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A VARIATIONAL APPROACH TO BIFURCATION IN REACTION-DIFFUSION SYSTEMS WITH SIGNORINI TYPE BOUNDARY CONDITIONS*

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Abstract. We consider a simple reaction-diffusion system exhibiting Turing's diffusion driven instability if supplemented with classical homogeneous mixed boundary conditions. We consider the case when the Neumann boundary condition is replaced by a unilateral condition of Signorini type on a part of the boundary and show the existence and location of bifurcation of stationary spatially non-homogeneous solutions. The nonsymmetric problem is reformulated as a single variational inequality with a potential operator, and a variational approach is used in a certain non-direct way.

Keywords: reaction-diffusion system, unilateral condition, variational inequality, local bifurcation, variational approach, spatial patterns

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1. Introduction

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitzian boundary $\partial\Omega$, and let Γ_D, Γ_N, Γ_U be pairwise disjoint parts of $\partial\Omega$,

$$\operatorname{mes}\Gamma_D > 0, \quad \operatorname{mes}\Gamma_U > 0, \quad \operatorname{mes}(\partial\Omega \setminus (\Gamma_D \cup \Gamma_N \cup \Gamma_U)) = 0 \tag{1.1}$$

(the $(m-1)$-dimensional Lebesgue measure). Our goal is to show on the basis of a simple variational approach the existence and location of bifurcations of nontrivial

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solutions of the system

$$d_1\Delta u + b_{11}u + b_{12}v + n(u) = 0 \qquad \text{in } \Omega \tag{1.2}$$
$$d_2\Delta v + b_{21}u + b_{22}v = 0$$

with unilateral boundary conditions, i.e.

$$\begin{cases} u = v = 0 \text{ on } \Gamma_D, \ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N, \ \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \setminus \Gamma_D, \\ u \geqslant 0, \ \dfrac{\partial u}{\partial \nu} \geqslant 0, \ u \cdot \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_U. \end{cases} \tag{1.3}$$

Simultaneously, similar results will be proved also for a "dual" problem with a nonlinearity n in the second equation and unilateral conditions for v instead of u. See Section 5 for further examples. It will be always assumed that the constant real matrix $B = (b_{ij})$ satisfies

$$b_{11} > 0 > b_{22}, \quad b_{12}b_{21} < 0, \tag{1.4}$$
$$\det B = b_{11}b_{22} - b_{12}b_{21} > 0, \quad b_{11} + b_{22} < 0,$$

n will be a function satisfying

$$n(0) = n'(0) = 0, \tag{1.5}$$

and $(d_1, d_2) \in \mathbb{R}^2_+ := (0,\infty) \times (0,\infty)$ will be real parameters (diffusion coefficients).

In order to explain the sense of our results and include it in the framework of the previous research, we must start our exposition with a more general reaction-diffusion system

$$u_t = d_1\Delta u + b_{11}u + b_{12}v + n_1(u,v) \qquad \text{in } (0,\infty) \times \Omega, \tag{1.6}$$
$$v_t = d_2\Delta v + b_{21}u + b_{22}v + n_2(u,v)$$

first with the classical mixed boundary conditions

$$u = v = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \setminus \Gamma_D. \tag{1.7}$$

Here, $n_j(0,0) = (\partial n_j/\partial u)(0,0) = (\partial n_j/\partial v)(0,0) = 0$. Clearly, $(0,0)$ is a solution of (1.6) with (1.3) as well as with (1.7).

In terms of models of chemical reactions, the first line in (1.4) means that our system is of an activator-inhibitor type (the case $b_{12} < 0 < b_{21}$) or of a positive feedback (substrate-depletion) type. See e.g. [4], [11], [16]. In the first case, u and v describe the concentration of the activator and inhibitor, respectively. It is well known that under the assumption (1.4), the trivial solution of the problem without any diffusion (i.e. of ODE's, $d_1 = d_2 = 0$) is stable but the trivial solution of the problem (1.6), (1.7) is stable only for parameters d_1, d_2 from a certain subset D_S (the domain of stability) of $\mathbb{R}^2_+$ and unstable for $d_1, d_2 \in D_U = \mathbb{R}^2_+ \setminus \overline{D}_S$ (the domain of instability). See Proposition 2.1, Remark 2.2 and Fig. 1 for details. Moreover, stationary but spatially nonhomogeneous solutions bifurcate at the border C_E between D_S and D_U (see e.g. [15], [17]). Such solutions describe spatial patterns in mathematical models in biology (see e.g. [4], [11], [16]). In fact, in applications there is usually a positive (not zero) spatially constant stationary solution $\bar{u}, \bar{v}$ and spatially nonhomogeneous stationary solutions bifurcate from $\bar{u}, \bar{v}$. However, this basic solution can be shifted to zero, which is done in our system. Let us note that any nontrivial solution of (1.6) with boundary conditions (1.7) or (1.3) is spatially nonhomogeneous due to the Dirichlet conditions on a part of the boundary.

In a series of papers (e.g. [1], [2], [6], [9], [12], [18]), an influence of unilateral conditions to this bifurcation was studied. (Usually only systems of activator-inhibitor type were discussed but in fact only the assumption (1.4) was used, i.e. the results were true also for systems of positive feedback type.) Roughly speaking, it was proved that if unilateral conditions are prescribed for the inhibitor v then, if an eigenfunction of the Laplacian satisfies a certain sign condition, bifurcation occurs even in the domain D_S, while if unilateral conditions are prescribed for the activator u then bifurcation is excluded in D_S and in some situations it is excluded even in C_E. However, the existence of a bifurcation in the last case has not been proved up to now, and is shown only in the current paper for the particular case $n_2 = 0$.

The goal of this paper is two-fold. One goal is to prove existence of bifurcation in D_U for the particular case of a nonlinearity only in the first equation and unilateral conditions for u. The other goal is to prove bifurcation in the domain D_S for the case of a nonlinearity only in the second equation and unilateral conditions for v even if all eigenfunctions of the Laplacian fail to satisfy the mentioned sign condition demanded in all previous papers. Both results are obtained by the same approach in a somewhat dual manner.

All previous results concerning bifurcation for our system with unilateral conditions for the inhibitor were based on topological methods. Namely, either on a certain homotopical joining of the variational inequality to the equation by a system of penalty problems and a transfer of the information about the existence of small nontrivial solutions from the equation to the inequality (e.g. [2], [6]), or on the direct use of the Leray-Schauder degree (a jump of the degree implies bifurcation, e.g. [18], [9]). As we already mentioned, for the proof it was always essential that certain eigenfunctions of the Laplacian satisfy a certain sign condition (in general, it must be in the interior or in a certain pseudo-interior of the cone related to the corresponding variational inequality), see e.g. [1], [9]. In the present variational approach we need no such assumption, cf. Remark 5.2. The price for the use of the variational approach is that we consider only some particular situations, and that we get no information concerning the character of the set of bifurcating solutions. However, in a particular case of situations like in Examples 5.2, 5.3, our method combined with the results [8], [19] can give a bifurcation for the case of general n_1, n_2 (Remark 3.2), and even the direction of bifurcation can be described.

In Section 2 we give an abstract formulation of our problem and summarize some facts necessary for the formulation of main results, which are given in an abstract form in Section 3 (Theorems 3.1 and 3.2). The proofs of the abstract results are given in Section 4. Section 5 is devoted to applications to unilateral boundary value problems; the first example covers the boundary conditions (1.3).

For the proof of existence of a bifurcation in D_U, we transfer for fixed $d_2 > 0$ the problem to a single variational inequality and prove by a variational approach that there is a bifurcation for a suitable d_1. The proof of existence of a bifurcation in D_S works in a dual manner by interchanging the roles of d_1 and d_2.

The idea to transfer our nonsymmetric problem for fixed d_2 to a single variational inequality comes already from [12], but up to now it has been always used only for the proof of nonexistence of critical points (and consequently, nonexistence of bifurcation). The dual approach, i.e. to transfer the nonsymmetric problem for fixed d_1 to a single variational inequality, was used in [5], also for the proof of nonexistence of bifurcation.

Unilateral boundary conditions can describe a certain regulation, e.g. by a unilateral membrane. Interpretation of the boundary conditions (1.3) (even in a more general form) and of unilateral conditions from Examples 5.2, 5.3 (the last section) is described e.g. in [1] and in Remark 5.3, respectively.

2. Motivation of abstract formulation, general remarks

Let us assume that n is a continuous function satisfying (1.5) and that there exists $c \in \mathbb{R}$ such that

$$|n(u)| \leqslant c(1 + |u|)^{q-1} \tag{2.1}$$

with some $q > 2$ or $2 < q < 2m/(m-2)$ in the case $m \leqslant 2$ or $m > 2$, respectively (in the case $m = 1$, one can even formally put $q = \infty$ and does not need to require (2.1)).

Let us introduce the real Hilbert space

$$\mathbb{H} = \{\varphi \in W^{1,2}(\Omega)\colon\ \varphi = 0 \text{ on } \Gamma_D \text{ in the sense of traces}\} \tag{2.2}$$

and the closed convex cone K with its vertex at the origin in $\mathbb{H}$,

$$K = \{\varphi \in H\colon\ \varphi \geqslant 0 \text{ on } \Gamma_U \text{ in the sense of traces}\}.$$

We equip $\mathbb{H}$ with the scalar product

$$\langle u, \varphi\rangle = \int_\Omega \nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x \quad (u,\varphi\in\mathbb{H}) \tag{2.3}$$

and the corresponding norm $\|\varphi\|^2 = \langle\varphi,\varphi\rangle$ which is equivalent to the usual Sobolev norm under the assumption (1.1), see e.g. [10]. Set

$$\langle U, W\rangle = \langle u, w\rangle + \langle v, z\rangle,\ \ \|U\|^2 = \|u\|^2 + \|v\|^2 \text{ for } U = (u, v),\ W = (w, z) \in \mathbb{H}\times\mathbb{H}.$$

Let us define operators $A, N\colon\ \mathbb{H}\to\mathbb{H}$ by

$$\langle Au, \varphi\rangle = \int_\Omega u(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } u,\varphi\in\mathbb{H}, \tag{2.4}$$

$$\langle N(u), \varphi\rangle = \int_\Omega n(u(x))\varphi(x)\,\mathrm{d}x \quad \text{for all } u,\varphi\in\mathbb{H}. \tag{2.5}$$

It follows from the compactness of the embedding $\mathbb{H}\hookrightarrow\hookrightarrow L^q(\Omega)$ and the continuity of the Nemyckij operator of $L^q(\Omega)$ into $L^{q^*}(\Omega)$, $1/q + 1/q^* = 1$ (see e.g. [10]) that under the assumption (2.1)

$$A \text{ is linear, symmetric, positive and compact,} \tag{2.6}$$

$$N \text{ is nonlinear, continuous and compact.} \tag{2.7}$$

Furthermore, under the conditions (1.5) and (2.1)

$$N \text{ is Fréchet differentiable at } 0,\ N(0) = 0,\ N'(0) = 0, \tag{2.8}$$

see e.g. [3]. Moreover, let us introduce the functional $G_N\colon\ \mathbb{H}\to\mathbb{R}$ by

$$G_N(u) = \int_\Omega\int_0^{u(x)} n(s)\,\mathrm{d}s\,\mathrm{d}x.$$

Under the assumptions (2.1), this functional is well defined, Fréchet differentiable and we have

$$G_N'(u) = N(u), \tag{2.9}$$

i.e. G_N is a potential of the operator N.

Now, we introduce a weak solution of the problem (1.2), (1.3) or (1.2), (1.7) as a couple (u, v) satisfying the variational inequality

$$\tag{2.10} \begin{cases} u \in K, \ v \in \mathbb{H}, \\ \langle d_1 u - b_{11} A u - b_{12} A v - N(u), \varphi - u \rangle \geqslant 0 \quad \text{for all } \varphi \in K, \\ d_2 v - b_{21} A u - b_{22} A v = 0, \end{cases}$$

or the equations

$$\tag{2.11} \begin{cases} u, v \in \mathbb{H}, \\ d_1 u - b_{11} A u - b_{12} A v - N(u) = 0, \\ d_2 v - b_{21} A u - b_{22} A v = 0, \end{cases}$$

respectively. The weak formulation of the linearized system

$$\tag{2.12} d_1 \Delta u + b_{11} u + b_{12} v = 0, \quad d_2 \Delta v + b_{21} u + b_{22} v = 0 \ \text{ in } \Omega$$

with (1.3) or with (1.7) is

$$\tag{2.13} \begin{cases} u \in K, \ v \in \mathbb{H}, \\ \langle d_1 u - b_{11} A u - b_{12} A v, \varphi - u \rangle \geqslant 0 \quad \text{for all } \varphi \in K, \\ d_2 v - b_{21} A u - b_{22} A v = 0, \end{cases}$$

or

$$\tag{2.14} \begin{cases} u, v \in \mathbb{H}, \\ d_1 u - b_{11} A u - b_{12} A v = 0, \\ d_2 v - b_{21} A u - b_{22} A v = 0, \end{cases}$$

respectively.

In parallel, we will consider the following dual situation with the variational inequality for v and nonlinearity dependent only on v, that means

$$\tag{2.15} \begin{cases} u \in \mathbb{H}, \ v \in K, \\ d_1 u - b_{11} A u - b_{12} A v = 0, \\ \langle d_2 v - b_{21} A u - b_{22} A v - N(v), \varphi - u \rangle \geqslant 0 \quad \text{for all } \varphi \in K, \end{cases}$$

the corresponding system of equations

$$\tag{2.16} \begin{cases} u, v \in \mathbb{H}, \\ d_1 u - b_{11} A u - b_{12} A v = 0, \\ d_2 v - b_{21} A u - b_{22} A v - N(v) = 0, \end{cases}$$

and the variational inequality with linearized operators

$$
\begin{cases} u \in \mathbb{H},\ v \in K, \\ d_1 u - b_{11} A u - b_{12} A v = 0, \\ \langle d_2 v - b_{21} A u - b_{22} A v, \varphi - u \rangle \geqslant 0 \quad \text{for all } \varphi \in K. \end{cases} \tag{2.17}
$$

In fact, we will formulate our main results in the abstract form for the variational inequalities (2.10) and (2.15) in a general real Hilbert space with a closed convex cone K with its vertex at the origin in $\mathbb{H}$, and with operators A, N satisfying (2.6), (2.7), (2.8), with N having a potential G_N, i.e. (2.9) holds.

In the sequel, with the exception of the last section, $\mathbb{H}$, K, A, N have this general meaning if nothing else is mentioned.

In some of the papers mentioned in Section 1, bifurcations with respect to (d_1, d_2) along general curves in $\mathbb{R}^2_+$ were studied. Our variational approach enables us to consider only particular cases of bifurcation in the direction d_1 or in the direction d_2 with fixed d_2 or d_1, respectively.

Definition 2.1. A parameter d_1 is a *bifurcation point* of (2.10) or (2.11) *with fixed* $d_2 = d_2^0$ if in any neighborhood of $(d_1, 0, 0)$ in $\mathbb{R} \times \mathbb{H} \times \mathbb{H}$ there is $(\tilde{d}_1, u, v)$ with $(u, v) \neq (0, 0)$ such that $(\tilde{d}_1, d_2^0, u, v)$ satisfies (2.10) or (2.11), respectively.

A parameter d_2 is a *bifurcation point* of (2.15) or (2.16) *with fixed* $d_1 = d_1^0$ if in any neighborhood of $(d_2, 0, 0)$ in $\mathbb{R} \times \mathbb{H} \times \mathbb{H}$ there is $(\tilde{d}_2, u, v)$ with $(u, v) \neq (0, 0)$ such that $(d_1^0, \tilde{d}_2, u, v)$ satisfies (2.15) or (2.16), respectively.

By a bifurcation point of the problem (1.2), (1.3) we mean always a bifurcation point of the problem (2.10) with A, N from (2.4), (2.5). Analogously for the other problems considered.

A *critical point* of the problem (2.13), (2.17) or (2.14) is a parameter $d = (d_1, d_2) \in \mathbb{R}^2_+$ for which the corresponding system has a solution $(u, v) \neq (0, 0)$. By a critical point of the problem (2.12), (1.3) we mean always a critical point of the problem (2.13) with A, N from (2.4), (2.5). Analogously for the other problems considered.

Remark 2.1. If d_1 is a bifurcation point of (2.10) with fixed $d_2 = d_2^0$ or d_2 is a bifurcation point of (2.15) with fixed $d_1 = d_1^0$ then (d_1, d_2^0) or (d_1^0, d_2) is a critical point of (2.13) or (2.17), respectively, cf. e.g. [2]. Of course, analogously for (2.11) and (2.16).

Remark 2.2. Let us consider the eigenvalue problem

$$
\begin{aligned} d_1 \Delta u + b_{11} u + b_{12} v &= \lambda u \\ d_2 \Delta v + b_{21} u + b_{22} v &= \lambda v \end{aligned} \qquad \text{in } \Omega \tag{2.18}
$$

with the boundary conditions (1.7). Let us recall that if $\operatorname{Re}\lambda \leqslant -\varepsilon < 0$ for all eigenvalues of the problem (2.18), (1.7) then the trivial solution of (1.6), (1.7) is linearly stable, and if there is at least one eigenvalue of (2.18), (1.7) satisfying $\operatorname{Re}\lambda > 0$ then the trivial solution of (1.6), (1.7) is linearly unstable (see e.g. [20]). Hence, the definition of the domain D_S and D_U of stability and instability below (related to the classical problem (1.6), (1.7)) is natural due to Proposition 2.1 below.

Notation 2.1. Let us denote by $0 < \kappa_1 < \kappa_2 \leqslant \kappa_3 \leqslant \ldots$ the characteristic values (i.e. reciprocals of eigenvalues) of the operator A, counted according to their multiplicity. Furthermore, let e_j $(j = 1, 2, \ldots)$ be a corresponding orthonormal system of eigenvectors. With each κ_j, we associate the hyperbola

$$\tilde{C}_j := \left\{d = (d_1, d_2) \in \mathbb{R}^2 \colon\ d_2 = \frac{b_{12}b_{21}/\kappa_j^2}{d_1 - b_{11}/\kappa_j} + \frac{b_{22}}{\kappa_j}\right\}$$

and denote by C_j the part of $\tilde{C}_j$ lying in the positive quadrant $\mathbb{R}^2_+$, i.e.

$$C_j := \left\{d = (d_1, d_2) \in \mathbb{R}^2_+ \colon\ d_2 = \frac{b_{12}b_{21}/\kappa_j^2}{d_1 - b_{11}/\kappa_j} + \frac{b_{22}}{\kappa_j}\right\}.$$

We denote by C_E the envelope of C_j $(j = 1, 2, \ldots)$ and define the domain of stability

$$D_S := \{d \in \mathbb{R}^2_+ \colon\ d \text{ lies to the right from } C_E, \text{ i.e. from all } C_j,\ j = 1, 2, \ldots\}$$

and the domain of instability

$$D_U := \{d \in \mathbb{R}^2_+ \colon\ d \text{ lies to the left from } C_E, \text{ i.e. from at least one } C_j\}$$

(see Fig. 1).

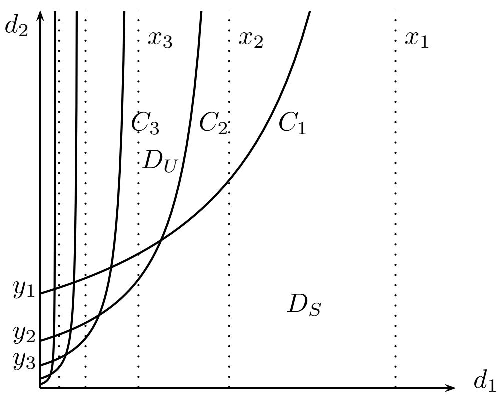

Figure 1. The system of hyperbolas C_j, their asymptotes x_j, their intersection y_j with the axis d_2, domains of stability D_S (to the right from the envelope C_E) and instability D_U (to the left from C_E).

For any $j = 1, 2, \ldots$, we will denote by $y_j := \det B/\kappa_j b_{11}$ the d_2-coordinate of the intersection point of C_j with the d_2-axis, that means

$$y_j \text{ is the positive real with } (0, y_j) \in C_j,$$

and by $x_j := b_{11}/\kappa_j$ the d_1-coordinate of the vertical asymptote of C_j.

For $(d_1, d_2) \in \bigcup_j \tilde{C}_j$, it will be convenient to define the index set

$$I(d_1, d_2) = \{j\colon\ (d_1, d_2) \in \tilde{C}_j\}.$$

Note that two hyperbolas C_j, C_k are either identical or intersect at exactly one point, and no more than two different hyperbolas intersect at any point. Hence, concerning the number of elements $|I(d_1, d_2)|$ there are essentially only two cases: If $(d_1, d_2) \in C_j$ lies on only one hyperbola, then $|I(d_1, d_2)|$ is the multiplicity of the characteristic value κ_j. If (d_1, d_2) lies on the intersection points of two hyperbolas $C_j \neq C_k$ then $|I(d_1, d_2)|$ is the sum of the multiplicities of κ_j and of κ_k.

Remark 2.3. In the case of the operator A from (2.4), κ_j is a characteristic value of A if and only if it is an eigenvalue of the boundary value problem

$$\begin{cases} -\Delta u = \kappa_j u \quad \text{in } \Omega, \\ u = 0 \text{ on } \Gamma_D, \quad \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \setminus \Gamma_D, \end{cases} \tag{2.19}$$

and the corresponding eigenvectors of A coincide with the eigenfunctions of (2.19).

The weak formulation of the eigenvalue problem (2.18), (1.7) is

$$\begin{aligned} d_1 u - b_{11} A u - b_{12} A v &= \lambda A u, \\ d_2 v - b_{21} A u - b_{22} A v &= \lambda A v \end{aligned} \tag{2.20}$$

with the operator A defined by (2.4). In this particular case the eigenvalues and the corresponding eigenfunctions of (2.20) and of (2.18), (1.7) coincide.

Proposition 2.1. *Let $\mathbb{H}$ be a real Hilbert space and let $A\colon \mathbb{H} \to \mathbb{H}$ be an operator satisfying* (2.6). *Assume that* (1.4) *is fulfilled. Then $\bigcup_{j=1}^{\infty} \tilde{C}_j$ is the set of all critical points of the problem* (2.14), *and for $(d_1, d_2) \in \bigcup_{j=1}^{\infty} \tilde{C}_j$ we have*

$$\left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{H} \times \mathbb{H}\colon\ (2.14) \text{ is fulfilled} \right\} = \operatorname{span} \left\{ \begin{pmatrix} e_j \\ b_{21}(d_2\kappa_j - b_{22})^{-1} e_j \end{pmatrix} :\ j \in I(d_1, d_2) \right\}.$$

If $d \in D_S$ then there is $\varepsilon > 0$ such that $\operatorname{Re}\lambda < -\varepsilon$ *for all eigenvalues of* (2.20) *and if $d \in D_U$ then there exists at least one positive eigenvalue of* (2.20).

P r o o f. For a particular case of the reaction-diffusion system in one space dimension see e.g. [15], [17], for the general case see e.g. [2], [9]. □

3. General results

In this section we will consider a general real Hilbert space $\mathbb{H}$ with the scalar product $\langle\cdot,\cdot\rangle$ and a closed convex cone K with its vertex at the origin in $\mathbb{H}$. We will discuss the variational inequalities (2.10) and (2.15) with general operators $A, N\colon \mathbb{H} \to \mathbb{H}$ satisfying (2.6), (2.7), (2.8) with N having a potential G_N, i.e. (2.9) holds. The condition (1.4) will be always assumed. The proofs of the results described here will be given in Section 4.

Theorem 3.1. *Let $d_2^0 > 0$ be such that there are $\xi_j \in \mathbb{R}$ with*

$$\sum_{j=1}^{\infty} \xi_j e_j \in K \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{d_2^0 b_{11} - \kappa_j^{-1} \det B}{d_2^0 \kappa_j - b_{22}} \xi_j^2 > 0. \tag{3.1}$$

Then the value

(3.2)

$$d_1^0 := \max\Biggl\{\sum_{j=1}^{\infty} \frac{d_2^0 b_{11} - \kappa_j^{-1} \det B}{d_2^0 \kappa_j - b_{22}} \xi_j^2 \Bigm/ \sum_{j=1}^{\infty} \xi_j^2 \colon\ (\xi_j) \in \ell_2 \setminus \{0\} \ \text{and} \ \sum_{j=1}^{\infty} \xi_j e_j \in K\Biggr\}$$

is the largest bifurcation point of the problem (2.10) *with fixed $d_2 = d_2^0$. Moreover, $(d_1^0, d_2^0) \in D_U \cup C_E$, and the case $(d_1^0, d_2^0) \in C_E$ occurs only if there is $(u, v) \neq (0,0)$ satisfying* (2.14) *with $(d_1, d_2) = (d_1^0, d_2^0)$ and $u \in K$. In the latter case, these (u, v) are exactly the nonzero solutions of* (2.13).

Let us emphasize that in the case $(d_1^0, d_2^0) \in C_E$ in Theorem 3.1 it can happen that the bifurcating solutions of (2.10) are not solutions of (2.11) because the bifurcating solutions of (2.11) (if they exist) need not satisfy $u \in K$.

For the proof in Section 4 we will introduce an operator S_{2,d_2^0} (see Proposition 4.2) such that the assumption (3.1) is equivalent to the existence of $u \in K$ satisfying $\langle S_{2,d_2^0} u, u\rangle > 0$, and (3.2) means

$$d_1^0 = \max_{u \in K \setminus \{0\}} \frac{\langle S_{2,d_2^0} u, u\rangle}{\|u\|^2}. \tag{3.3}$$

Corollary 3.1. *Let $d_2^0 > 0$ be such that the assumptions of Theorem 3.1 are fulfilled. Let $d_1^{\max} > 0$ denote the unique number with $(d_1^{\max}, d_2^0) \in C_E$. If $(d_1, d_2) = (d_1^{\max}, d_2^0)$ satisfies*

$$\sum_{j \in I(d_1,d_2)} \alpha_j e_j \notin K \quad \text{for all nontrivial } (\alpha_j)_{j \in I(d_1,d_2)}, \tag{3.4}$$

then $(d_1, d_2^0) \in D_U$ for all bifurcation points d_1 of (2.10) with fixed $d_2 = d_2^0$.

P r o o f. The assumption (3.4) means in view of Proposition 2.1 that for $(d_1, d_2) = (d_1^{\max}, d_2^0)$ there is no solution $(u, v) \neq (0, 0)$ of (2.14) with $u \in K$. Hence, it follows from the last assertion of Theorem 3.1 that the case $d_1^0 = d_1^{\max}$ is excluded, and therefore $d_1^0 < d_1^{max}$. □

Theorem 3.1 implies in particular that problem (2.10) has at least one bifurcation point with fixed $d_2 = d_2^0$ for every $d_2^0 > y_1$:

R e m a r k 3.1. Hypothesis (3.1) is satisfied for $d_2^0 > y_1$ if there is $u \in K \setminus \{0\}$. More generally, condition (3.1) is satisfied for $d_2^0 > y_{j_0}$ if

$$\text{there is } u \in K \setminus \{0\} \quad \text{with} \quad u \perp \{e_j \colon j < j_0\}. \tag{3.5}$$

Indeed, we can write u uniquely in the form $u = \sum\limits_{j=1}^{\infty} \xi_j e_j$, and in view of $u \perp \{e_j \colon j < j_0\}$, we must have $\xi_j = 0$ for $j < j_0$. The coefficients

$$c_j(d_2) := \frac{d_2 b_{11} - \kappa_j^{-1} \det B}{d_2 \kappa_j - b_{22}}$$

in (3.1) are strictly increasing in d_2 on $[0, \infty)$, because $c_j'(d) > 0$ for $d > 0$ under the assumption (1.4), and $y_{j_0} \geqslant y_j$ for $j \geqslant j_0$. Hence,

$$c_j(d_2^0) > c_j(y_{j_0}) \geqslant c_j(y_j) = 0 \quad \text{for } j \geqslant j_0.$$

The assumption (3.5) as well as the conditions given below will be verified in concrete examples in Section 5.

R e m a r k 3.2. In particular, if $N = 0$ then Theorem 3.1 yields the existence of a critical point of problem (2.13). In the case of particular variational inequalities (e.g. when the cone K is given by a finite number of isolated obstacles as in Examples 5.2, 5.3 in the last section), such a critical point is also a bifurcation point even for our system with general N_1, N_2 (not only $N_2 = 0$) if certain simplicity assumptions are fulfilled. Moreover, bifurcating solutions form a smooth branch if N_1, N_2 are smooth, see [19]. In this case also the bifurcation direction can be described, see [8].

The following result is the "dual" version of Theorem 3.1 for problem (2.15).

Theorem 3.2. *Let* $d_1^0 > 0$ *satisfy* $d_1^0 \notin \{x_j\colon\ j = 1, 2, \ldots\}$. *Assume that there are* $\xi_j \in \mathbb{R}$ *with*

$$\sum_{j=1}^{\infty} \xi_j e_j \in K \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{\kappa_j^{-1} \det B - d_1^0 b_{22}}{b_{11} - d_1^0 \kappa_j}\, \xi_j^2 > 0. \tag{3.6}$$

Then the value

$$d_2^0 := \max\Big\{ \sum_{j=1}^{\infty} \frac{\kappa_j^{-1} \det B - d_1^0 b_{22}}{b_{11} - d_1^0 \kappa_j}\, \xi_j^2 \Big/ \sum_{j=1}^{\infty} \xi_j^2\colon\ (\xi_j) \in \ell_2 \setminus \{0\} \text{ and } \sum_{j=1}^{\infty} \xi_j e_j \in K \Big\} \tag{3.7}$$

is the largest bifurcation point of problem (2.15) *with fixed* $d_1 = d_1^0$. *Moreover,* $d_1^0 < x_1$ *and* $d_2^0 \leqslant d_2^{\max} := \max\Big\{ d_2 > 0\colon\ (d_1^0, d_2) \in \bigcup_j C_j \Big\}$, *and the case* $d_2^0 = d_2^{\max}$ *occurs only if there is a solution* $(u, v) \neq (0, 0)$ *of* (2.14) *with* $(d_1, d_2) = (d_1^0, d_2^0)$ *and* $v \in K$. *In the latter case, these* (u, v) *are exactly the nonzero solutions of* (2.17).

Corollary 3.2. *Let* $d_1^0 > 0$ *be such that the assumptions of Theorem 3.2 are fulfilled (and thus* $d_1^0 < x_1$*). Let* $d_2^{\max} = \max\Big\{ d_2\colon\ (d_1^0, d_2) \in \bigcup_j C_j \Big\}$. *If* (3.4) *holds with* $(d_1, d_2) = (d_1^0, d_2^{\max})$, *then* $d_2^0 < d_2^{\max}$ *in Theorem 3.2.*

In particular, if $d_1^0 \geqslant x_2$ *and* (3.4) *holds with* $(d_1, d_2) = (d_1^0, d_2^{\max})$, *then* $(d_1^0, d_2^0) \in D_S$.

Remark 3.3. Hypothesis (3.6) is satisfied for $d_1^0 \in (0, x_{j_0})$ if

$$\text{there is } u \in K \setminus \{0\} \text{ with } u \in \operatorname{span}\{e_j\colon\ j = 1, \ldots, j_0\}. \tag{3.8}$$

Indeed, writing $u = \sum\limits_{j=1}^{\infty} \xi_j e_j$ with $\xi_j = 0$ for $j > j_0$, it suffices to observe that the coefficients in (3.6) are positive for $j \leqslant j_0$, because the nominator is positive due to the assumption (1.4), and the denominator is positive in view of $d_1^0 < x_{j_0} \leqslant x_j = \kappa_j^{-1} b_{11}$.

Remark 3.3 is "dual" to Remark 3.1, nevertheless condition (3.8) is much more restrictive than (3.5) because the orthogonal complement to $\{e_j\colon\ j < j_0\}$ is infinite-dimensional while the space $\operatorname{span}\{e_j\colon\ j \leqslant j_0\}$ has only a finite dimension. In particular, while Remark 3.1 implies that Theorem 3.1 applies for at least some $d_2^0 > 0$ (if $K \neq \{0\}$), Remark 3.3 does not give an analogous consequence for Theorem 3.2. However, this consequence is also true for Theorem 3.2, but it requires a different argument. Indeed, if there is $u \in K \setminus \{0\}$ then the completeness of the basis (e_j) implies that there is at least one j_0 with $\langle u, e_{j_0}\rangle \neq 0$. For these j_0, we can use the following observation:

R e m a r k 3.4. Let $j_0 \in \{1, 2, \ldots\}$ be such that

$$\text{there is } u \in K \text{ with } \langle u, e_{j_0}\rangle \neq 0. \tag{3.9}$$

Then there is $\varepsilon_{j_0} > 0$ such that hypothesis (3.6) holds for every $d_1^0 \in [x_{j_0} - \varepsilon_{j_0}, x_{j_0})$.

Indeed, writing $u = \sum_{j=1}^{\infty} \xi_j e_j$, we have $\xi_{j_0} = \langle u, e_{j_0}\rangle \neq 0$. Put $I := \{j\colon\ \kappa_j = \kappa_{j_0}\}$. The corresponding series in (3.6) has the form $S_1(d_1) + S_2(d_1)$ with

$$S_1(d_1) := \sum_{j\in I} \frac{\kappa_j^{-1}\det B - d_1 b_{22}}{b_{11} - d_1\kappa_j}\,\xi_j^2, \qquad S_2(d_1) := \sum_{\substack{j=1\\ j\notin I}}^{\infty} \frac{\kappa_j^{-1}\det B - d_1 b_{22}}{b_{11} - d_1\kappa_j}\,\xi_j^2.$$

Now observe that $S_2(d_1)$ remains bounded for d_1 close to $\kappa_{j_0}^{-1} b_{11} = x_{j_0}$, and that $S_1(d_1) \to \infty$ as d_1 approaches x_{j_0} from the left due to (1.4). Hence, for all $d_1 < x_{j_0}$ sufficiently close to x_{j_0} the series in (3.6) is strictly positive.

4. Proof of the main results

The proofs are based on the following well-known variational principle. Given a mapping $G\colon\ \mathbb{H} \to \mathbb{H}$ with $G(0) = 0$, we call $\lambda_0 \in \mathbb{R}$ a bifurcation point of the variational inequality

$$u \in K, \quad \langle \lambda u - G(u), \varphi - u\rangle \geqslant 0 \text{ for all } \varphi \in K, \tag{4.1}$$

if every neigborhood of $(\lambda_0, 0) \in \mathbb{R} \times \mathbb{H}$ contains (λ, u) satisfying (4.1) with $u \neq 0$. If $S\colon\ \mathbb{H} \to \mathbb{H}$ is linear, we say that λ is an eigenvalue of

$$u \in K, \quad \langle \lambda u - Su, \varphi - u\rangle \geqslant 0 \text{ for all } \varphi \in K, \tag{4.2}$$

if (4.2) has a solution $u \neq 0$; we call each such u a corresponding eigenvector.

Proposition 4.1. *Let $G\colon\ \mathbb{H} \to \mathbb{H}$ be a compact potential operator satisfying $G(0) = 0$. Suppose that G is Fréchet differentiable at 0, $S := G'(0)$, and that there is $u_0 \in K$ with $\langle Su_0, u_0\rangle > 0$. Then the maximum*

$$\lambda_0 := \max_{u\in K\setminus\{0\}} \frac{\langle Su, u\rangle}{\|u\|^2} \tag{4.3}$$

exists and is the largest bifurcation point of (4.1) *and the largest eigenvalue of* (4.2). *Moreover, the eigenvectors of* (4.2) *are exactly those u for which the maximum in* (4.3) *is assumed.*

P r o o f. The result follows from [14, Theorem 1] or [23, Theorem 64.A], where it is formulated in terms of bilinear forms. Let us verify the assumptions of the last theorem. Let g be a potential of G with $g(0) = 0$. The compactness of $g' = G$ implies the weak sequential continuity of g and the compactness of S, i.e. also the compactness of the bilinear form $b(u,u) := \langle Su, u\rangle$ (see e.g. [22, Corollary 41.9] and [21, Corollary 21.33]). Since g is twice Fréchet differentiable at 0 with $g(0) = 0$ and $g'(0) = 0$, we conclude from [23, Corollary 64.4] that all hypotheses of [23, Theorem 64.A] are satisfied with $F(u) = \|u\|^2$ and $a(u,v) := \langle u, v\rangle$ (and G in place of g). The latter implies that the maximum in (4.3) exists and is the largest bifurcation point of (4.1) and the largest eigenvalue of (4.2). The last assertion of Proposition 4.1 follows from the last statement of [14, Theorem 1 (c)]. $\square$

The crucial hypothesis of Proposition 4.1 is of course that the operator g needs to have a potential, which is not the case for the operators occurring in (2.10) and (2.15) if we interpret the equation in an obvious manner on the product space $\mathbb{H}\times\mathbb{H}$. However, we can equivalently rewrite the equations in the space $\mathbb{H}$ in the required form when we consider only d_1 or only d_2 as a bifurcation parameter (cf. [12] for this idea).

Proposition 4.2. *For fixed $d_2 > 0$ the problem* (2.10) *is equivalent to the problem*

$$u \in K, \quad \langle d_1 u - S_{2,d_2} u - N(u), \varphi - u\rangle \geqslant 0 \quad \text{for all } \varphi \in K, \tag{4.4}$$

$$v = (d_2 I - b_{22} A)^{-1} b_{21} A u, \tag{4.5}$$

with the operator

$$S_{2,d_2} := b_{11} A u + b_{12} A (d_2 I - b_{22} A)^{-1} b_{21} A.$$

P r o o f. Since $d_2/b_{22} < 0$ by (1.4) and the operator A has only positive eigenvalues, the inverse in (4.5) exists. Hence, we can uniquely solve the last equation of (2.10) for v, which is equivalent to (4.5). Inserting this value into the inequality of (2.10), we obtain (4.4). $\square$

Interchanging the roles of d_1 and d_2 in Proposition 4.2, we can similarly rewrite the problem (2.15):

Proposition 4.3. *For fixed $d_1 > 0$ with $d_1 \notin \{x_j \colon j = 1, 2, \ldots\}$ the problem* (2.15) *is equivalent to the problem*

$$v \in K, \quad \langle d_2 v - S_{1,d_1} v - N(v), \varphi - v\rangle \geqslant 0 \quad \text{for all } \varphi \in K, \tag{4.6}$$

$$u = (d_1 I - b_{11} A)^{-1} b_{12} A v, \tag{4.7}$$

with the operator

$$S_{1,d_1} := b_{21}A(d_1I - b_{11}A)^{-1}b_{12}A + b_{22}A.$$

P r o o f. Since $d_1 \neq x_j = b_{11}\kappa_j^{-1}$, we have $d_1 \neq b_{11}\lambda$ for every eigenvalue λ of A, and so the inverse in the last equation exists. Hence, we can solve the first equation in (2.15) for u and obtain (4.7). Inserting this into the second equation, we obtain (4.6). □

The same calculation shows the corresponding statements for (2.14):

Proposition 4.4. *With the notation of Propositions* 4.2, 4.3 *the problem* (2.14) *is equivalent to the equation* $S_{2,d_2}u = d_1u$ *with* (4.5), *and also equivalent to* $S_{1,d_1}u = d_2u$ *with* (4.7).

We recall the well-known variational characterization of the largest eigenvalue of a symmetric compact operator, and the characterization of the corresponding eigenvectors.

Proposition 4.5. *Let* $S\colon \mathbb{H} \to \mathbb{H}$ *be linear, symmetric, and compact. Then the largest eigenvalue of* S *is given by*

$$\lambda_0 := \max_{u\in\mathbb{H}\setminus\{0\}} \frac{\langle Su, u\rangle}{\|u\|^2}, \tag{4.8}$$

and this maximum is attained exactly in all the corresponding eigenvectors.

Corollary 4.1. *If the number* λ_0 *of Proposition* 4.1 *is simultaneously the largest eigenvalue of the operator* S, *then there is* $u_0 \in K \setminus \{0\}$ *with* $Su_0 = \lambda_0u_0$.

P r o o f. If u_0 is an eigenvector of (4.2) corresponding to λ_0 then the maximum in (4.3) is attained at u_0 by Proposition 4.1. If λ_0 is the largest eigenvalue of the operator S then Proposition 4.5 implies that also the maximum in (4.8) is assumed at u_0 and $Su_0 = \lambda_0u_0$. □

P r o o f of Theorem 3.1. In view of Proposition 4.2, we only have to study bifurcation points of the problem (4.4) (with fixed $d_2 = d_2^0$). However, this problem has exactly the form (4.1) with $\lambda = d_1$, $G(u) = S_{2,d_2}u + N(u)$. Using operator calculus, we can write $S_{2,d_2^0} = f_{2,d_2^0}(A)$, where the function f_{2,d_2^0} is defined on the spectrum of A by

$$f_{2,d_2^0}(\lambda) := b_{11}\lambda + \frac{b_{12}b_{21}\lambda^2}{d_2^0 - b_{22}\lambda} = \frac{d_2^0b_{11}\lambda - (\det B)\lambda^2}{d_2^0 - b_{22}\lambda}.$$

Note that G has a potential, since $S := S_{2,d_2^0} = f_{2,d_2^0}(A)$ is symmetric and since N has a potential G_N by hypothesis. Moreover, (2.8) implies $G(0) = 0$ and $G'(0) = S$.

Every $u \in K$ can be uniquely written in the form $u = \sum\limits_{j=1}^{\infty} \xi_j e_j$ with $\xi_j \in \mathbb{R}$. By Parseval's identity, we have $(\xi_j) \in \ell_2$, $\|u\|^2 = \sum\limits_{j=1}^{\infty} \xi_j^2$, and by the rules of calculus for symmetric operators, we have

$$\langle S_{2,d_2^0}u, u\rangle = \langle f_{2,d_2^0}(A)u, u\rangle = \left\langle \sum_{j=1}^{\infty} f_{2,d_2^0}(\kappa_j^{-1})\xi_j e_j, \sum_{i=1}^{\infty} \xi_i e_i \right\rangle = \sum_{j=1}^{\infty} f_{2,d_2^0}(\kappa_j^{-1})\xi_j^2,$$

which is exactly the series occurring in (3.1) and (3.2). In particular, (3.1) means that $\langle S_{2,d_2^0}u_0, u_0\rangle > 0$ for $u_0 = \sum\limits_{j=1}^{\infty} \xi_j e_j$, and (3.2) means exactly (3.3). Hence, d_1^0 is the largest bifurcation point of (2.10) with fixed $d_2 = d_2^0$ by Proposition 4.1.

Propositions 2.1 and 4.4 imply that the eigenvalues of $S = S_{2,d_2^0}$ are exactly those d_1 with $(d_1, d_2^0) \in \bigcup\limits_j C_j$. Hence, putting $d_1^{\max} = \max\Big\{d_1 \in \mathbb{R}\colon (d_1, d_2^0) \in \bigcup\limits_j \tilde{C}_j\Big\}$, we have $d_1^0 \leqslant d_1^{\max}$, i.e. $(d_1^0, d_2^0) \in D_U \cup C_E$, and the equality $d_1^0 = d_1^{\max}$ (i.e. $(d_1^0, d_2^0) \in C_E$)) holds if and only if d_1^0 is the maximal eigenvalue of S. Corollary 4.1 and Proposition 4.2 used for $N = 0$ imply that if this is true then all solutions of (2.13) (with $(d_1, d_2) = (d_1^0, d_2^0)$) satisfy also (2.14), and the last assertions of Theorem 3.1 follow. □

P r o o f of Theorem 3.2. Using operator calculus, we can write $S_{1,d_1^0} = f_{1,d_1^0}(A)$ with the function f_{1,d_1^0} defined on the spectrum of A by

$$f_{1,d_1^0}(\lambda) := \frac{b_{12}b_{21}\lambda^2}{d_1^0 - b_{11}\lambda} + b_{22}\lambda = \frac{(\det B)\lambda^2 - d_1^0 b_{22}\lambda}{b_{11}\lambda - d_1^0}.$$

Then the proof is analogous to the proof of Theorem 3.1 by applying Proposition 4.1 to the problem (4.6), and observing that for $u = \sum\limits_{j=1}^{\infty} \xi_j e_j \in K$ the expression

$$\langle S_{1,d_1^0}u, u\rangle = \sum_{j=1}^{\infty} f_{1,d_1^0}(\kappa_j^{-1})\xi_j^2$$

is exactly the series occurring in (3.6) and (3.7). By an analogous reasoning as in the proof of Theorem 3.1, we obtain $d_2^0 \leqslant \tilde{d}_2^{\max} := \max\Big\{d_2 \in \mathbb{R}\colon (d_1^0, d_2) \in \bigcup\limits_j \tilde{C}_j\Big\}$ and that $\tilde{d}_2^{\max}$ is the maximal eigenvalue of $S := S_{1,d_1^0}$. Since $d_2^0 > 0$, we must have

$\tilde{d}_2^{\max} > 0$. The operator S_{1,d_1^0} has only negative eigenvalues if $d_1^0 \geqslant x_1$ due to the form of the hyperbolas C_n and Proposition 4.4, which in turn implies $d_1^0 < x_1$ and $\tilde{d}_2^{\max} = \max\Big\{d_2 > 0\colon\ (d_1^0, d_2) \in \bigcup\limits_j C_j\Big\} = d_2^{\max}$. The proof of the last assertion is again similar to that of the last assertion of Theorem 3.1. □

5. Application to unilateral boundary value problems

Throughout this section, we consider a domain Ω as in the introduction, a function n satisfying (1.5) and (2.1), the Hilbert space $\mathbb{H}$ from (2.2) with the scalar product (2.3) and the corresponding norm $\|\cdot\|$, and the operators defined by (2.4), (2.5). The condition (1.4) will be always assumed. The problem (2.11), (2.14), or (2.20) is a weak formulation of (1.2), (2.12), or (2.18), respectively, with boundary conditions (1.7). Similarly, (2.16) is the weak formulation of the problem

$$\begin{aligned} d_1\Delta u + b_{11}u + b_{12}v &= 0 \\ d_2\Delta v + b_{21}u + b_{22}v + n(v) &= 0 \end{aligned} \quad \text{in } \Omega \tag{5.1}$$

with boundary conditions (1.7). The characteristic values and eigenvectors κ_j and e_j of the operator A are the eigenvalues and eigenfunctions of the problem (2.19), see Section 2. Hence, Proposition 2.1 remains valid if we replace (2.14) by (2.12), (1.7), and (2.20) by (2.18), (1.7). Speaking about solutions of boundary value problems we have always in mind weak solutions.

We will apply our abstract results to concrete choices of the cone K corresponding to different unilateral boundary conditions. As we have already mentioned in Section 2, the conditions (2.6), (2.7), (2.8) are automatically fulfilled and need not be repeated.

Example 5.1. Suppose that Γ_D, Γ_N, Γ_U^1, Γ_U^2 are pairwise disjoint nonempty parts of $\partial\Omega$,

$$\operatorname{mes}\Gamma_D > 0, \quad \operatorname{mes}(\Gamma_U^1 \cup \Gamma_U^2) > 0, \quad \operatorname{mes}(\partial\Omega \setminus (\Gamma_D \cup \Gamma_N \cup \Gamma_U^1 \cup \Gamma_U^2)) = 0.$$

We consider the system (1.2) with unilateral boundary conditions

$$\begin{cases} u = v = 0 \quad \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_N, \quad \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_D, \\ u \geqslant 0, \ \dfrac{\partial u}{\partial \nu} \geqslant 0, \ u \cdot \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_U^1, \\ u \leqslant 0, \ \dfrac{\partial u}{\partial \nu} \leqslant 0, \ u \cdot \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_U^2, \end{cases} \tag{5.2}$$

or the system (5.1) with unilateral boundary conditions

$$(5.3)\qquad \begin{cases} u = v = 0 \quad \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_D, \quad \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_N, \\ v \geqslant 0, \ \dfrac{\partial v}{\partial \nu} \geqslant 0, \ v \cdot \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_U^1, \\ v \leqslant 0, \ \dfrac{\partial v}{\partial \nu} \leqslant 0, \ v \cdot \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_U^2. \end{cases}$$

It is natural to introduce a weak solution of (1.2), (5.2) or (5.1), (5.3) as a solution of (2.10) or (2.15), respectively, with the operators A, N from (2.4), (2.5) and with the closed convex cone

$$K = \{\varphi \in \mathbb{H}\colon \ \varphi \geqslant 0 \text{ on } \Gamma_U^1, \ \varphi \leqslant 0 \text{ on } \Gamma_U^2 \text{ in the sense of traces}\}.$$

Hypotheses (3.5) or (3.8) mean in this example that

$$(5.4)\qquad \text{there is a nontrivial } u = \sum_{j=j_0}^{\infty} \xi_j e_j \text{ with } u \geqslant 0 \text{ on } \Gamma_U^1 \text{ and } u \leqslant 0 \text{ on } \Gamma_U^2$$

or

$$(5.5)\qquad \text{there is a nontrivial } u = \sum_{j=1}^{j_0} \xi_j e_j \text{ with } u \geqslant 0 \text{ on } \Gamma_U^1 \text{ and } u \leqslant 0 \text{ on } \Gamma_U^2,$$

respectively. Hypothesis (3.9) means that

$$(5.6)\qquad \text{there is } u \in \mathbb{H} \text{ with } \langle u, e_{j_0} \rangle \neq 0 \text{ such that } u \geqslant 0 \text{ on } \Gamma_U^1, \text{ and } u \leqslant 0 \text{ on } \Gamma_U^2.$$

The assumption (3.4) is fulfilled if

$$(5.7)\qquad \begin{gathered} \sum_{j \in I(d_1,d_2)} \alpha_j e_j \text{ changes sign in } \Gamma_U^1 \text{ or in } \Gamma_U^2 \\ \text{or is nonzero with constant sign on a set } \Gamma \\ \text{with } \operatorname{mes}(\Gamma \cap \Gamma_U^i) > 0 \text{ for } i = 1, 2 \\ \text{for all nontrivial } (\alpha_j)_{j \in I(d_1,d_2)}. \end{gathered}$$

Hence, the following assertion follows by using Theorem 3.1, Corollary 3.1, and Remark 3.1.

Theorem 5.1. *Let $d_2^0 > y_{j_0}$ where j_0 is such that (5.4) holds. Then the problem (1.2), (5.2) has a bifurcation point with fixed $d_2 = d_2^0$. Moreover, if additionally for the unique $d_1^{\max} > 0$ with $(d_1^{\max}, d_2^0) \in C_E$ the hypothesis (5.7) is satisfied with $(d_1, d_2) = (d_1^{\max}, d_2^0)$, then $(d_1, d_2^0) \in D_U$ for every bifurcation point d_1 of (1.2), (5.2) with fixed $d_2 = d_2^0$.*

In general, it can happen that the largest bifurcating point in the situation of the last theorem is d_1^0 with $(d_1^0, d_2^0) \in C_E$ but the corresponding bifurcating solutions of (1.2), (5.2) are not solutions of (1.2), (1.7) because the bifurcating solutions of (1.2), (1.7) (if they exist) need not satisfy $u \in K$ (cf. the text after Theorem 3.1).

Theorem 3.2 implies in view of Remark 3.3 or 3.4 the following results.

Theorem 5.2. *Let j_0 be such that* (5.5) *holds. Then for any $d_1^0 \in (0, x_{j_0})$ the problem* (5.1), (5.3) *has a bifurcation point with fixed $d_1 = d_1^0$.*

Theorem 5.3. *Let j_0 be such that* (5.6) *holds. Then there is $\varepsilon_{j_0} > 0$ such that for all $d_1^0 \in [x_{j_0} - \varepsilon_{j_0}, x_{j_0})$ the problem* (5.1), (5.3) *has a bifurcation point with fixed $d_1 = d_1^0$.*

For $j_0 = 1$, we get the following modification which needs a proof.

Theorem 5.4. *If $\operatorname{mes} \Gamma_U^k > 0$ for both $k = 1, 2$ then there is $\varepsilon > 0$ such that for all $d_1^0 \in [x_1 - \varepsilon, x_1)$ the problem* (5.1), (5.3) *has a bifurcation point with fixed $d_1 = d_1^0$, and $(d_1^0, d_2) \in D_S$ for every such bifurcation point d_2.*

Proof. Since the eigenfunction e_1 is simple and does not change sign, we have (5.7) for all (d_1, d_2) with $I(d_1, d_2) = \{1\}$. In particular, (3.4) holds if $(d_1, d_2) = (d_1^0, d_2^{\max})$ with $d_1^0 \in (x_1 - \varepsilon, x_1)$ (ε small) and $d_2^{\max} > 0$ being the unique number with $(d_1^0, d_2^{\max}) \in C_1$. There is a function $u \in \mathbb{H}$ with $\langle u, e_1 \rangle \neq 0$, $u \geqslant 0$ on Γ_U^1 and $u \leqslant 0$ on Γ_U^2. This u satisfies (5.6) with $j_0 = 1$, and so Remark 3.4 implies that the hypotheses of Theorem 3.1 and of Corollary 3.2 are satisfied. Our assertion follows. □

Remark 5.1. If $\operatorname{mes} \Gamma_U^1 = 0$ or if $\operatorname{mes} \Gamma_U^2 = 0$ then, since e_1 does not change sign, the hypothesis (5.7) and thus (3.4) is not fulfilled for $(d_1, d_2) \in C_1$.

Remark 5.2. All previously known results about the existence of a bifurcation point of (5.1), (5.3) in D_S have had the hypothesis that

$$\text{there is } u = \sum_{j \in I(d_1, d_2)} \xi_j e_j \text{ with } u > 0 \text{ on } \Gamma_U^1 \text{ and } u < 0 \text{ on } \Gamma_U^2$$

for certain $(d_1, d_2) \in \bigcup_{j=1}^{\infty} C_j$. Theorem 5.4 applies in particular if this hypothesis is violated for all $(d_1, d_2) \in \bigcup_{j=1}^{\infty} C_j$.

E x a m p l e 5.2. Let I_+ and I_- be two finite sets of indices, at least one of them nonempty. Let Γ_D, Γ_N, and Γ_i $(i \in I_+ \cup I_-)$ be pairwise disjoint parts of $\partial\Omega$ such that

$$\operatorname{mes}\Gamma_D > 0, \quad \operatorname{mes}\Gamma_i > 0, \quad \operatorname{mes}\Bigl(\partial\Omega \setminus \Bigl(\Gamma_D \cup \Gamma_N \cup \bigcup_{i\in I_+\cup I_-} \Gamma_i\Bigr)\Bigr) = 0.$$

We consider the system (1.2) with unilateral boundary conditions

$$(5.8)\qquad \begin{cases} u = v = 0 \quad \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_N, \quad \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_D, \\ \displaystyle\int_{\Gamma_i} u \,\mathrm{d}x \geqslant 0, \ \dfrac{\partial u}{\partial \nu} = \text{const} \geqslant 0, \ \int_{\Gamma_i} u \,\mathrm{d}x \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_i, \ i \in I_+, \\ \displaystyle\int_{\Gamma_i} u \,\mathrm{d}x \leqslant 0, \ \dfrac{\partial u}{\partial \nu} = \text{const} \leqslant 0, \ \int_{\Gamma_i} u \,\mathrm{d}x \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_i, \ i \in I_-, \end{cases}$$

or the system (5.1) with unilateral boundary conditions

$$(5.9)\qquad \begin{cases} u = v = 0 \quad \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_D, \quad \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_N, \\ \displaystyle\int_{\Gamma_i} v \,\mathrm{d}x \geqslant 0, \ \dfrac{\partial v}{\partial \nu} = \text{const} \geqslant 0, \ \int_{\Gamma_i} v \,\mathrm{d}x \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_i, \ i \in I_+, \\ \displaystyle\int_{\Gamma_i} v \,\mathrm{d}x \leqslant 0, \ \dfrac{\partial v}{\partial \nu} = \text{const} \leqslant 0, \ \int_{\Gamma_i} v \,\mathrm{d}x \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_i, \ i \in I_-. \end{cases}$$

It is natural to introduce a weak solution of (1.2), (5.8) or (5.1), (5.9) as a solution of (2.10) or (2.15), respectively, with the operators A, N from (2.4), (2.5), and with the closed convex cone

$$K = \Bigl\{\varphi \in \mathbb{H}\colon \int_{\Gamma_i} \varphi \,\mathrm{d}x \geqslant 0 \text{ for all } i \in I_+, \ \int_{\Gamma_i} \varphi \,\mathrm{d}x \leqslant 0 \text{ for all } i \in I_-\Bigr\}.$$

More precisely, u, v is a solution of (2.10) if and only if $u, v \in \mathbb{H}$, $\Delta u, \Delta v \in L_2(\Omega)$, the equation (1.2) holds a.e. in Ω and (5.8) is fulfilled; an analogous statement holds for (2.15). See [7, Observation 5.2] for details.

Hypotheses (3.5) or (3.8) mean in this example that

$$\text{there is a nontrivial } u = \sum_{j=j_0}^{\infty} \xi_j e_j \text{ such that } \sum_{j=j_0}^{\infty} \xi_j \int_{\Gamma_i} e_j \,\mathrm{d}x \text{ is}$$
$$\text{nonnegative for all } i \in I_+ \text{ and nonpositive for all } i \in I_-,$$

or

$$\text{there is a nontrivial } u = \sum_{j=1}^{j_0} \xi_j e_j \text{ such that } \sum_{j=1}^{j_0} \xi_j \int_{\Gamma_i} e_j \, \mathrm{d}x \text{ is}$$
$$\text{nonnegative for all } i \in I_+ \text{ and nonpositive for all } i \in I_-,$$

respectively. Hypothesis (3.9) means

$$\text{there is } u \in \mathbb{H} \text{ with } \langle u, e_{j_0} \rangle \neq 0 \text{ such that } \int_{\Gamma_i} u \, \mathrm{d}x \text{ is}$$
$$\text{nonnegative for all } i \in I_+ \text{ and nonpositive for all } i \in I_-.$$

The condition (3.4) is fulfilled if

$$\begin{gathered}\text{for any nontrivial } (\alpha_j)_{j \in I(d_1,d_2)} \text{ there are}\\ i_1, i_2 \text{ lying both in } I_+ \text{ or both in } I_- \text{ such that}\\ \operatorname{sgn} \sum_{j \in I(d_1,d_2)} \alpha_j \int_{\Gamma_{i_1}} e_j \, \mathrm{d}x = -\operatorname{sgn} \sum_{j \in I(d_1,d_2)} \alpha_j \int_{\Gamma_{i_2}} e_j \, \mathrm{d}x\\ \text{or there are } i_1 \in I_+,\ i_2 \in I_- \text{ such that}\\ \operatorname{sgn} \sum_{j \in I(d_1,d_2)} \alpha_j \int_{\Gamma_{i_1}} e_j \, \mathrm{d}x = \operatorname{sgn} \sum_{j \in I(d_1,d_2)} \alpha_j \int_{\Gamma_{i_2}} e_j \, \mathrm{d}x.\end{gathered} \tag{5.10}$$

If $I_+ \neq \emptyset \neq I_-$ then (5.10) automatically holds for $(d_1, d_2) \in C_1 \setminus \bigcup_{j=2}^{\infty} C_j$ due to the positivity and simplicity of the first eigenfunction e_1 of (2.19).

Summarizing these facts to obtain assertions concerning all situations mentioned on the basis of the abstract results of Section 3 is left to the reader.

Example 5.3. Analogously to Example 5.1 or 5.2, we can consider also a regulation in the interior of the domain Ω described by the closed convex cone

$$K = \{\varphi \in \mathbb{H} \colon \ \varphi \geqslant 0 \text{ on } G_U^1, \ \ \varphi \leqslant 0 \text{ on } G_U^2$$

or

$$K = \left\{\varphi \in \mathbb{H} \colon \int_{G_i} \varphi \, \mathrm{d}x \geqslant 0 \text{ for all } i \in I_+, \ \int_{G_i} \varphi \, \mathrm{d}x \leqslant 0 \text{ for all } i \in I_- \right\},$$

where G_U^1, G_U^2 or G_i $(i \in I_+ \cup I_i)$ are pairwise disjoint subsets of Ω. It is straightforward to change the respective conditions (5.4)–(5.10) to these situations.

Remark 5.3. Let us suppose that the system (1.2) describes a coexistence of two populations with densities u, v. Let us consider a given nontrivial solution (u, v) of (1.2), (5.8) corresponding to some (d_1, d_2). Hence, (u, v) is a nontrivial equilibrium of the original evolution system (1.6) with (5.8). Then the integral $\int_{\Gamma_j} u \, \mathrm{d}x$ determines the amount of the species u which can be removed (e.g. harvested) in the region Γ_j to keep the equilibrium. It is assumed that the amount harvested is the same at all places of Γ_j. Analogously for the integral $\int_{G_j} u \, \mathrm{d}x$ from Example 5.3. The boundary conditions from Example 5.1 can describe unilateral membranes, see e.g. [1].

Remark 5.4. Example 5.2 and the second case of Example 5.3 fit into the theory developed in [8], [19]. It follows that if the critical point (d_1, d_2) obtained from Theorem 3.1 or 3.2 used for the particular case $n = 0$ is such that the nontrivial solution of the problem (2.12), (5.8) or (2.12), (5.9) is unique up to a positive multiple and satisfies certain activity conditions then a smooth branch of nontrivial solutions of the problem

$$\begin{aligned} &\sigma_1(s)\Delta u + b_{11}u + b_{12}v + n_1(u, v) = 0, \\ &\sigma_2(s)\Delta v + b_{21}u + b_{22}v + n_2(u, v) = 0 \end{aligned} \tag{5.11}$$

with the boundary conditions (5.8) or (5.9), respectively, bifurcates at a point s_B with $(\sigma_1(s_B), \sigma_2(s_B)) = (d_1, d_2)$, where σ is a smooth curve in $\mathbb{R}^2_+$ containing (d_1, d_2). The bifurcation direction for such situations is described in [8]. Cf. also [13] where an application of the result [19] is given to (5.11), (5.9).

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