilon^{p/q}}{p}\Big(\int_0^T \|\nabla u\|_p^p\Big).\end{aligned}$$

Thus, choosing $\varepsilon > 0$ such that $p^{-1}q\varepsilon^{p/q} < 1$, we have the a priori estimate

$$\frac{1}{m}\|u(t)\|_{m'}^{m'} + \Big(1 - \frac{q\varepsilon^{p/q}}{p}\Big)\int_0^t \|\nabla u\|_p^p \leqslant \frac{1}{m}\|u(0)\|_{m'}^{m'} + \frac{p-q}{p\varepsilon^{p/(p-q)}}\Big(\int_0^T C^{p/(p-q)}\Big)$$

valid for a.e. $t \in (0,T)$, and weak convergence of approximate solutions u_k can be concluded like in the case of an inhomogeneity. For $q = p$ the same conclusion is valid due to $C \in L^\infty(0,T)$ as long as $\|C\|_\infty < 1$.

However, to guarantee existence of weak solutions we have to prove solvability of the approximate problem and convergence $f(\cdot, u_k(\cdot)) \rightharpoonup f(\cdot, u(\cdot))$ in $L^{p'}(0,T;X_p')$ for the approximate solutions u_k.

Solvability of the approximate problem: As in the case of an inhomogeneity, for $p, m \geqslant 2$ the approximate problem can be solved locally in time in the sense of Carathéodory by Picard iterations, because the local Lipschitz condition formulated in assumption 3.1 (iii) implies

$$\int_0^T \|F(s,\cdot,u(\cdot)) - F(s,\cdot,v(\cdot))\|_{p'}\,\mathrm{d}s \leqslant \Big(\int_0^T L(s)^{p'}\,\mathrm{d}s\Big)^{1/p'}\Big(\int_0^T \|\nabla u - \nabla v\|_p^p\,\mathrm{d}s\Big)^{1/p}$$

for $u, v \in X_p \cap L^{m'}$ near u_0. Therefore, locally in time an approximate solution can be found by Picard iterations.

Weak convergence of the nonlinear term: Because of

$$\begin{aligned}\int_0^T \|F(t,u(t))\|_{p'}^{p'}\,\mathrm{d}t &\leqslant \int_0^T C(t)^{p'}\|\nabla u(t)\|_p^{(q-1)p'}\,\mathrm{d}t \\ &\leqslant \Big(\int_0^T C^{p/(p-q)}\Big)^{(p-q)/(p-1)}\Big(\int_0^T \|\nabla u\|_p^p\Big)^{(q-1)/(p-1)}\end{aligned}$$

(and analogously for $q = p$ and $C \in L^\infty$) from boundedness of u_k in $L^p(0,T;X_p)$ also boundedness of $F(\cdot, u_k(\cdot))$ in $L^{p'}(0,T;L^{p'})$ follows, thus $F(\cdot,u_k(\cdot))$ has a weak limit $(F(\cdot,u(\cdot)))_{\mathrm{ex}} \in L^{p'}(0,T;L^{p'})$.

To show the identity $(F(\cdot,u(\cdot)))_{\mathrm{ex}} = F(\cdot,u(\cdot))$, use the strong convergence $\varphi(u_k) \to \varphi(u)$ in $L^1(0,T;L^m)$ (at least on bounded subdomains $\Omega' \subset \Omega$): Due to this strong convergence there is a subsequence such that $\varphi(u_k)$ converges pointwise almost everywhere to $\varphi(u)$. Hence, u_k also converges pointwise almost everywhere to u. By continuity of F in u we have $F(t,x,u_k(t,x)) \to F(t,x,u(t,x))$ for a.e. $(t,x) \in (0,T) \times \Omega$.

Further, by the above estimate the functions $F(\cdot,\cdot,u_k(\cdot,\cdot))$ are uniformly bounded in $L^{p'}(0,T;L^{p'})$, and as $F(\cdot,\cdot,u(\cdot,\cdot)) \in L^{p'}(0,T;L^{p'})$ is a.e. the pointwise limit of these functions, by the weak dominated convergence theorem it is a weak limit. But the expected value $(f(\cdot,u(\cdot)))_{\mathrm{ex}}$ is also a weak limit of $F(\cdot,u_k(\cdot))$, and hence by uniqueness of weak limits we have $(F(\cdot,u(\cdot)))_{\mathrm{ex}} = F(\cdot,u(\cdot))$.

Now let us discuss an example where the conditions (3.1) and (3.2) on F are valid.

Example. If $p < n$, then due to the choice of X_p by Sobolev embeddings we have $X_p \subset L^{p^*}$ in the case of an arbitrary domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary, and $X_p \cap L^{m'} \subset L^{p^*}$ in the case of a domain $\Omega \subset \mathbb{R}^n$ with Neumann boundary satisfying the cone condition.

Thus the conditions (3.1) and (3.2) are valid, if F is a Carathéodory function satisfying $\|F(t,\cdot,u(\cdot)) - F(t,\cdot,u(\cdot))\|_{p'} \leqslant L(t)\|u-v\|_{p^*}$ for a function $L \in L^{p'}(0,T)$ and $\|F(t,\cdot,u(\cdot))\|_{p'} \leqslant C(t)\|u\|_{p^*}^{q-1}$ for an index $1 \leqslant q \leqslant p$ and a function $C \in L^{p/(p-q)}(0,T)$ ($\|C\|_\infty < 1$ in the case $q = p$).

Particularly, if $|F(t,x,u) - F(t,x,v)| \leqslant L(t)K(x)|u-v|$, then

$$\|F(t,\cdot,u(\cdot)) - F(t,\cdot,v(\cdot)\|_{p'}^{p'} \leqslant L(t)^{p'}\|K\|_{p^*p'/(p^*-p')}^{p'}\|u-v\|_{p^*}^{p'}$$

and hence assumption (3.1) is satisfied if $K \in L^{p^*p'/(p^*-p')}$ and $L \in L^{p'}(0,T)$.

Further, if $|F(t,x,u)| \leqslant C(t)D(x)|u|^a$, then

$$\|F(t,\cdot,u(\cdot))\|_{p'}^{p'} \leqslant C(t)^{p'}\|D\|_{p^*p'/(p^*-ap')}^{p'}\|u\|_{p^*}^{ap'}$$

and hence assumption (3.2) is satisfied if $0 \leqslant a \leqslant p-1$, $D \in L^{p^*p'/(p^*-ap')}$ and $C \in L^{p/(p-a-1)}(0,T)$ ($\|C\|_\infty\|D\|_{p^*p'/(p^*-ap')} < 1$ in the case $a = p-1$).

If $p > n$ (and similarly for $p = n$), then $X_p \cap L^{m'}$ is embedded into L^∞. In particular, if $|F(t,x,u) - F(t,x,v)| \leqslant L(t)K(x)|u-v|$, then

$$\|F(t,\cdot,u(\cdot)) - F(t,\cdot,v(\cdot)\|_{p'}^{p'} \leqslant L(t)^{p'}\|K\|_{p'}^{p'}\|u-v\|_\infty^{p'}$$

and hence assumption (3.1) is satisfied if $K \in L^{p'}$ and $L \in L^{p'}(0,T)$.

Further, if $|F(t,x,u)| \leqslant C(t)D(x)|u|^a$, then

$$\|F(t,\cdot,u(\cdot))\|_{p'}^{p'} \leqslant C(t)^{p'}\|D\|_{p'}^{p'}\|u\|_\infty^{ap'}$$

and hence assumption (3.2) is satisfied if $0 \leqslant a \leqslant p-1$, $D \in L^{p'}$ and $C \in L^{p/(p-a-1)}(0,T)$ ($\|C\|_\infty \|D\|_{p'} < 1$ in the case $a = p-1$).

Corollary 3.2. *Let $m, p \geqslant 2$. If F is as in the above example, i.e., F is Lipschitz in u and does not grow faster than $1 + |u|^{p-1}$, then the equation $\dot{\varphi(u)} = \Delta_p u + \operatorname{div}(F(u)) + f_{\text{inhom}}$ has a weak solution.*

If X_p is embedded into a space of functions Z, then also equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u) = \Delta_p u + f(u) + f_{\text{inhom}}$$

with a non-monotone nonlinearity f interpreted as a map from Z to Z' can be considered. Again, $f(t,x,u)$ has to be assumed a Carathéodory function which satisfies a Lipschitz condition and a growth condition analogously to assumption 3.1. The Lipschitz condition guarantees the solvability of the approximate equation, and due to the growth condition the nonlinearity can be compensated by the a priori estimates. This allows to prove the existence of weak solutions as before.

Example. Consider the case $p < n$ as before. Because $X_p \cap L^{m'}$ is embedded into L^{p^*}, consider a Carathéodory function $f(t,x,u)$ satisfying the Lipschitz condition

$$\|f(t,\cdot,u(\cdot)) - f(t,\cdot,v(\cdot))\|_{(p^*)'} \leqslant L(t)\|u-v\|_{p^*}$$

for a function $L \in L^{p'}(0,T)$, and the growth condition

$$\|f(t,\cdot,u(\cdot))\|_{(p^*)'} \leqslant C(t)\|u\|_{p^*}^{q-1}$$

for an index $1 \leqslant q \leqslant p$ and a function $C \in L^{p/(p-q)}(0,T;\mathbb{R})$ ($\|C\|_\infty < 1$ in the case $q = p$). Then existence of weak solutions can be proved as before.

Particularly, if $|f(t,x,u) - f(t,x,v)| \leqslant L(t)K(x)|u-v|$, then

$$\|f(t,\cdot,u(\cdot)) - f(t,\cdot,v(\cdot)\|_{(p^*)'}^{(p^*)'} \leqslant L(t)^{(p^*)'}\|K\|_{p^*(p^*)'/(p^*-(p^*)')}^{(p^*)'}\|u-v\|_{p^*}^{(p^*)'}$$

and hence the Lipschitz condition is satisfied if $K \in L^{p^*(p^*)'/(p^*-(p^*)')}$ and $L \in L^{p'}(0,T)$.

Further, if $|f(t,x,u)| \leqslant C(t)D(x)|u|^a$, then

$$\|f(t,\cdot,u(\cdot))\|_{(p^*)'}^{(p^*)'} \leqslant C(t)^{(p^*)'}\|D\|_{p^*(p^*)'/(p^*-a(p^*)')}^{(p^*)'}\|u\|_{p^*}^{a(p^*)'}$$

and hence the growth condition is satisfied if $0 \leqslant a \leqslant p-1$, $D \in L^{p^*(p^*)'/(p^*-a(p^*)')}$ and $C \in L^{p/(p-a-1)}(0,T)$ ($\|C\|_\infty \|D\|_{p^*(p^*)'/(p^*-a(p^*)')} < 1$ in the case $a = p-1$).

Similar results hold for $p \geqslant n$.

Corollary 3.3. *Let $m, p \geqslant 2$. If f is as in the above example, i.e., f is Lipschitz in u and does not grow faster than $1 + |u|^{p-1}$, then the equation $(\mathrm{d}/\mathrm{d}t)\varphi(u) = \Delta_p u + f(u) + f_{\mathrm{inhom}}$ has a weak solution.*

Note that also equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u) = \Delta_p u + \operatorname{div}(F(u)) + f(u) + f_{\mathrm{inhom}}$$

can be discussed by a similar method.

3.2. Monotone nonlinearities. Finally, let us assume that there is a monotone part of the nonlinearity, i.e. let us consider an equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u) = \Delta_p u - f_{\mathrm{mon}}(u) + f_{\mathrm{inhom}}$$

with a function f_{mon} interpreted as a monotone operator.

For simplicity, let us assume that f_{mon} is like $f_{\mathrm{mon}}(u) := u^{q-1}$ for $q \geqslant 2$, i.e. we assume that f_{mon} is a Carathéodory function satisfying the monotonicity condition $(f(u) - f(v))(u - v) \geqslant 0$, and that for some $q \geqslant 2$ a Lipschitz condition $|f_{\mathrm{mon}}(u) - f_{\mathrm{mon}}(v)| \leqslant C|u - v|(|u| + |v|)^{q-2}$, a growth condition $|f_{\mathrm{mon}}(u)| \leqslant C|u|^{q-1}$ and an ellipticity condition $\langle f_{\mathrm{mon}}(u), u\rangle \geqslant c\|u\|_q^q$ is satisfied. Then f_{mon} induces a monotone operator from L^q to $L^{q'}$.

Further, the notion of a weak solution changes, as by ellipticity the a priori estimate

$$\frac{1}{m}\|u(t)\|_{m'}^{m'} + \frac{1}{p'}\int_0^t \|\nabla u\|_p^p + c\int_0^t \|u\|_q^q \leqslant \frac{1}{m}\|u(0)\|_{m'}^{m'} + \frac{1}{p'}\int_0^T \|f_{\mathrm{inhom}}\|_{X_p'}^{p'}$$

is valid and implies that solutions u automatically belong to the space $L^p(0,T;X_p) \cap L^\infty(0,T;L^{m'}) \cap L^q(0,T;L^q)$.

Thus, a weak solution should be an element u of this space such that $\varphi(u) \in L^\infty(0,T;L^m)$ has a weak derivative $(\mathrm{d}/\mathrm{d}t)\varphi(u) \in L^{p'}(0,T;X_p') + L^{q'}(0,T;L^{q'})$ which satisfies the equation in the integral sense like in (1.2). In the same way as for the p-Laplacian, the Lipschitz condition implies existence of approximate solutions u_k via Picard iterations, from the growth condition the existence of a weak limit of $f_{\mathrm{mon}}(u_k)$ can be deduced, and the monotonicity lemma allows to prove that this weak limit coincides with $f_{\mathrm{mon}}(u)$ with the weak limit u of u_k. Hence, existence of weak solutions follows.

Additional monotone nonlinearities allow to compensate worse non-monotone nonlinearities. In fact, consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u) = \Delta_p u - f_{\mathrm{mon}}(u) + f_{\mathrm{nonmon}}(u) + f_{\mathrm{inhom}}.$$

The function f_{nonmon} need not be interpreted as an element of $L^{p'}(0,T;X_p')$ anymore, it also can be interpreted as an element of $L^{q'}(0,T;L^{q'})$, and then the a priori bound of $\|u\|_q^q$ can be used to compensate the nonlinearity.

Example. Regardless of whether X_p is embedded into a certain function space or not, if f_{mon} is a monotone nonlinearity as before and $f_{\mathrm{inhom}} \in L^{p'}(0,T;X_p') + L^{q*}(0,T;L^{q'})$, then existence of a weak solution can be proved by the former methods in the case of a Carathéodory function f_{nonmon} satisfying the Lipschitz condition

$$\|f_{\mathrm{nonmon}}(t,\cdot,u(\cdot)) - f_{\mathrm{nonmon}}(t,\cdot,v(\cdot))\|_{q'} \leqslant L(t)\|u-v\|_q$$

for a function $L \in L^{q'}(0,T)$ and the growth condition

$$\|f_{\mathrm{nonmon}}(t,\cdot,u(\cdot))\|_{q'} \leqslant C(t)\|u\|_q^{r-1}$$

for an index $1 \leqslant r \leqslant q$ and a function $C \in L^{q/(q-r)}(0,T;\mathbb{R})$ ($\|C\|_\infty < c$ in the case $r = q$).

Particularly, if $|f_{\mathrm{nonmon}}(t,x,u) - f_{\mathrm{nonmon}}(t,x,v)| \leqslant L(t)K(x)|u-v|$, then

$$\|f_{\mathrm{nonmon}}(t,\cdot,u(\cdot)) - f_{\mathrm{nonmon}}(t,\cdot,v(\cdot)\|_{q'}^{q'} \leqslant L(t)^{q'}\|K\|_{q/(q-2)}^{q'}\|u-v\|_q^{q'}$$

and hence the Lipschitz condition is satisfied if $K \in L^{q/(q-2)}$ ($K \in L^\infty$ for $q = 2$) and $L \in L^{q'}(0,T)$.

Further, if $|f_{\mathrm{nonmon}}(t,x,u)| \leqslant C(t)D(x)|u|^a$, then

$$\|f_{\mathrm{nonmon}}(t,\cdot,u(\cdot))\|_{q'}^{q'} \leqslant C(t)^{q'}\|D\|_{q/(q-a-1)}^{q'}\|u\|_q^{aq'}$$

and hence the growth condition is satisfied if $0 \leqslant a \leqslant q-1$, $D \in L^{q/(q-a-1)}$ and $C \in L^{q/(q-a-1)}(0,T)$ ($\|C\|_\infty\|D\|_\infty < c$ in the case $a = q-1$).

Corollary 3.4. *Let $m, p \geqslant 2$, let f_{mon} be a monotone nonlinearity of order q and let $f_{\mathrm{inhom}} \in L^{p'}(0,T;X_p') + L^{q'}(0,T;L^{q'})$ be an inhomogeneity. If f_{nonmon} is as in the above example, i.e. f_{nonmon} is Lipschitz in u and does not grow faster than $1+|u|^{q-1}$, then for $m, p \geqslant 2$ the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u) = \Delta_p u - f_{\mathrm{mon}}(u) + f_{\mathrm{nonmon}}(u) + f_{\mathrm{inhom}}$$

has a weak solution u in the space $L^p(0,T;X_p) \cap L^\infty(0,T;L^{m'}) \cap L^q(0,T;L^q)$.

Finally, note that equations with more than one monotone part can be discussed by similar methods, and that the last result can also be combined with the results previously obtained for non-monotone nonlinearities.

Appendix A. Banach space valued functions on intervals

In the following sections we present a short introduction to the Sobolev theory of Banach space valued functions on intervals, including initial values, the energy identity and a compactness criterion.

A.1. Weak derivatives. For two locally integrable functions $u, \dot{u} \in L^1_{\mathrm{loc}}(0,T;X')$ the following properties are equivalent:

- $\int_0^T \langle \dot{u}, v\rangle = -\int_0^T \langle u, \dot{v}\rangle$ holds for every $v \in C^1_c(0,T;X)$, i.e. $(\mathrm{d}/\mathrm{d}t)u = \dot{u}$ holds on $(0,T)$ in the sense of distributions with values in X'.
- $\int_0^T \langle \dot{u}, x\rangle\varphi = -\int_0^T \langle u, x\rangle\dot{\varphi}$ holds for each $x \in X$ and $\varphi \in C^1_c(0,T;\mathbb{R})$, i.e. $(\mathrm{d}/\mathrm{d}t)\langle u, x\rangle = \langle \dot{u}, x\rangle$ holds on $(0,T)$ for each $x \in X$ in the sense of scalar distributions.
- $\int_0^T \dot{u}\varphi = -\int_0^T u\dot{\varphi}$ holds as an equation in X' for each $\varphi \in C^1_c(0,T;\mathbb{R})$.
- $u(t) - u(s) = \int_s^t \dot{u}$ holds as an equation in X' for a.e. $s, t \in (0,T)$, i.e. u is a primitive of $\dot{u}$.

In each case we say that u is weakly differentiable with weak derivative $\dot{u}$. Note that a function $u \in L^1_{\mathrm{loc}}(0,T;X')$ has at most one weak derivative $\dot{u} \in L^1_{\mathrm{loc}}(0,T;X')$, and by the last property a weakly differentiable u is automatically continuous. Further, as $L^p_{\mathrm{loc}}(0,T;X') \subset L^1_{\mathrm{loc}}(0,T;X')$ holds for every $1 \leqslant p \leqslant \infty$, it also makes sense to say that a function $u \in L^{q'}_{\mathrm{loc}}(0,T;X')$ has weak derivative $\dot{u} \in L^{p'}_{\mathrm{loc}}(0,T;X')$. Consequently, if X is reflexive and $u \in L^{q'}_{\mathrm{loc}}(0,T;X')$ has weak derivative $\dot{u} \in L^{p'}_{\mathrm{loc}}(0,T;X')$ $(p, q < \infty)$, then

$$\int_0^T \langle \dot{u}, v\rangle = -\int_0^T \langle u, \dot{v}\rangle$$

is valid for every $v \in L^p_{\mathrm{loc}}(0,T;X)$ with support in $[s,t]$ for $s, t \in (0,T)$ and weak derivative $\dot{v} \in L^q_{\mathrm{loc}}(0,T;X)$.

We also need the following generalization: Let X, Y be Banach spaces with dense intersection $X \cap Y$,[1] then $u \in L^{q'}_{\mathrm{loc}}(0,T;Y')$ is said to have weak derivative $\dot{u} \in L^{p'}_{\mathrm{loc}}(0,T;X')$, if one of the following equivalent conditions is valid:

[1] More precisely, we assume that X, Y are subspaces of a space Z such that their intersection $X \cap Y$ within Z is dense in X with respect to $\|\cdot\|_X$ and dense in Y with respect to $\|\cdot\|_Y$, and we equip $X \cap Y$ with the norm $\|z\|_{X\cap Y} := \|z\|_X + \|z\|_Y$, so that it becomes a Banach space. Note that $X \cap Y$ depends on the embedding of X, Y into Z.

- $\int_0^T \langle \dot u, v\rangle = -\int_0^T \langle u, \dot v\rangle$ holds for every $v \in C_c^1(0,T;X\cap Y)$, i.e. $(\mathrm{d}/\mathrm{d}t)u = \dot u$ holds on $(0,T)$ in the sense of distributions with values in $X \cap Y$.
- $\int_0^T \langle \dot u, z\rangle\varphi = -\int_0^T \langle u, z\rangle\dot\varphi$ holds for every $z \in X\cap Y$ and $\varphi \in C_c^1(0,T;\mathbb{R})$, i.e. $(\mathrm{d}/\mathrm{d}t)\langle u, z\rangle = \langle \dot u, z\rangle$ holds on $(0,T)$ for every $z \in X\cap Y$ in the sense of scalar distributions.
- $\int_0^T \dot u\varphi = -\int_0^T u\dot\varphi$ holds as an equation in $(X\cap Y)'$ for each $\varphi \in C_c^1(0,T;\mathbb{R})$.
- $u(t) - u(s) = \int_s^t \dot u$ holds as an equation in $(X\cap Y)'$ for a.e. $s,t \in (0,T)$, i.e. u is a primitive of $\dot u$.

A function $u \in L^{q'}_{\mathrm{loc}}(0,T;Y')$ has at most one weak derivative $\dot u \in L^{p'}_{\mathrm{loc}}(0,T;X')$, because $X\cap Y$ is dense in X. Further, if X, Y are reflexive and $u \in L^{q'}_{\mathrm{loc}}(0,T;Y')$ has weak derivative $\dot u \in L^{p'}_{\mathrm{loc}}(0,T;X')$ $(p,q<\infty)$, then

$$\int_0^T \langle \dot u, v\rangle = -\int_0^T \langle u, \dot v\rangle$$

is valid for every $v \in L^p_{\mathrm{loc}}(0,T;X)$ with support in $[s,t]$ for $s,t \in (0,T)$ and weak derivative $\dot v \in L^q_{\mathrm{loc}}(0,T;Y)$.

It is important to point out that the validity of an equation in $(X\cap Y)'$ is a much weaker condition than equality in Y'. Especially, the continuity of a weakly differentiable u into $(X\cap Y)'$ guaranteed by the last condition is not so useful as continuity of u into Y' would be.

A.2. Initial values. Let us see how to incorporate initial values in the definition of weak derivatives. Therefore, we have to assume more than local integrability, because else we do not have any information about the behaviour near 0. Denote by $C_c^1([0,T],\cdot)$ the space of continuously differentiable functions on $[0,T]$ whose derivatives have compact support in $(0,T)$, and by $L^p_{\mathrm{loc}}([0,T);\cdot)$ the space of functions which are p-integrable over compact intervals in $[0,T)$. A function $u \in L^{q'}_{\mathrm{loc}}(0,T;Y')$ is said to have weak derivative $\dot u \in L^{p'}_{\mathrm{loc}}([0,T);X')$ and initial value $u(0) \in Y'$, if one of the following equivalent conditions is valid:

- $\int_0^T \langle \dot u, v - v(T)\rangle = -\int_0^T \langle u - u(0), \dot v\rangle$ holds for every $v \in C_c^1([0,T];X\cap Y)$ with final value $v(T)$.
- $\int_0^T \langle \dot u, z\rangle(\varphi - \varphi(T)) = -\int_0^T \langle u - u(0), z\rangle\dot\varphi$ holds for each $z \in X\cap Y$ and $\varphi \in C_c^1([0,T];\mathbb{R})$ with final value $\varphi(T)$.
- $\int_0^T \dot u(\varphi - \varphi(T)) = -\int_0^T (u - u(0))\dot\varphi$ holds as an equation in $(X\cap Y)'$ for each $\varphi \in C_c^1([0,T];\mathbb{R})$ with final value $\varphi(T)$.
- $u(t) - u(0) = \int_0^t \dot u$ holds as an equation in $(X\cap Y)'$ for a.e. $t \in (0,T)$.

Note that a function $u \in L^{q'}_{\mathrm{loc}}(0,T;Y')$ with weak derivative $\dot u \in L^{p'}_{\mathrm{loc}}([0,T);X')$ automatically is absolutely continuous on $[0,T)$ when viewed as a function into $(X\cap Y)'$

and has a unique initial value $u(0) \in (X \cap Y)'$. But maybe this $u(0)$ cannot be extended to an element of Y', thus having an initial value $u(0) \in Y'$ is a nontrivial condition.

Further, if X, Y are reflexive and $u \in L^{q'}(0,T;Y')$ has weak derivative $\dot{u} \in L^{p'}(0,T;X')$ $(p, q < \infty)$ and initial value $u(0) \in Y'$, then

$$\int_0^T \langle \dot{u}, v - v(T) \rangle = -\int_0^T \langle u - u(0), \dot{v} \rangle$$

is valid for every $v \in L^p(0,T;X)$ with weak derivative $\dot{v} \in L^q(0,T;Y)$ and final value $v(T) \in X$.

A.3. Energy identity. The energy identity is a useful tool in the study of evolution equations, as the fundamental a priori estimate for weak solutions of differential equations $(\mathrm{d}/\mathrm{d}t)\varphi(u) = Au$ can be deduced from it.

Assume that Y is a uniformly smooth Banach space and consider the convex C^1-functional $\Phi\colon Y \to \mathbb{R}$, $\Phi(y) := q^{-1}\|y\|_Y^q$. Its derivative $\varphi\colon Y \to Y'$ is a monotone operator called the semi-inner product on Y. It satisfies $\langle \varphi(y), y \rangle = \|y\|_Y^q$, $\|\varphi(y)\|_{Y'} = \|y\|_Y^{q-1}$ and induces a map $\varphi\colon L^q_{\mathrm{loc}}(0,T;Y) \to L^{q'}_{\mathrm{loc}}(0,T;Y')$.

Let $u \in L^p_{\mathrm{loc}}(0,T;X) \cap L^q_{\mathrm{loc}}(0,T;Y)$ be such that $\varphi(u) \in L^{q'}_{\mathrm{loc}}(0,T;Y')$ is weakly differentiable with weak derivative $(\mathrm{d}/\mathrm{d}t)\varphi(u) \in L^{p'}_{\mathrm{loc}}(0,T;X')$. Then the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{q'}\|u(t)\|_Y^q = \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\varphi(u), u \Big\rangle$$

holds in the sense of scalar distributions on $(0,T)$. Moreover, if $u \in L^p(0,T;X) \cap L^q(0,T;Y)$ and $u(0) \in Y$ are such that $\varphi(u) \in L^{q'}(0,T;Y')$ is weakly differentiable with weak derivative $(\mathrm{d}/\mathrm{d}t)\varphi(u) \in L^{p'}(0,T;X')$ and initial value $\varphi(u(0)) \in Y'$, then the energy identity also holds on $[0,T)$.

From the energy identity we obtain

$$\frac{1}{q'}\|u(t)\|_Y^q = \frac{1}{q'}\|u(s)\|_Y^q + \int_s^t \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\varphi(u), u \Big\rangle$$

for a.e. $s, t \in (0,T)$. Estimate the last term by $\|(\mathrm{d}/\mathrm{d}t)\varphi(u)\|_{L^{p'}(0,T;X')}\|u\|_{L^p(0,T;X)}$ and integrate the inequality wih respect to s to obtain

$$\frac{1}{q'}\|u(t)\|_Y^q \leqslant \frac{1}{q'}\|u\|_{L^q(0,T;Y)}^q + T\Big\|\frac{\mathrm{d}}{\mathrm{d}t}\varphi(u)\Big\|_{L^{p'}(0,T;X')}\|u\|_{L^p(0,T;X)}$$

for a.e. $t \in (0,T)$. This inequality proves $u \in L^\infty(0,T;Y)$ and thus also $\varphi(u) \in L^\infty(0,T;Y')$.

Further, the existence of a representative of u which is a continuous map from $[0,T)$ to Y can be deduced, if there exists a sequence of functions $u_n \in C^\infty(0,T;X\cap Y)$ with $u_n \to u$ in $L^p(0,T;X)\cap L^q(0,T;Y)$, $u_n(0)\to u(0)$ in Y and $(\mathrm{d}/\mathrm{d}t)\varphi(u_n)\to (\mathrm{d}/\mathrm{d}t)\varphi(u)$ in $L^{p'}(0,T;X')$. In fact, the above inequality implies that the space of functions $u \in L^p(0,T;X)\cap L^q(0,T;Y)$ such that $\varphi(u)$ has weak derivative $(\mathrm{d}/\mathrm{d}t)\varphi(u)\in L^{p'}(0,T;X')$ is embedded into $L^\infty(0,T;Y)$. Thus the sequence u_n of continuous functions is a Cauchy sequence in $L^\infty(0,T;Y)$ and converges uniformly to a continuous function. But as u_n has also the limit u, the function u has a representative in $C([0,T),Y)$.

If Y is a Hilbert space and $q=2$, such a sequence u_n can easily be obtained by mollification. In fact, in this case the map φ is the usual linear identification of Y with Y', and due to linearity for a mollification u_n of u also the derivative $\dot{u}_n$ is a mollification of $\dot{u}$. However, in the general setting it is more difficult to obtain such a sequence u_n.

A.4. Compactness criterion. Compact subsets in the space $L^p(0,T;X)$ of p-integrable Banach valued functions on intervals have been characterized by [13, Theorem 1]:

Theorem A.1. *Let $1\leqslant p<\infty$. A subset $U\subset L^p(0,T;X)$ is relatively compact iff*

(i) *for each $0<s<t<T$ the subset $\{\int_s^t u\colon\ u\in U\}$ of X is relatively compact and*
(ii) *$\lim_{h\searrow 0}\int_0^{T-h}\|u(t+h)-u(t)\|_X^p\,\mathrm{d}t=0$ holds uniformly in $u\in U$.*

The first condition is called the space-criterion, space-compactness or uniform integrability, the second condition is called the time-criterion, time-compactness or equiintegrability. In presence of a compact operator, space-compactness need not be tested due to the following nonlinear compactness lemma proved by [11] via a method similar to the proof of [2, Lemma 1.9]:

Lemma A.2. *Let X, Y be Banach spaces and suppose that a (nonlinear) map $\varphi\colon X\cap Y\to Y'$ is compact, i.e. bounded subsets of $X\cap Y$ are mapped by φ to relatively compact subsets of Y'. Let U be a bounded subset of $L^p(0,T;X\cap Y)$ which is mapped by φ to a bounded subset in $L^r(0,T;Y')$, $r>1$, and assume that*

$$\lim_{h\searrow 0}\int_0^{T-h}\|\varphi(u(t+h))-\varphi(u(t))\|_{Y'}\,\mathrm{d}t=0$$

holds uniformly in $u\in U$. Then $\varphi(U)$ is relatively compact in $L^1(0,T;Y')$.

B.1. Compactness of the map φ. In this section we discuss compactness of the (nonlinear) map $\varphi(u) := u^{m'-1}$ as a map from $X_p \cap L^{m'}$ to L^m. Note that φ can be realized as a map $\varphi\colon (X_p \cap L^{m'})(\Omega) \to L^m(\Omega')$ for an arbitrary subdomain $\Omega' \subset \Omega$ by applying φ to a function u on Ω and restricting the resulting function $\varphi(u)$ to Ω'.

Lemma B.1. *Let $p, m \geqslant 2$, then the map $\varphi(u) := u^{m'-1}$ is compact*

- *as a map $\varphi\colon (X_p \cap L^{m'})(\Omega) \to L^m(\Omega')$ in the case that X_p is the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\nabla u\|_p$ (corresponding to Dirichlet boundary data) and $\Omega' \subset \Omega$ is an arbitrary bounded subdomain;*
- *as a map $\varphi\colon (X_p \cap L^{m'})(\Omega) \to L^m(\Omega')$ in the case that $(X_p \cap L^{m'})(\Omega)$ is the space of functions $u \in L^{m'}(\Omega)$ having distributional derivative $\nabla u \in L^p(\Omega, \mathbb{R}^n)$ (corresponding to Neumann boundary data), Ω satisfies the cone condition and $\Omega' \subset \Omega$ is an arbitrary bounded subdomain.*

P r o o f. By the Rellich-Kondrachov compactness theorem (see [1, Theorem 6.3]), in both cases the embedding of $(X_p \cap L^{m'})(\Omega)$ into $L^{m'}(\Omega')$ is compact (because for $p, m \geqslant 2$ in the case $p < n$ we have $m' \leqslant 2 < p^*$). Further, for $m \geqslant 2$ the map φ is continuous from $L^{m'}$ to L^m due to the elementary inequality

$$|a^{m'-1} - b^{m'-1}| \leqslant C|a-b|^{m'-1}$$

valid for real numbers $a, b \in \mathbb{R}$ in the case $m' < 2$ (proved e.g. in [4]). Indeed, this elementary inequality implies

$$\int_\Omega |\varphi(u) - \varphi(v)|^m \leqslant C \int |u-v|^{(m'-1)m} = C \int |u-v|^{m'},$$

so that $u_k \to u$ in $L^{m'}$ implies the convergence $\varphi(u_k) \to \varphi(u)$ in L^m. Thus if $u_k \in X_p \cap L^{m'}$ is uniformly bounded, then due to the compact embedding there is a subsequence converging strongly in $L^{m'}$ to some u, and by continuity of φ also $\varphi(u_k)$ converges strongly to $\varphi(u)$ in L^m. Hence φ is compact. $\square$

R e m a r k B.2. Compactness of φ can be proved directly also for other parameters m, p, e.g. for $m < 2$ and $m' < p$.

In fact, the proof of [1, Theorem 6.3] can be modified in the following way: To show that the image of a bounded set of functions $u \in (X_p \cap L^m)(\Omega)$ under φ restricted to Ω' is compact in $L^m(\Omega')$, on the one hand it has to be proved that

$$\int_{\Omega' \setminus \Omega_j} |\varphi(u)|^m = \int_{\Omega' \setminus \Omega_j} |u|^{m'}$$

becomes small for large j, where Ω_j is a certain exhaustion of Ω. But this follows easily from the boundedness of u in $(X_p \cap L^{m'})(\Omega)$. Then also

$$\int_{\Omega'\setminus\Omega_j} |\varphi(u)(x+h) - \varphi(u)(x)|^m \,\mathrm{d}x \leqslant C \int_{\Omega'\setminus\Omega_j} |u(x+h)|^{m'} + |u(x)|^{m'} \,\mathrm{d}x$$

becomes small for large j.

Hence it merely has to be shown that

$$\int_{\Omega'} |\varphi(u)(x+h) - \varphi(u)(x)|^m \,\mathrm{d}x$$

becomes small for small h. However,

$$\begin{aligned}
&\int_{\Omega_j} |\varphi(u)(x+h) - \varphi(u)(x)|^m \,\mathrm{d}x \\
&\quad\leqslant \int_{\Omega_j} \left(\int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}t} \varphi(u)(x+th) \right| \mathrm{d}t \right)^m \mathrm{d}x \leqslant \int_{\Omega_j} \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}t} \varphi(u)(x+th) \right|^m \mathrm{d}t \,\mathrm{d}x \\
&\quad\leqslant |h| \int_{\Omega_{2j}} |\varphi'(u)(y)|^m |\nabla u(y)|^m \,\mathrm{d}y = |h|(m'-1)^m \int_{\Omega'} |u|^{m'(2-m)} |\nabla u|^m \\
&\quad\leqslant |h|(m'-1)^m \|u\|_{m',\Omega'}^{m'(2-m)} \|\nabla u\|_{m',\Omega'}^m
\end{aligned}$$

holds and becomes small for small h, as $\|\nabla u\|_{m',\Omega'} \leqslant \|\nabla u\|_{p,\Omega'}$ due to $m' < p$ and the boundedness of Ω'. Thus also for parameters $m < 2$ and $m' < p$ the map φ is compact.

B.2. Monotonicity. In the course of proof of the existence of weak solutions, monotonicity was extensively used. Especially, we applied the following well-known monotonicity lemma, which is proved e.g. in [15, Lemma 3.2.2]:

Lemma B.3. *Let $A\colon X \to X'$ be a nonlinear monotone, bounded and hemicontinuous operator from a real reflexive Banach space X to X'. If $u_n \rightharpoonup u$ in X, $Au_n \rightharpoonup (Au)_{\mathrm{ex}}$ weakly in X' and $\limsup_{n\to\infty} \langle Au_n, u_n\rangle_X \leqslant \langle (Au)_{\mathrm{ex}}, u\rangle$, then $Au = (Au)_{\mathrm{ex}}$.*

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