Wen Zhang; Jianwen Zhang

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GLOBAL EXISTENCE OF SOLUTIONS FOR THE 1-D RADIATIVE AND REACTIVE VISCOUS GAS DYNAMICS*

Wen Zhang, Jianwen Zhang, Xiamen

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Abstract. In this paper, we prove the existence of a global solution to an initial-boundary value problem for 1-D flows of the viscous heat-conducting radiative and reactive gases. The key point here is that the growth exponent of heat conductivity is allowed to be any nonnegative constant; in particular, constant heat conductivity is allowed.

Keywords: reactive and radiative gas, global solution, global a priori estimates

MSC 2010: 35Q30, 35L65, 35N10, 76N15

1. Introduction

The purpose of this paper is to study mathematically an initial-boundary value problem in the bounded domain $\Omega := (0,1)$ for the 1-D flows of compressible, viscous and heat-conducting gases in the case that both the reactive processes of combustion and the radiative effects at high temperature are taken into account. The system under consideration reads (see [20], [6], [7])

$$\varrho_t + (\varrho u)_x = 0, \tag{1.1}$$

$$\varrho(u_t + uu_x) + p_x = (\nu u_x)_x, \tag{1.2}$$

$$\varrho(e_t + ue_x) + pu_x = (\kappa\theta_x)_x + \nu u_x^2 + \lambda\varrho\varphi z, \tag{1.3}$$

$$\varrho(z_t + uz_x) = (d\varrho z_x)_x - \varrho\varphi z \tag{1.4}$$

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with the initial-boundary conditions

$$(\varrho, u, \theta, z)(x, 0) = (\varrho_0, u_0, \theta_0, z_0)(x), \tag{1.5}$$

$$(u, \theta_x, z_x)\big|_{x=0} = (u, \theta_x, z_x)\big|_{x=1} = 0. \tag{1.6}$$

Here, the density $\varrho = \varrho(x, t)$, the velocity $u = u(x, t)$, the absolute temperature $\theta = \theta(x, t)$ and the fraction of reactant $z = z(x, t)$ are unknown functions. The positive constants ν, d, and λ are the viscosity coefficient, the species diffusion, and the difference in heat between the reactant and the product, respectively; while κ is the heat conductivity satisfying a certain condition given below.

The function $\varphi = \varphi(\theta)$, describing the intensity of chemical reaction, mimics the simplest one-order Arrhenius kinetics (cf. [3], [4], [20]):

$$\varphi(\theta) = K\theta^{\beta}\mathrm{e}^{-A/\theta} \geqslant 0, \tag{1.7}$$

where the positive constants A and K are the activation energy and the coefficient of rate of the reactant, respectively; and β is a nonnegative physical number.

The pressure $p = p(\varrho, \theta)$ and the internal energy $e = e(\varrho, \theta)$ are related with the density and the temperature by the equations of state (cf. [15], [22]):

$$p(\varrho, \theta) = R\varrho\theta + \frac{1}{3}a\theta^4, \qquad e(\varrho, \theta) = c_V\theta + a\theta^4/\varrho, \tag{1.8}$$

where the positive constants R, c_V, and a are the perfect gas constant, the specific heat at constant volume, and the Stefan-Boltzmann constant, respectively. The state functions in (1.8) include the perfect polytropic gas contribution proportional to θ and the Stefan-Boltzmann radiative contribution proportional to θ^4.

Since the pioneering work of Kazhikhov and Shelukhin (cf. [13]), there has been a lot of literature on the study of global existence and large-time behavior of solutions for the 1-D models of compressible, viscous and heat-conductive fluids. In particular, the non-radiative ($a = 0$) but reactive flows for one-dimensional compressible, viscous and heat-conductive gases were considered in [2], [3], [4], [14], [19], [21]. The special case $d = 0$ (the binary-mixture case) was treated in [5], [23]. The 1-D non-reactive but radiative flows were studied in [8]. Recently, the combined radiative and reactive case which is the most general and important case has been extensively studied, see, for example, [7], [9], [10], [18] among others.

As was pointed out in [7], [8], to prove the global existence of strong/classical solutions, the following growth condition on the heat conductivity,

$$\kappa_1(1 + \theta)^q \leqslant \kappa(\varrho, \theta) \leqslant \kappa_2(1 + \theta)^q \quad \text{for some } q > 0, \tag{1.9}$$

plays a mathematically very important role in the proof of a priori estimates, see, for instance, [7], [8], [18] for $q \geqslant 4$, [10] for $q \geqslant 2$, and [9] for $q \geqslant 1$. Although condition (1.9) with $q \geqslant 3$ is physically reasonable when the radiative effects at high temperature are involved (cf. [15]), it is indeed a "regularized" condition from the mathematical point of view. The reason lies in the fact that condition (1.9) leads to some additional estimates of temperature from the starting energy-entropy estimates, for example, $\|\theta(t)\|_{L^\infty} \in L^{q+4}(0,T)$ and $\theta^{(q-2)/2}\theta_x \in L^2(Q_T)$, and hence, the nonlinear radiation terms can be well controlled. As a result, the estimates of the derivatives for the solutions and the pointwise upper bound for temperature can be obtained in a similar manner to that in [12], [11] by using these additional estimates. In view of this observation, it is thus mathematically interesting to study the optimal growth condition of heat conductivity for the problem (1.1)–(1.8). "Optimal" here means that whether or not there exists a uniquely global strong/classical solution to the problem (1.1)–(1.8) when the growth condition (1.9) is satisfied by any nonnegative exponent q, especially, when the heat conductivity κ is only a positive constant. This is our main purpose in this paper. Precisely, we shall prove

Theorem 1.1. *Let the heat conductivity κ be a positive constant, and let $\beta \in [0,8]$. Assume that for some positive constant C_0 we have*

$$C_0^{-1} \leqslant \varrho_0(x), \quad \theta_0(x) \leqslant C_0, \quad 0 \leqslant z_0(x) \leqslant 1, \tag{1.10}$$

$$\|(\varrho_0, u_0, \theta_0, z_0)\|_{H^2(\Omega)} \leqslant C_0. \tag{1.11}$$

Then for any given $T > 0$, the initial-boundary value problem (1.1)–(1.8) *has a uniquely global solution (ϱ, u, θ, z) defined over $Q_T := \Omega \times (0,T)$ such that*

$$C^{-1} \leqslant \varrho(x,t), \quad \theta(x,t) \leqslant C, \quad 0 \leqslant z(x,t) \leqslant 1 \qquad \forall\, (x,t) \in \bar{Q}_T, \tag{1.12}$$

$$\|(\varrho, u, \theta, z)(t)\|_{H^2(\Omega)} + \|(\varrho_t, u_t, \theta_t, z_t)(t)\|_{L^2(0,T;H^1(\Omega))} \leqslant C, \tag{1.13}$$

where C is a generic positive constant depending on T.

The global existence of solutions will be proved by combining the local existence theorem and the global a priori estimates. The local existence of solutions can be shown in a standard way as in [1], [16], [17]. So, to prove the global existence, it suffices to prove the global a priori estimates stated in (1.12) and (1.13). This will be done in Section 2. Compared with those works in which various growth conditions on the heat conductivity were assumed, the a priori global estimates of solutions to the problem considered are more complicated, and some new ideas have to be developed. This is mainly due to the effects of radiative pressure, consequently, the boundedness of $\|\varrho_x(t)\|_{L^2}$ cannot be obtained directly from the standard energy-entropy estimates (cf. Lemma 2.1) and the pointwise upper-lower bounds of ϱ (cf. Lemma 2.2) at the stage when κ is only a positive constant. The boundedness of $\|\varrho_x(t)\|_{L^2}$ plays an important role in the proof of boundedness of θ and the global estimates of high-order derivatives for the physical quantities. To overcome this difficulty, the key step here is to derive some higher integrability of θ in terms of the norm $\|u_{xx}\|_{L^2(Q_T)}$ and to bound $\|\varrho_x(t)\|_{L^2}$ in terms of the norms $\|\theta\|_{L^\infty(Q_T)}$ and $\|u_{xx}\|_{L^2(Q_T)}$ (see Lemmas 2.4 and 2.5). This method is different from those used in the previous works mentioned above.

The analysis in the paper also allows the growth power q in (1.9) to be any positive constant. More precisely, similarly to the proof of Theorem 1.1, we can show

Theorem 1.2. *Let* (1.10), (1.11), *and* $\beta \in [0,8]$ *be satisfied. Assume that* $\kappa = \kappa(\varrho,\theta)$ *is strictly positive, continuously differentiable on* $\mathbb{R}^+ \times \mathbb{R}^+$, *and for any* $q > 0$ *let*

$$\kappa_1(1+\theta)^q \leqslant \kappa(\varrho,\theta) \leqslant \kappa_2(1+\theta)^q \quad \forall\, \varrho \in (0,+\infty), \tag{1.14}$$

$$|\kappa_\varrho(\varrho,\theta)| \leqslant \kappa_2(1+\theta)^q, \quad |\kappa_\theta(\varrho,\theta)| \leqslant \kappa_2(1+\theta)^{q-1}, \tag{1.15}$$

where κ_1 *and* κ_2 *are positive constants. Then problem* (1.1)–(1.8) *has a uniquely global solution* (ϱ,u,θ,z) *satisfying* (1.12), (1.13).

Theorem 1.2 will be proved in Section 3. It is clear that Theorem 1.2 covers the most physically interesting case of thermal radiation (i.e. $q = 3$), and extends thus the existence results in [7], [9], [18]. Moreover, by virtue of (1.12) and (1.13) one can easily deduce Hölder estimates of the solutions to problem (1.1)–(1.8) (see, e.g. [18]).

We remark here that the results of Theorems 1.1 and 1.2 hold for more general Arrhenius-type law where the rate function of chemical reaction has the form (cf. [20])

$$f(\varrho,\theta,z)z^m = \begin{cases} 0 & \text{for } 0 \leqslant \theta < \theta_I, \\ c_0\varrho^{m-1}\theta^r \mathrm{e}^{-c_1/(\theta-\theta_I)} z^m & \text{for } \theta > \theta_I, \end{cases}$$

where $c_0, c_1 > 0$, $r \leqslant 4$, $m \geqslant 1$ is the kinetics order, and $\theta_I \geqslant 0$ is the ignition temperature (ignition is ignored if $\theta_I = 0$), since the pointwise boundedness of z and ϱ given in (2.1) and (2.6) is still valid in this situation. So, the existence result for the higher-order kinetics case treated in [10] with $q \geqslant 2$ being assumed, is extended to any growth exponent $q \geqslant 0$. We also note that the condition $\beta \in [0,8]$ improves the result in [18] where $\beta \in [0, 13/2]$ was assumed.

2. Proof of Theorem 1.1

In this section, we prove our main result stated in Theorem 1.1. For this purpose, we need to establish some a priori global estimates for the solution and its derivatives. For simplicity, throughout the rest of this paper we use the same letter C to denote generic positive constants possibly depending on T. As usual, we begin with the standard energy-entropy estimates, which can be found in [4], [7], [19], [18], by using the Lagrangian coordinates.

Lemma 2.1. *For any fixed $T > 0$ and $t \in [0, T]$ we have*

$$0 \leqslant z(x,t) \leqslant 1 \quad \forall\,(x,t) \in [0,1] \times [0,T], \tag{2.1}$$

$$\int_0^1 \varrho(x,t)\,\mathrm{d}x = \int_0^1 \varrho_0(x)\,\mathrm{d}x > 0, \tag{2.2}$$

$$\int_0^1 [\varrho(\theta + u^2 + z) + \theta^4](x,t)\,\mathrm{d}x \leqslant C, \tag{2.3}$$

$$\int_0^1 \varrho z(x,t)\,\mathrm{d}x + \int_0^t \int_0^1 \varrho\varphi z\,\mathrm{d}x\,\mathrm{d}s \leqslant C, \tag{2.4}$$

$$\int_0^t \int_0^1 \Big(\frac{\kappa\theta_x^2}{\theta^2} + \frac{u_x^2 + \varrho\varphi z}{\theta}\Big)\,\mathrm{d}x\,\mathrm{d}s \leqslant C. \tag{2.5}$$

With help of Lemma 2.1, similarly to the proof in [13], [16], it is easy to establish

Lemma 2.2. *For any fixed $T > 0$ we have*

$$C^{-1} \leqslant \varrho(x,t) \leqslant C \quad \forall\,(x,t) \in [0,1] \times [0,T], \tag{2.6}$$

$$\int_0^T \|\theta(t)\|_{L^\infty}^4\,\mathrm{d}t + \int_0^T \int_0^1 \theta^8(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{2.7}$$

As an immediate consequence of Lemma 2.2, we find

Lemma 2.3. *For any fixed $T > 0$ we have*

$$\sup_{t\in[0,T]} \int_0^1 (u^2 + z^2)(x,t)\,\mathrm{d}x + \int_0^T \int_0^1 (u_x^2 + z_x^2)\,\mathrm{d}x\,\mathrm{d}t \leqslant C, \tag{2.8}$$

$$\int_0^T (\|u(t)\|_{L^\infty}^4 + \|z(t)\|_{L^\infty}^4)\,\mathrm{d}t + \int_0^T \int_0^1 (u^6 + z^6)\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{2.9}$$

P r o o f. Multiplying (1.2) and (1.4) by u and z in $L^2(Q_t)$ ($Q_t := (0,1) \times (0,t)$), respectively, by virtue of Lemma 2.2 we obtain (2.8) after integrating by parts.

Observing that

$$\|u(t)\|_{L^\infty}^2 \leqslant 2\int_0^1 |u||u_x|\,\mathrm{d}x \leqslant C\Big(\int_0^1 u_x^2\,\mathrm{d}x\Big)^{1/2},$$

we immediately deduce from (2.8) that $\|u(t)\|_{L^\infty}^4 \in L^1(0,T)$. Consequently,

$$\int_0^T\int_0^1 u^6(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^T \|u(t)\|_{L^\infty}^4 \cdot \int_0^1 u^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C.$$

Analogous estimates for z can be shown in the same manner. $\square$

To continue, we need some further estimates on temperature, which will be used later to bound the radiation terms and to prove the pointwise boundedness of temperature. For simplicity, we set

$$\Theta := \|\theta\|_{L^\infty(Q_T)} = \sup_{t\in[0,T]} \|\theta(t)\|_{L^\infty}.$$

We shall prove

Lemma 2.4. *For any given $T>0$ and $t\in[0,T]$ we have*

$$\int_0^1 \theta^8(x,t)\,\mathrm{d}x + \int_0^t\int_0^1 \kappa\theta^3\theta_x^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\Big(\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2}, \tag{2.10}$$

$$\int_0^1 \theta^{11}(x,t)\,\mathrm{d}x + \int_0^t\int_0^1 \kappa\theta^6\theta_x^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\Big(\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{15/16}. \tag{2.11}$$

P r o o f. To prove this lemma, we first rewrite (1.3) as follows, using (1.1) and (1.8):

$$a(\theta^4)_t + c_V\varrho\theta_t + c_V\varrho u\theta_x + a(u\theta^4)_x + \Big(R\varrho\theta + \frac{a}{3}\,\theta^4\Big)u_x = (\kappa\theta_x)_x + \nu u_x^2 + \lambda\varrho\varphi z. \tag{2.12}$$

Multiplying (2.12) by θ^4 and integrating the resulting equation by parts over Q_t, we have by Lemmas 2.1–2.3 and the fact that $\varphi \leqslant C\theta^\beta$ that (keeping in mind that $\beta\in[0,8]$)

$$\begin{aligned}\int_0^1 \theta^8(x,t)\,\mathrm{d}x &+ \int_0^t\int_0^1 \kappa\theta^3\theta_x^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C + C\int_0^t\int_0^1 \Big(\theta^8|u_x| + \theta^4u_x^2 + \theta^{12}\Big)\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C + C\int_0^t (\|\theta(s)\|_{L^\infty}^8 + \|u_x(s)\|_{L^\infty}^2)\,\mathrm{d}s,\end{aligned} \tag{2.13}$$

where we have also used the Cauchy-Schwarz inequality and the identity

$$\int_0^t\int_0^1 (u\theta^4)_x\theta^4\,\mathrm{d}x\,\mathrm{d}s = -4\int_0^t\int_0^1 u\theta^7\theta_x\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_0^t\int_0^1 \theta^8 u_x\,\mathrm{d}x\,\mathrm{d}s.$$

To deal with $\|\theta(s)\|_{L^\infty}^8$, we first utilize (2.3) and Hölder's inequality to infer that

$$\|\theta(s)\|_{L^\infty}^{9/2} \leqslant C + C\int_0^1 \theta^2\theta^{3/2}|\theta_x|\,\mathrm{d}x \leqslant C + C\Big(\int_0^1 \theta^3\theta_x^2\,\mathrm{d}x\Big)^{1/2},$$

and hence,

$$\int_0^t \|\theta(s)\|_{L^\infty}^9\,\mathrm{d}s \leqslant C + C\int_0^t\int_0^1 \theta^3\theta_x^2\,\mathrm{d}x\,\mathrm{d}s.$$

This, together with Young's inequality, gives

$$\int_0^t \|\theta(s)\|_{L^\infty}^8\,\mathrm{d}s \leqslant \varepsilon\int_0^t \|\theta(s)\|_{L^\infty}^9\,\mathrm{d}s + \frac{C}{\varepsilon} \leqslant C\varepsilon\int_0^t\int_0^1 \theta^3\theta_x^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C}{\varepsilon}. \tag{2.14}$$

On the other hand, it follows from (2.8) that

$$\begin{aligned}\int_0^t \|u_x(s)\|_{L^\infty}^2\,\mathrm{d}s &\leqslant C\int_0^t \|u_x(s)\|_{L^2}\|u_{xx}(s)\|_{L^2}\,\mathrm{d}s\\ &\leqslant C\Big(\int_0^t \|u_{xx}(s)\|_{L^2}^2\,\mathrm{d}s\Big)^{1/2}.\end{aligned} \tag{2.15}$$

So, putting (2.14), (2.15) into (2.13) and taking $\varepsilon > 0$ small enough, one immediately gets (2.10).

Similarly, multiplying (2.12) by θ^7 and integrating it by parts, we get

$$\begin{aligned}\int_0^1 \theta^{11}(x,t)\,\mathrm{d}x &+ \int_0^t\int_0^1 \kappa\theta^6\theta_x^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant C + C\int_0^t\int_0^1 (\theta^{11}|u_x| + \theta^7u_x^2 + \theta^{15})\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant C + C\int_0^t\int_0^1 (\theta^7u_x^2 + \theta^{15})\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{2.16}$$

In view of (2.10), we have for any $s \in (0,t)$ that

$$\int_0^1 \theta^7(x,s)\,\mathrm{d}x \leqslant \Big(\int_0^1 \theta^8(x,s)\,\mathrm{d}x\Big)^{7/8} \leqslant C + C\Big(\int_0^t \|u_{xx}(s)\|_{L^2}^2\,\mathrm{d}s\Big)^{7/16},$$

from which and (2.15) we deduce that

$$\begin{aligned}
(2.17) \qquad \int_0^t \int_0^1 \theta^7 u_x^2 \,\mathrm{d}x \,\mathrm{d}s &\leqslant \int_0^t \|u_x(s)\|_{L^\infty}^2 \cdot \int_0^1 \theta^7(x,s) \,\mathrm{d}x \\
&\leqslant C + C\Big(\int_0^t \int_0^1 u_{xx}^2 \,\mathrm{d}x \,\mathrm{d}s\Big)^{15/16}.
\end{aligned}$$

Moreover, due to the estimates

$$\int_0^t \int_0^1 \theta^{15} \,\mathrm{d}x \,\mathrm{d}s \leqslant C \int_0^t \|\theta(s)\|_{L^\infty}^{11} \cdot \int_0^1 \theta^4(x,s) \,\mathrm{d}x \,\mathrm{d}s \leqslant C \int_0^t \|\theta(s)\|_{L^\infty}^{11} \,\mathrm{d}s$$

and

$$\int_0^t \|\theta(s)\|_{L^\infty}^{12} \,\mathrm{d}s \leqslant C + C \int_0^t \Big(\int_0^1 \theta^2 \theta^3 |\theta_x| \,\mathrm{d}x\Big)^2 \mathrm{d}s \leqslant C + C \int_0^t \int_0^1 \theta^6 \theta_x^2 \,\mathrm{d}x \,\mathrm{d}s,$$

it is easy to see that

$$\begin{aligned}
(2.18) \qquad \int_0^t \int_0^1 \theta^{15} \,\mathrm{d}x \,\mathrm{d}s \leqslant C \int_0^t \|\theta(s)\|_{L^\infty}^{11} \,\mathrm{d}s &\leqslant C\Big(\int_0^t \|\theta(s)\|_{L^\infty}^{12} \,\mathrm{d}s\Big)^{11/12} \\
&\leqslant C + C\Big(\int_0^t \int_0^1 \theta^6 \theta_x^2 \,\mathrm{d}x \,\mathrm{d}s\Big)^{11/12}.
\end{aligned}$$

Plugging (2.17) and (2.18) into (2.16) and using the Cauchy-Schwarz inequality, we immediately arrive at (2.11). Therefore, the proof of Lemma 2.4 is complete. □

As aforementioned in the introduction, due to the effect of radiative pressure, we cannot obtain the boundedness of $\|\varrho_x(t)\|_{L^2}$ directly from Lemmas 2.1–2.3 when the heat conductivity is only a positive constant. Instead, we shall bound $\|\varrho_x(t)\|_{L^2}$ by $\Theta = \|\theta\|_{L^\infty(Q_T)}$ and $\|u_{xx}\|_{L^2(Q_T)}$, which is crucial for the estimates of derivatives of the solutions.

Lemma 2.5. *For any fixed $T > 0$ we have*

$$\begin{aligned}
(2.19) \qquad \sup_{t\in[0,T]} \int_0^1 \varrho_x^2(x,t) \,\mathrm{d}x + \int_0^T \int_0^1 \theta \varrho_x^2 \,\mathrm{d}x \,\mathrm{d}t & \\
\leqslant C(1+\Theta) + C\Big(\int_0^T \int_0^1 u_{xx}^2 \,\mathrm{d}x \,\mathrm{d}t\Big)^{15/16}&.
\end{aligned}$$

P r o o f. By virtue of (1.1) and (2.6), we can rewrite (1.2) in the form

$$\Big[\varrho\Big(\nu\frac{\varrho_x}{\varrho^2} + u\Big)\Big]_t + \Big[\varrho u\Big(\nu\frac{\varrho_x}{\varrho^2} + u\Big)\Big]_x = -\Big(R\varrho\theta + \frac{a}{3}\,\theta^4\Big)_x$$

which, multiplied by $(\nu\varrho_x/\varrho^2+u)$ in L^2, results in

$$L := \frac{1}{2}\int_0^1 \varrho\Big(\nu\frac{\varrho_x}{\varrho^2}+u\Big)^2(x,s)\,\mathrm{d}x\Big|_0^t + \nu R\int_0^t\int_0^1 \frac{\theta}{\varrho^2}\,\varrho_x^2\,\mathrm{d}x\,\mathrm{d}s$$
$$= \int_0^t\int_0^1\Big[\Big(-R\varrho\theta_x - \frac{4a}{3}\,\theta^3\theta_x\Big)\Big(\nu\frac{\varrho_x}{\varrho^2}+u\Big) - R\theta u\varrho_x\Big]\,\mathrm{d}x\,\mathrm{d}s := \sum_{i=1}^3 R_i, \tag{2.20}$$

where the left-hand side can be bounded as follows, using (2.3) and (2.6),

$$L \geqslant C^{-1}\Big(\int_0^1 \varrho_x^2(x,t)\,\mathrm{d}x + \int_0^t\int_0^1 \theta\varrho_x^2\,\mathrm{d}x\,\mathrm{d}s\Big) - C.$$

To bound the right-hand side, we first make use of (2.3) and (2.5)–(2.7) to deduce

$$\begin{aligned}
|R_1| &:= \Big|\int_0^t\int_0^1 R\varrho\theta_x\Big(\nu\frac{\varrho_x}{\varrho^2}+u\Big)\,\mathrm{d}x\,\mathrm{d}s\Big| \\
&\leqslant \delta\int_0^t\int_0^1 \theta\varrho_x^2\,\mathrm{d}x\,\mathrm{d}s + C\delta^{-1}\int_0^t\int_0^1 \frac{\theta_x^2}{\theta}\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1\Big(\theta^2u^2+\frac{\theta_x^2}{\theta^2}\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant \delta\int_0^t\int_0^1 \theta\varrho_x^2\,\mathrm{d}x\,\mathrm{d}s + C\delta^{-1}(1+\Theta), \qquad 0<\delta<1.
\end{aligned}$$

Secondly, using (2.6)–(2.8), (2.10), and (2.11), we obtain from the Cauchy-Schwarz inequality that

$$\begin{aligned}
|R_2| &:= \Big|\int_0^t\int_0^1 \frac{4a}{3}\,\theta^3\theta_x\Big(\nu\frac{\varrho_x}{\varrho^2}+u\Big)\,\mathrm{d}x\,\mathrm{d}s\Big| \\
&\leqslant \delta\int_0^t\int_0^1 \theta\varrho_x^2\,\mathrm{d}x\,\mathrm{d}s + C\delta^{-1}\int_0^t\int_0^1 \theta^5\theta_x^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1 \theta^4|u_x|\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant \delta\int_0^t\int_0^1 \theta\varrho_x^2\,\mathrm{d}x\,\mathrm{d}s + C\delta^{-1}\int_0^t\int_0^1 (\theta^3+\theta^6)\theta_x^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1 (\theta^8+u_x^2)\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C + \delta\int_0^t\int_0^1 \theta\varrho_x^2\,\mathrm{d}x\,\mathrm{d}s + C\delta^{-1}\Big\{1+\Big(\int_0^t\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{15/16}\Big\},
\end{aligned}$$

since integrating by parts yields

$$\int_0^t\int_0^1 \theta^3\theta_x u\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{4}\int_0^t\int_0^1 \theta^4u_x\,\mathrm{d}x\,\mathrm{d}s \leqslant C\int_0^t\int_0^1 \theta^4|u_x|\,\mathrm{d}x\,\mathrm{d}s.$$

Finally, it follows from (2.7) and (2.8) that

$$|R_3| := \Big|\int_0^t\int_0^1 R\theta u\varrho_x\,\mathrm{d}x\,\mathrm{d}s\Big| \leqslant \delta\int_0^t\int_0^1 \theta\varrho_x^2\,\mathrm{d}x\,\mathrm{d}s + C\delta^{-1}.$$

Thus, putting the estimates of L, R_i $(i=1,2,3)$ into (2.20) and choosing $\delta>0$ small enough, we obtain the desired estimates indicated in (2.19) by taking the super-norm over $t\in[0,T]$. □

In order to estimate $\|u_x(t)\|_{L^2}$ and $\|u_{xx}\|_{L^2(Q_T)}$, we first need to prove

Lemma 2.6. *For any fixed $T > 0$ we have*

$$\sup_{t\in[0,T]} \int_0^1 u^4(x,t)\,\mathrm{d}x + \int_0^T\int_0^1 u^2u_x^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\Big(\sup_{t\in[0,T]}\int_0^1 u_x^2(x,t)\,\mathrm{d}x\Big)^{1/2}. \tag{2.21}$$

Proof. Multiplying (1.2) by u^3 and integrating the resulting equation by parts over Q_t, one infers from (2.6), (2.7), and the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\int_0^1 u^4(x,t)\,\mathrm{d}x + \int_0^t\int_0^1 u^2u_x^2\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad \leqslant C + C\int_0^t\int_0^1 (\theta+\theta^4)u^2|u_x|\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad \leqslant C + \frac12\int_0^t\int_0^1 u^2u_x^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_0^1 (1+\theta^8)u^2\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad \leqslant C + \frac12\int_0^t\int_0^1 u^2u_x^2\,\mathrm{d}x\,\mathrm{d}s + C\sup_{t\in[0,T]}\|u(t)\|_{L^\infty}^2 \\
&\qquad \leqslant C + \frac12\int_0^t\int_0^1 u^2u_x^2\,\mathrm{d}x\,\mathrm{d}s + C\Big(\sup_{t\in[0,T]}\int_0^1 u_x^2(x,t)\,\mathrm{d}x\Big)^{1/2}.
\end{aligned}$$

This completes the proof of (2.21). $\square$

Lemma 2.7. *For any fixed $T > 0$ we have*

$$\sup_{t\in[0,T]}\int_0^1 u_x^2(x,t)\,\mathrm{d}x + \int_0^T\int_0^1 (u_t^2+u_{xx}^2)\,\mathrm{d}x\,\mathrm{d}s \leqslant C(1+\Theta^2), \tag{2.22}$$

$$\int_0^T \|u_x(t)\|_{L^\infty}^2\,\mathrm{d}t \leqslant C(1+\Theta), \quad \int_0^T\int_0^1 u_x^4\,\mathrm{d}x\,\mathrm{d}t \leqslant C(1+\Theta^3). \tag{2.23}$$

Proof. Multiplying (1.2) by u_t and integrating it by parts over $(0,1)\times(0,t)$, we deduce from (2.5)–(2.9), (2.11), (2.19), (2.21), and the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\int_0^1 u_x^2(x,t)\,\mathrm{d}x + \int_0^t\int_0^1 u_t^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \leqslant C + C\int_0^t\int_0^1 (u^2u_x^2 + \theta^2\varrho_x^2 + \theta^6\theta_x^2 + \theta_x^2)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \leqslant C\Big(\sup_{t\in[0,T]}\int_0^1 u_x^2(x,t)\,\mathrm{d}x\Big)^{1/2} + C\Big(\int_0^T\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{15/16} + C(1+\Theta^2),
\end{aligned} \tag{2.24}$$

where we have used (2.7) and (2.19) to get

$$\begin{aligned}\int_0^t\int_0^1 \theta^2\varrho_x^2\,\mathrm{d}x\,\mathrm{d}s &\leqslant \int_0^t \|\theta(s)\|_{L^\infty}^2\cdot\int_0^1 \varrho_x^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant C(1+\Theta)+C\Big(\int_0^T\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{15/16}.\end{aligned}$$

Equation (1.2) also implies that

$$\begin{aligned}\int_0^T\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s &\leqslant C\int_0^T\int_0^1 (u^2u_x^2+\theta^6\theta_x^2+\theta^2\varrho_x^2+\theta_x^2+u_t^2)\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant C\Big(\sup_{t\in[0,T]}\int_0^1 u_x^2(x,t)\,\mathrm{d}x\Big)^{1/2}+C\Big(\int_0^T\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{15/16}\\ &\quad+C(1+\Theta^2)+C\int_0^T\int_0^1 u_t^2\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

which, together with (2.24) and Young's inequality, gives

$$\int_0^T\int_0^1 u_{xx}^2\,\mathrm{d}x\,\mathrm{d}s\leqslant C(1+\Theta^2)+C\Big(\sup_{t\in[0,T]}\int_0^1 u_x^2(x,t)\,\mathrm{d}x\Big)^{1/2}. \tag{2.25}$$

Hence, adding (2.25) to (2.24) and taking the super-norm over $t\in[0,T]$, we conclude from Young's inequality that

$$\sup_{t\in[0,T]}\int_0^1 u_x^2(x,t)\,\mathrm{d}x+\int_0^T\int_0^1 (u_t^2+u_{xx}^2)\,\mathrm{d}x\,\mathrm{d}s\leqslant C(1+\Theta^2),$$

which proves (2.22). As an immediate result, using (2.8) and Hölder's inequality, we also have

$$\begin{aligned}\int_0^T \|u_x(t)\|_{L^\infty}^2\,\mathrm{d}t &\leqslant C+C\int_0^T \|u_x(t)\|_{L^2}\|u_{xx}(t)\|_{L^2}\,\mathrm{d}t\\ &\leqslant C+C\Big(\int_0^T \|u_{xx}(t)\|_{L^2}^2\,\mathrm{d}t\Big)^{1/2}\leqslant C(1+\Theta)\end{aligned}$$

and

$$\int_0^T\int_0^1 u_x^4\,\mathrm{d}x\,\mathrm{d}t\leqslant C\int_0^T \|u_x(t)\|_{L^\infty}^2\cdot\int_0^1 u_x^2\,\mathrm{d}x\,\mathrm{d}t\leqslant C(1+\Theta^3).$$

The proof of (2.23) is therefore complete. □

To complete the proof of Lemmas 2.4–2.7, we are now in a position to estimate Θ. In the previous works (see, e.g., [7], [9], [18], ...), the proof of boundedness of θ depends heavily on the growth condition of heat conductivity with various positive exponents q which play an important role in handling the strongly nonlinear radiation terms ($\sim \theta^4$). In the present paper, we can remove such growth constraint on κ by using the previous lemmas.

Lemma 2.8. *We have $\theta(x,t) \leqslant C$ for all $(x,t) \in \bar{Q}_T$. Furthermore,*

$$\sup_{t\in[0,T]} \int_0^1 \theta_x^2(x,t)\,\mathrm{d}x + \int_0^T\int_0^1 (1+\theta^3)\theta_t^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{2.26}$$

P r o o f. If we multiply (2.12) by θ_t in $L^2(Q_t)$ and integrate by parts, we have by (2.1), (2.6), (2.7), and the Cauchy-Schwarz inequality that

$$\begin{aligned}\int_0^1 \theta_x^2(x,t)\,\mathrm{d}x &+ \int_0^t\int_0^1 (1+\theta^3)\theta_t^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C + C\int_0^t\int_0^1 (u^2\theta_x^2 + u^2\theta^3\theta_x^2 + \theta^5u_x^2 + \theta^2u_x^2 + u_x^4 + \theta^{(2\beta-3)_+})\,\mathrm{d}x\,\mathrm{d}s \\ &:= C + C\sum_{i=1}^6 R_i,\end{aligned} \tag{2.27}$$

where (1.7) and (2.7) were used to get that

$$\int_0^t\int_0^1 \theta^\beta|\theta_t|\,\mathrm{d}x\,\mathrm{d}s \leqslant \begin{cases} \varepsilon\displaystyle\int_0^t\int_0^1 \theta_t^2\,\mathrm{d}x\,\mathrm{d}s + C\varepsilon^{-1}, & \beta\in[0,4],\\ \varepsilon\displaystyle\int_0^t\int_0^1 \theta^3\theta_t^2\,\mathrm{d}x\,\mathrm{d}s + C\varepsilon^{-1}\int_0^t\int_0^1 \theta^{2\beta-3}\,\mathrm{d}x\,\mathrm{d}s, & \beta\in[4,8],\end{cases}$$

for any small $\varepsilon\in(0,1)$. The right-hand side of (2.27) can be estimated term by term as follows. First, using (2.5), (2.8), (2.22), and Sobolev's inequality, we find

$$R_1 \leqslant \Theta^2 \sup_{t\in[0,T]} \|u(t)\|_{L^\infty}^2 \int_0^t\int_0^1 \frac{\theta_x^2}{\theta^2}\,\mathrm{d}x\,\mathrm{d}s \leqslant C\Theta^2 \sup_{t\in[0,T]} \|u_x(t)\|_{L^2} \leqslant C(1+\Theta^3).$$

Secondly, by virtue of (2.10) and (2.22) we have

$$R_2 \leqslant \sup_{t\in[0,T]} \|u(t)\|_{L^\infty}^2 \int_0^t\int_0^1 \theta^3\theta_x^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C(1+\Theta^2).$$

Thirdly, it follows from (2.7) and (2.22) that

$$R_3 + R_4 \leqslant C(1+\Theta)\int_0^t (1+\|\theta(s)\|_{L^\infty}^4)\cdot\int_0^1 u_x^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C(1+\Theta^3).$$

Finally, recalling that $0 \leqslant \beta \leqslant 8$, we have by (2.23) and (2.7) that

$$R_5 + R_6 \leqslant C(1+\Theta^3) + C\Theta^5 \int_0^t \int_0^1 \theta^8 \,\mathrm{d}x \,\mathrm{d}s \leqslant C(1+\Theta^5).$$

Putting the estimates of R_i $(i = 1, \ldots, 6)$ into (2.27), we get

$$\sup_{t\in[0,T]} \int_0^1 \theta_x^2(x,t) \,\mathrm{d}x + \int_0^T \int_0^1 (\theta_t^2 + \theta^3 \theta_t^2) \,\mathrm{d}x \,\mathrm{d}s \leqslant C(1+\Theta^5). \tag{2.28}$$

To deal with the high-order term Θ^5, we first make use of (2.11) and (2.22) to see that

$$\sup_{t\in[0,T]} \int_0^1 \theta^{11}(x,t) \,\mathrm{d}x \leqslant C(1+\Theta^2). \tag{2.29}$$

On the other hand, due to

$$\|\theta(t)\|_{L^\infty}^{13/2} \leqslant C + C \int_0^1 \theta^{11/2} |\theta_x| \,\mathrm{d}x \leqslant C + C \int_0^1 (\theta^{11} + \theta_x^2)(x,t) \,\mathrm{d}x,$$

we deduce from (2.28) and (2.29) that

$$\Theta^{13/2} \leqslant C(1+\Theta^5)$$

which, combined with Young's inequality, immediately leads to

$$\Theta \leqslant C, \quad \text{i.e.,} \quad \theta(x,t) \leqslant C \ \text{ for all } \ (x,t) \in \bar{Q}_T.$$

Therefore, combining this with (2.28), we complete the proof of Lemma 2.8. $\square$

As an immediate consequence of Lemma 2.8, by (1.1) and Sobolev's inequality we infer from Lemmas 2.5 and 2.7

Lemma 2.9. *For any fixed $T > 0$ we have*

$$\sup_{t\in[0,T]} (\|\varrho_x(t)\|_{L^2} + \|\varrho_t(t)\|_{L^2} + \|u(t)\|_{L^\infty} + \|u_x(t)\|_{L^2}) \leqslant C, \tag{2.30}$$

$$\int_0^T \|u_x(t)\|_{L^\infty}^2 \,\mathrm{d}t + \int_0^T \int_0^1 (u_t^2 + u_{xx}^2 + u_x^4) \,\mathrm{d}x \,\mathrm{d}s \leqslant C. \tag{2.31}$$

In view of Lemmas 2.1–2.9, we can prove a positive lower bound for θ and estimates of higher-order derivatives for (u, θ, z) in a very standard way.

Lemma 2.10. *For any fixed $T>0$ and $t\in[0,T]$ the pair (u,θ,z) satisfies*

$$\int_0^1 (u_t^2+u_{xx}^2)(x,t)\,\mathrm{d}x+\int_0^t\int_0^1 u_{xt}^2\,\mathrm{d}x\,\mathrm{d}s\leqslant C, \tag{2.32}$$

$$\theta(x,t)\geqslant C^{-1},\quad \forall\,(x,t)\in\bar{Q}_T, \tag{2.33}$$

$$\int_0^1 (z_x^2+z_t^2+z_{xx}^2)(x,t)\,\mathrm{d}x+\int_0^t\int_0^1 z_{xt}^2\,\mathrm{d}x\,\mathrm{d}s\leqslant C, \tag{2.34}$$

$$\int_0^1 (\theta_t^2+\theta_{xx}^2)(x,t)\,\mathrm{d}x+\int_0^t\int_0^1 \theta_{xt}^2\,\mathrm{d}x\,\mathrm{d}s\leqslant C. \tag{2.35}$$

Proof. We only sketch the proof. Differentiating (1.2) with respect to t, multiplying it by u_t, and integrating the resulting equation over Q_t, by Lemmas 2.1–2.9 we find

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \varrho u_t^2(x,s)\,\mathrm{d}x\bigg|_0^t+\nu\int_0^t\int_0^1 u_{xt}^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_0^t\int_0^1\Big[(\varrho u)_x(u_t^2+uu_xu_t)-\varrho u_xu_t^2-\Big(R\varrho\theta+\frac{a}{3}\,\theta^4\Big)_{xt}u_t\Big]\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_0^t\int_0^1\Big[-(\varrho u)(u_t^2+uu_xu_t)_x-\varrho u_xu_t^2+\Big(R\varrho\theta+\frac{a}{3}\,\theta^4\Big)_{t}u_{xt}\Big]\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\leqslant\varepsilon\int_0^t\int_0^1 u_{xt}^2\,\mathrm{d}x\,\mathrm{d}s+C\varepsilon^{-1}+C\varepsilon^{-1}\int_0^t(1+\|u_x(s)\|_{L^\infty})\cdot\int_0^1 u_t^2\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}$$

from which we obtain (2.32) by taking $\varepsilon>0$ appropriately small and applying Gronwall's lemma. Note that the boundedness of $\|u_{xx}(t)\|_{L^2}$ follows from (1.2).

In view of the state equations (1.8), equation (1.3) can be written as

$$\theta_t+u\theta_x+\frac{u_xp_\theta}{\varrho e_\theta}\,\theta-\frac{\kappa}{\varrho e_\theta}\,\theta_{xx}=\frac{\nu u_x^2+\lambda\varrho\varphi z}{\varrho e_\theta}\geqslant 0.$$

Define the parabolic operator

$$\mathcal{L}(f):=\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+\frac{u_xp_\theta}{\varrho e_\theta}f-\frac{\kappa}{\varrho e_\theta}\frac{\partial^2 f}{\partial x^2},$$

where the coefficients are bounded due to (2.6), (2.32), and Lemmas 2.8–2.9. Thus, if we choose $\tilde{\theta}(t):=\inf\limits_{x\in[0,1]}\theta_0(x)\mathrm{e}^{-Kt}$ with K suitably large as a compared function, then we obtain (2.33) by applying the standard comparison argument.

Multiplying (1.4) by $\varrho^{-1} z_{xx}$ and integrating it over $(0,1)\times(0,t)$, one gets

$$\begin{aligned}\int_0^1 z_x^2(x,t)\,\mathrm{d}x + \int_0^t\int_0^1 z_{xx}^2\,\mathrm{d}x\,\mathrm{d}s &\leqslant C + C\int_0^t\int_0^1 \varrho_x^2 z_x^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant C + C\int_0^t \|z_x(s)\|_{L^2}\|z_{xx}(s)\|_{L^2}\,\mathrm{d}s\\ &\leqslant \frac{1}{2}\int_0^t\int_0^1 z_{xx}^2\,\mathrm{d}x\,\mathrm{d}s + C,\end{aligned}$$

from which and Sobolev's inequality we easily get

$$\sup_{t\in[0,T]} \|z_x(t)\|_{L^2}^2 + \int_0^T \|z_x(t)\|_{L^\infty}^2\,\mathrm{d}t + \int_0^T\int_0^1 z_{xx}^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C.$$

Using this and the estimate $\|\varrho_t(t)\|_{L^2} \leqslant C$ in (2.30), we can prove (2.34) in a manner similar to the proof of (2.32). The estimates of θ indicated in (2.35) can be shown in exactly the same way by using Lemmas 2.1–2.9 and (2.32)–(2.34). □

As the final step, we still need to estimate the second order derivatives of density.

Lemma 2.11. *For any given $T>0$ and $t\in[0,T]$ we have*

$$\int_0^1 (\varrho_{xx}^2 + \varrho_{xt}^2)(x,t)\,\mathrm{d}x \leqslant C.$$

P r o o f. As in Lemma 2.5, it follows from (1.1) and (1.2) that

$$\nu\varrho_{xt} + R\varrho\theta\varrho_x = -\nu(2\varrho_x u_x + u\varrho_{xx}) - \varrho^2(u_t + uu_x) - \varrho\Big(R\varrho\theta_x + \frac{4a}{3}\,\theta^3\theta_x\Big)$$

which, differentiated with respect to x, results in

$$\begin{aligned}\nu\varrho_{xxt} + R\varrho\theta\varrho_{xx} = &-\nu(3\varrho_{xx}u_x + 2\varrho_x u_{xx} + u\varrho_{xxx}) - 2\varrho\varrho_x(u_t + uu_x)\\ &-\varrho^2(u_{xt} + u_x^2 + uu_{xx}) - R(3\varrho\varrho_x\theta_x + \varrho_x^2\theta + \varrho^2\theta_{xx})\\ &-\frac{4a}{3}\,(\varrho_x\theta^3\theta_x + 3\varrho\theta^2\theta_x^2 + \varrho\theta^3\theta_{xx}).\end{aligned}$$

Multiplying this by ϱ_{xx} and integrating it over $(0,1)\times(0,t)$, we have by Lemmas 2.1–2.10 and the Cauchy-Schwarz inequality that

$$\int_0^1 \varrho_{xx}^2(x,t)\,\mathrm{d}x + \int_0^t\int_0^1 \varrho_{xx}^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C + C\int_0^t (1+\|u_x(s)\|_{L^\infty})\cdot\int_0^1 \varrho_{xx}^2\,\mathrm{d}x\,\mathrm{d}s,$$

where we have also used Sobolev's inequality and the estimates

$$\begin{aligned}
\int_0^t \|\varrho_x(s)\|_{L^\infty}^2 \,\mathrm{d}s &\leqslant C + C\int_0^t \|\varrho_x(s)\|_{L^2}\|\varrho_{xx}(s)\|_{L^2}\,\mathrm{d}s \\
&\leqslant C + C\int_0^t \|\varrho_{xx}(s)\|_{L^2}^2\,\mathrm{d}s, \\
\int_0^t\int_0^1 u\varrho_{xx}\varrho_{xxx}\,\mathrm{d}x\,\mathrm{d}s &= -\frac{1}{2}\int_0^t\int_0^1 u_x\varrho_{xx}^2\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C\int_0^t \|u_x(s)\|_{L^\infty}\cdot\int_0^1 \varrho_{xx}^2\,\mathrm{d}x\,\mathrm{d}s, \\
\int_0^t\int_0^1 \varrho_x^2|\varrho_{xx}|\,\mathrm{d}x\,\mathrm{d}s &\leqslant C\int_0^t\int_0^1 \varrho_{xx}^2\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t \|\varrho_x(s)\|_{L^\infty}^2\cdot\int_0^1 \varrho_x^2\,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C + C\int_0^t\int_0^1 \varrho_{xx}^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Thanks to (2.31), we have that $\|u_x(s)\|_{L^\infty} \in L^1(0,T)$. Thus, by virtue of Gronwall's lemma we obtain $\|\varrho_{xx}(t)\|_{L^2}^2 \leqslant C$, from which and (1.1) it also follows that $\|\varrho_{xt}(t)\|_{L^2}^2 \leqslant C$. The proof of Lemma 2.11 is therefore complete, and so is that of Theorem 1.1. □

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. To this end, we first notice that under the assumptions of Theorem 1.2, the estimates established in Lemmas 2.1–2.7 remain valid. So, to prove Theorem 1.2, we only need to prove the estimates analogous to those in Lemma 2.8 under the growth conditions (1.14) and (1.15). To do so, we begin with the observation

$$\int_0^T \|\theta(t)\|_{L^\infty}^{q+4}\,\mathrm{d}t + \int_0^T\int_0^1 \theta^{q+8}\,\mathrm{d}x\,\mathrm{d}t \leqslant C, \quad q > 0, \tag{3.1}$$

which follows from (1.14), (2.3), (2.5), and (2.6). The combination of (2.5), (2.10), and (2.22) also gives

$$\int_0^T\int_0^1 (1+\theta)^{q+3}\theta_x^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C(1+\Theta), \quad \Theta := \|\theta\|_{L^\infty(Q_T)}. \tag{3.2}$$

Similarly to the derivation of (2.27), multiplying (2.12) by $\kappa\theta_t$ in $L^2(Q_t)$ gives

$$\begin{aligned}
&\int_0^t\int_0^1 (\kappa\theta_x)(\kappa\theta_t)_x\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_0^1 (1+\theta)^{q+3}\theta_t^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\leqslant C\int_0^t\int_0^1 (1+\theta)^q(u^2\theta_x^2 + u^2\theta^3\theta_x^2 + \theta^5u_x^2 + \theta^2u_x^2 + u_x^4 + \theta^{(2\beta-3)_+})\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned} \tag{3.3}$$

where the right-hand side can be estimated in the same manner as in Lemma 2.8 as follows:

$$\text{RHS of (3.3)} \leqslant C(1+\Theta^q)(1+\Theta^5) \leqslant C(1+\Theta^{q+5}). \tag{3.4}$$

The first term on the left-hand side of (3.3) can be written as

$$\begin{aligned} L_1 &= \int_0^t \int_0^1 (\kappa\theta_x)(\kappa\theta_t)_x \,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^t \int_0^1 (\kappa\theta_x)[(\kappa\theta_x)_t + \kappa_\varrho \varrho_x \theta_t - \kappa_\varrho \varrho_t \theta_x] \,\mathrm{d}x\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^1 (\kappa\theta_x)^2(x,s)\,\mathrm{d}x\Big|_0^t + \int_0^t \int_0^1 \kappa\theta_x(\kappa_\varrho \varrho_x \theta_t + u\kappa_\varrho \varrho_x \theta_x + \varrho\kappa_\varrho u_x \theta_x)\,\mathrm{d}x\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^1 (\kappa\theta_x)^2(x,s)\,\mathrm{d}x\Big|_0^t + \sum_{i=1}^3 L_1^i. \end{aligned} \tag{3.5}$$

Due to (2.19) and (2.22), we have ($k_+ := \max\{k, 0\}$)

$$\begin{aligned} |L_1^1| &\leqslant \varepsilon \int_0^t \int_0^1 (1+\theta)^{q+3}\theta_t^2 \,\mathrm{d}x\,\mathrm{d}s + C\varepsilon^{-1}\int_0^t \|(\kappa\theta_x)(s)\|_{L^\infty}^2 \int_0^1 (1+\theta)^{q-3}\varrho_x^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant \varepsilon \int_0^t \int_0^1 (1+\theta)^{q+3}\theta_t^2 \,\mathrm{d}x\,\mathrm{d}s + C\varepsilon^{-1}(1+\Theta^{2+(q-3)_+})\int_0^t \|(\kappa\theta_x)(s)\|_{L^\infty}^2\,\mathrm{d}s, \end{aligned}$$

while, due to (2.6), (2.8), (2.19), and (2.22), we obtain

$$\begin{aligned} |L_1^2 + L_1^3| &\leqslant C\int_0^t \|\kappa\theta_x(s)\|_{L^\infty}^2 \int_0^1 (|u||\varrho_x| + |u_x|)\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C\int_0^t \|\kappa\theta_x(s)\|_{L^\infty}^2 \left(\int_0^1 (\varrho_x^2 + u_x^2)\,\mathrm{d}x\right)^{1/2}\mathrm{d}s \\ &\leqslant C(1+\Theta)\int_0^t \|\kappa\theta_x(s)\|_{L^\infty}^2\,\mathrm{d}s. \end{aligned}$$

So, taking $\varepsilon > 0$ small enough, we infer from (3.3)–(3.5) that

$$\begin{aligned} \int_0^1 (1+\theta)^{2q}\theta_x^2(x,t)\,\mathrm{d}x &+ \int_0^t \int_0^1 (1+\theta)^{q+3}\theta_t^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant C(1+\Theta^{q+5}) + C(1+\Theta^{2+(q-3)_+})\int_0^t \|(\kappa\theta_x)(s)\|_{L^\infty}^2\,\mathrm{d}s. \end{aligned} \tag{3.6}$$

Noticing that $\|u(t)\|_{L^\infty}^2 \leqslant \|u(t)\|_{L^2}\|u_x(t)\|_{L^2} \leqslant C(1+\Theta)$ and $\beta \in [0,8]$, by virtue of (2.22), (2.23), (3.1), and (3.2) we deduce from (2.12) that

$$\begin{aligned}
&\int_0^t\int_0^1 (1+\theta)^{q-3}|(\kappa\theta_x)_x|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C\int_0^t\int_0^1 (1+\theta)^{q-3}[(1+\theta)^6\theta_t^2 + (\theta^2+\theta^8)u_x^2 + u^2(1+\theta^6)\theta_x^2 + u_x^4 + \theta^{2\beta}] \,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C\int_0^t\int_0^1 (1+\theta)^{q+3}\theta_t^2 \,\mathrm{d}x\,\mathrm{d}s + C(1+\Theta)\int_0^t \|(1+\theta)(t)\|_{L^\infty}^{q+4}\cdot\int_0^1 u_x^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + C\sup_{t\in[0,T]}\|u(t)\|_{L^\infty}^2\int_0^t\int_0^1 (1+\theta)^{q+3}\theta_x^2 \,\mathrm{d}x\,\mathrm{d}s + C(1+\Theta^{(q-3)_+})\int_0^t\int_0^1 u_x^4 \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + C(1+\Theta^5)\int_0^t\int_0^1 (1+\theta)^{q+8} \,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C\int_0^t\int_0^1 (1+\theta)^{q+3}\theta_t^2 \,\mathrm{d}x\,\mathrm{d}s + C(1+\Theta^5+\Theta^{3+(q-3)_+}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
(3.7)\quad &\int_0^t \|(\kappa\theta_x)(s)\|_{L^\infty}^2 \,\mathrm{d}s \leqslant C\int_0^t\int_0^1 |(\kappa\theta_x)||(\kappa\theta_x)_x| \,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C\Big(\int_0^t\int_0^1 (1+\theta)^{q+3}\theta_x^2 \,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^t\int_0^1 (1+\theta)^{q-3}|(\kappa\theta_x)_x|^2 \,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2} \\
&\leqslant C(1+\Theta^{1/2})\Big(\int_0^t\int_0^1 (1+\theta)^{q-3}|(\kappa\theta_x)_x|^2 \,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2} \\
&\leqslant C(1+\Theta^{1/2})\Big(\int_0^t\int_0^1 (1+\theta)^{q+3}\theta_t^2 \,\mathrm{d}x\,\mathrm{d}s\Big)^{1/2} \\
&\quad + C\Big(1+\Theta^3+\Theta^{(4+(q-3)_+)/2}\Big),
\end{aligned}$$

where we have used (3.2). Inserting (3.7) into (3.6) and using Young's inequality, we find

$$\begin{aligned}
(3.8)\quad &\int_0^1 (1+\theta)^{2q}\theta_x^2(x,t) \,\mathrm{d}x + \int_0^t\int_0^1 (1+\theta)^{q+3}\theta_t^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\leqslant C\big(1+\Theta^{q+5}+\Theta^{5+2(q-3)_+}+\Theta^{(8+3(q-3)_+)/2}\big).
\end{aligned}$$

Thanks to (2.11) and (2.22), we have

$$\|\theta(t)\|_{L^\infty}^{q+13/2} \leqslant C + C\int_0^1 \theta^{11/2}\theta^q\theta_x \,\mathrm{d}x \leqslant C + C(1+\Theta)\Big(\int_0^1 \theta^{2q}\theta_x^2 \,\mathrm{d}x\Big)^{1/2},$$

which, combined with (3.8), yields

$$\begin{aligned}(3.9) \qquad \Theta^{2q+13} = \sup_{t\in[0,T]} \|\theta(t)\|_{L^\infty}^{2q+13} &\leqslant C + C(1+\Theta^2) \sup_{t\in[0,T]} \int_0^1 \theta^{2q}\theta_x^2 \,\mathrm{d}x \\ &\leqslant C\big(1+\Theta^{q+7}+\Theta^{7+2(q-3)_+}+\Theta^{(12+3(q-3)_+)/2}\big).\end{aligned}$$

It is easy to check that $2q+13 > \max\{q+7, 7+2(q-3)_+, 6+3(q-3)_+/2\}$ for any $q>0$. Therefore, applying Young's inequality again, we deduce from (3.9) that $\Theta = \|\theta\|_{L^\infty(Q_T)} \leqslant C$. This, together with (3.8), leads to

$$(3.10) \qquad \sup_{t\in[0,T]} (\|\theta(t)\|_{L^\infty} + \|\theta_x(t)\|_{L^2}) + \int_0^T\int_0^1 (1+\theta)^{q+3}\theta_t^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C.$$

With help of (3.10), we can obtain all the desired estimates by following arguments similar to those in the proof of Lemmas 2.9–2.11. The proof of Theorem 1.2 is thus complete. □

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Authors' address: *W. Zhang*, *J. Zhang* (corresponding author), School of Mathematical Sciences, Xiamen University, Xiamen 361005, P. R. China, e-mail: `jwzhang@xmu.edu.cn`.