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Zhendong Gu; Daochun Sun

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THE GROWTH OF DIRICHLET SERIES

Zhendong Gu, Daochun Sun, Guangzhou

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Abstract. We define Knopp-Kojima maximum modulus and the Knopp-Kojima maximum term of Dirichlet series on the right half plane by the method of Knopp-Kojima, and discuss the relation between them. Then we discuss the relation between the Knopp-Kojima coefficients of Dirichlet series and its Knopp-Kojima order defined by Knopp-Kojima maximum modulus. Finally, using the above results, we obtain a relation between the coefficients of the Dirichlet series and its Ritt order. This improves one of Yu Jia-Rong's results, published in Acta Mathematica Sinica *21* (1978), 97–118. We also give two examples to show that the condition under which the main result holds can not be weakened.

Keywords: Dirichlet series, order, abscissa of convergence

MSC 2010: 30B50

1. Introduction and main result

Consider the Dirichlet series

$$f(s)=\sum_{n=0}^{+\infty}a_n\mathrm{e}^{-\lambda_n s},$$

where $s=\sigma+\mathrm{i}t$ denotes the complex variable, $\{a_n\}$ is a sequence of complex numbers, and $0=\lambda_0<\lambda_1<\lambda_2<\ldots<\lambda_n\uparrow+\infty$. Following Bohr [2], we define the quantities

$$\sigma_c=\inf\Big\{\sigma\in\mathbb{R}\colon\ \sum a_n\mathrm{e}^{-\lambda_n\sigma}\text{ converges.}\Big\},$$
$$\sigma_a=\inf\Big\{\sigma\in\mathbb{R}\colon\ \sum |a_n|\mathrm{e}^{-\lambda_n\sigma}\text{ converges.}\Big\},$$
$$\sigma_u=\inf\Big\{\sigma\in\mathbb{R}\colon\ \sum a_n\mathrm{e}^{-\lambda_n(\sigma+\mathrm{i}t)}\text{ converges uniformly on }\mathbb{R}.\Big\}.$$

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When $\sigma_u = -\infty$, $f(s)$ is an entire function. In this case, S. Mandelbrojt [4], M. Blambert [1], Yu Chia-Yung [14] have studied the relation between the growth of $f(s)$ and the coefficients. J. Ritt [6], S. Izumi [5], and K. Sugimura [7] have given formulas determining the order and the type of $f(s)$ in terms of a_n under an additional condition imposed upon $\{\lambda_n\}$. C. Tanaka [8] improved these formulas.

When $\sigma_u = 0$, by the method of J. Ritt [6], Yu Chia-Yung [15], [13] defined the order and type of $f(s)$ under the conditions

$$\overline{\lim_{n\to+\infty}} \frac{\ln|a_n|}{\lambda_n} = 0 \quad \text{and} \quad \overline{\lim_{n\to+\infty}} \frac{n}{\lambda_n} < +\infty,$$

and obtained some results between the growth of $f(s)$ and the coefficients, which extends some of G. Valiron's results [9]. In this paper, we improve one of his results.

Put

$$\Delta = \overline{\lim_{k\to+\infty}} \frac{\ln^+ \ln(p_k+1)}{\ln k}, \quad \sigma_0 = \overline{\lim_{n\to+\infty}} \frac{\ln|a_n|}{\lambda_n},$$

where p_k is given by $[k, k+1) \cap \{\lambda_n\} = \{\lambda_{n_k}, \lambda_{n_k+1}, \ldots, \lambda_{n_k+p_k}\}$, $k \in \mathbb{N}$. Moreover, let

$$M(\sigma) = \sup\{|f(\sigma + \mathrm{i}t)|\colon\ t \in \mathbb{R}\}.$$

Our main result is the following theorem.

Theorem 1. *Consider the Dirichlet series $f(s)$ with frequencies $\{\lambda_n\}$, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \uparrow +\infty$. If $\sigma_0 = 0$ and $\Delta = 0$, then*

$$\overline{\lim_{\sigma\to0^+}} \frac{\ln^+\ln^+ M(\sigma)}{-\ln\sigma} = \varrho \Leftrightarrow \overline{\lim_{n\to+\infty}} \frac{\ln^+\ln^+|a_n|}{\ln\lambda_n} = \begin{cases} \dfrac{\varrho}{\varrho+1}, & \varrho < +\infty; \\ 1, & \varrho = +\infty. \end{cases}$$

By Theorem 1, we deduce Yu Chia-Yung's result [15], [13] as Corollary 1. Then we give Example 1 to show that the condition $\Delta = 0$ is much less restrictive than the condition $\overline{\lim\limits_{n\to+\infty}} n/\lambda_n < +\infty$, which implies that the Dirichlet series acts more or less like a power series. More precisely, we give Example 2 to show that the condition $\Delta = 0$ cannot be replaced by $\Delta < +\infty$.

2. Lemmas

Throughout this section, $f(s)$ is a Dirichlet series with frequencies $\{\lambda_n\}$ as in the introduction. To give our lemmas, we define some symbols by the method of Knopp-Kojima [3]. For each $k \in \mathbb{N}$, when

$$[k, k+1) \cap \{\lambda_n\} = \{\lambda_{n_k}, \lambda_{n_k+1}, \ldots, \lambda_{n_k+p_k}\} \neq \emptyset, \tag{1}$$

put

$$A_k = \max\left\{\left|\sum_{j=0}^{p} a_{n_k+j}\right| :\ 0 \leqslant p \leqslant p_k\right\}; \quad A_k^* = \sum_{j=0}^{p_k} |a_{n_k+j}|;$$

$$\overline{A}_k = \sup_{0\leqslant p\leqslant p_k, t\in\mathbb{R}} \left|\sum_{j=0}^{p} a_{n_k+j}\mathrm{e}^{-\mathrm{i}t\lambda_{n_k+j}}\right|;$$

when $[k, k+1) \cap \{\lambda_n\} = \emptyset$, put $\ln A_k = \ln A_k^* = \ln \overline{A}_k = -\infty$. Then we have formulas [3], [10] for the abscissas $\sigma_c, \sigma_u, \sigma_a$ in terms of $A_k, \overline{A}_k, A_k^*$,

$$\sigma_c = \overline{\lim_{k\to+\infty}} \frac{\ln A_k}{k}; \quad \sigma_u = \overline{\lim_{k\to+\infty}} \frac{\ln \overline{A}_k}{k}; \quad \sigma_a = \overline{\lim_{k\to+\infty}} \frac{\ln A_k^*}{k}.$$

When $\sigma_u < +\infty$, for any $\sigma > \sigma_u$ put

$$\overline{M}_u(\sigma) = \sup\left\{\left|\sum_{j=0}^{n} a_j \mathrm{e}^{-\lambda_j(\sigma+\mathrm{i}t)}\right| :\ n \in \mathbb{N}, t \in \mathbb{R}\right\};$$

$$\overline{m}(\sigma) = \max\{\overline{A}_k \mathrm{e}^{-k\sigma} :\ k \in \mathbb{N}\};$$

$$\varrho_u = \overline{\lim_{\sigma\to0^+}} \frac{\ln^+ \ln^+ \overline{M}_u(\sigma)}{-\ln\sigma}; \quad \varrho_\mu = \overline{\lim_{\sigma\to0^+}} \frac{\ln^+ \ln^+ \overline{m}(\sigma)}{-\ln\sigma}.$$

Lemma 1. *Suppose $\sigma_u < +\infty$, then*

(I) $\overline{m}(\sigma) \leqslant 4\mathrm{e}^{|\sigma|}\overline{M}_u(\sigma)$ $(\sigma > \sigma_u)$;

(II) *if* $\sigma_u = 0$, $\varepsilon > 0$, *then* $\overline{M}_u(\sigma) \leqslant \overline{m}((1-\varepsilon)\sigma)/(1-\mathrm{e}^{-\varepsilon\sigma})$ $(\sigma > 0)$;

(III) *if* $\sigma_u = 0$, *then* $\varrho_u = \varrho_\mu$.

P r o o f. Take $p \in \mathbb{N}$ such that $n_k + p < n_{k+1}$, where n_k is defined by (1). Using Abel's transformation, we obtain

$$\begin{aligned}\sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-\mathrm{i}t\lambda_j} &= \sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_j}\mathrm{e}^{\sigma\lambda_j} \\ &= \sum_{j=n_k}^{n_k+p-1} \sum_{q=n_k}^{j} a_q \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_q}(\mathrm{e}^{\sigma\lambda_j} - \mathrm{e}^{\sigma\lambda_{j+1}}) + \sum_{q=n_k}^{n_k+p} a_q \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_q}\mathrm{e}^{\sigma\lambda_{n_k+p}}.\end{aligned}$$

Noting that

$$\left|\sum_{q=n_k}^{j} a_q \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_q}\right| \leqslant 2\overline{M}_u(\sigma),$$

we conclude that

$$\overline{A}_k \leqslant 2\overline{M}_u(\sigma)|\mathrm{e}^{\sigma\lambda_{n_k}} - \mathrm{e}^{\sigma\lambda_{n_k+p}}| + 2\mathrm{e}^{\sigma\lambda_{n_k+p}}\overline{M}_u(\sigma) \leqslant 4\overline{M}_u(\sigma)\mathrm{e}^{(k+\mathrm{sgn}\sigma)\sigma}.$$

This gives (I).

Now we prove (II). Suppose $n_k + p < n_{k+1}$. Using Abel's transformation, we arrive at

$$\sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_j} = \sum_{j=n_k}^{n_k+p-1} \sum_{q=n_k}^{j} a_q \mathrm{e}^{-\mathrm{i}t\lambda_q}(\mathrm{e}^{-\sigma\lambda_j} - \mathrm{e}^{-\sigma\lambda_{j+1}}) + \sum_{q=n_k}^{n_k+p} a_q \mathrm{e}^{-\mathrm{i}t\lambda_q}\mathrm{e}^{-\sigma\lambda_{n_k+p}}.$$

So, when $\sigma > 0$,

$$\left|\sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_j}\right| \leqslant \overline{A}_k \sum_{j=n_k}^{n_k+p-1} (\mathrm{e}^{-\sigma\lambda_j} - \mathrm{e}^{-\sigma\lambda_{j+1}}) + \overline{A}_k \mathrm{e}^{-\sigma\lambda_{n_k+p}} = \overline{A}_k \mathrm{e}^{-\sigma\lambda_{n_k}} \leqslant \overline{A}_k \mathrm{e}^{-\sigma k}.$$

Therefore,

$$\left|\sum_{j=0}^{n_k+p} a_j \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_j}\right| \leqslant \sum_{j=0}^{k} \overline{A}_j \mathrm{e}^{-\sigma j} = \sum_{j=0}^{k} \overline{A}_j \mathrm{e}^{-(1-\varepsilon)\sigma j}\mathrm{e}^{-j\varepsilon\sigma} \leqslant \overline{m}((1-\varepsilon)\sigma)\sum_{j=0}^{k} \mathrm{e}^{-j\varepsilon\sigma} \leqslant \frac{\overline{m}((1-\varepsilon)\sigma)}{1-\mathrm{e}^{-\varepsilon\sigma}}.$$

This gives (II).

Since $\ln^+\ln^+\overline{m}(\sigma) \leqslant \ln^+\ln^+\frac{1}{4}\mathrm{e}^{-\sigma}\overline{m}(\sigma) + \ln^+\ln^+ 4\mathrm{e}^{\sigma} + \ln 2$, we have

$$\varlimsup_{\sigma\to 0^+} \frac{\ln^+\ln^+\overline{m}(\sigma)}{-\ln\sigma} \leqslant \varlimsup_{\sigma\to 0^+} \frac{\ln^+\ln^+\frac{1}{4}\mathrm{e}^{-\sigma}\overline{m}(\sigma)}{-\ln\sigma}.$$

On the other hand,

$$\varlimsup_{\sigma\to 0^+} \frac{\ln^+\ln^+\dfrac{\overline{m}((1-\varepsilon)\sigma)}{1-\mathrm{e}^{-\varepsilon\sigma}}}{-\ln\sigma} \leqslant \varlimsup_{\sigma\to 0^+} \frac{\ln^+\ln^+\overline{m}((1-\varepsilon)\sigma)}{-\ln\sigma} + \varlimsup_{\sigma\to 0^+} \frac{\ln^+\ln^+(1-\mathrm{e}^{-\varepsilon\sigma})^{-1}}{-\ln\sigma} = \varlimsup_{\sigma\to 0^+} \frac{\ln^+\ln^+\overline{m}(\sigma)}{-\ln\sigma}.$$

Thus (III) is proved. □

Lemma 2. *If $\sigma_u = 0$, then*

$$\varlimsup_{k\to+\infty} \frac{\ln^+\ln^+\overline{M}_u(\sigma)}{-\ln\sigma} = \varrho_u \Leftrightarrow \varlimsup_{k\to+\infty} \frac{\ln^+\ln^+\overline{A}_k}{\ln k} = \begin{cases} \dfrac{\varrho_u}{\varrho_u+1}, & \varrho_u < +\infty; \\ 1, & \varrho_u = +\infty. \end{cases}$$

P r o o f. Consider the case $\varrho_u < +\infty$. We prove the necessity of the right-hand side condition. By Lemma 1(III), for all $\varepsilon > 0$, when $\sigma > 0$ is sufficiently small,

$$\overline{m}(\sigma) < \exp\Big\{\Big(\frac{1}{\sigma}\Big)^{\varrho_u+\varepsilon}\Big\}.$$

Since

$$\min\Big\{k\sigma + \Big(\frac{1}{\sigma}\Big)^{\varrho_u+\varepsilon}:\ \sigma > 0\Big\} = (\varrho_u+\varepsilon+1)\Big(\frac{k}{\varrho_u+\varepsilon}\Big)^{(\varrho_u+\varepsilon)/(\varrho_u+\varepsilon+1)},$$

it follows that for sufficiently large $k \in \mathbb{N}$,

$$\overline{A}_k \leqslant \exp\Big\{(\varrho_u+\varepsilon+1)\Big(\frac{k}{\varrho_u+\varepsilon}\Big)^{(\varrho_u+\varepsilon)/(\varrho_u+\varepsilon+1)}\Big\}.$$

So, as $\varepsilon \to 0$,

$$\varlimsup_{k\to+\infty} \frac{\ln^+\ln^+\overline{A}_k}{\ln k} \leqslant \frac{\varrho_u}{\varrho_u+1}.$$

As for the converse, suppose that $\varlimsup\limits_{k\to+\infty} \ln^+\ln^+\overline{A}_k/\ln k < \varrho_u/(\varrho_u+1)$. There exist $0 < \varrho_u' < \varrho_u$ such that for any $k \in \mathbb{N}$,

$$\overline{A}_k < \exp(k^{\varrho_u'/(\varrho_u'+1)}).$$

Since

$$\max\{(k^{\varrho_u'/(\varrho_u'+1)} - k\sigma):\ k \geqslant 0\} = \frac{1}{\varrho_u'+1}\Big(\frac{\varrho_u'}{\varrho_u'+1}\frac{1}{\sigma}\Big)^{\varrho_u'},$$

we have

$$\overline{A}_k e^{-k\sigma} < \exp\Big\{\frac{1}{\varrho_u'+1}\Big(\frac{\varrho_u'}{\varrho_u'+1}\frac{1}{\sigma}\Big)^{\varrho_u'}\Big\}.$$

Thus

$$\overline{m}(\sigma) \leqslant \exp\Big\{\frac{1}{\varrho_u'+1}\Big(\frac{\varrho_u'}{\varrho_u'+1}\frac{1}{\sigma}\Big)^{\varrho_u'}\Big\}.$$

Hence, by Lemma 1(III),

$$\varlimsup_{k\to+\infty} \frac{\ln^+\ln^+\overline{M}_u(\sigma)}{-\ln\sigma} \leqslant \varrho_u' < \varrho_u,$$

which contradicts the left-hand side condition of the theorem. Thus we have proved the necessity of the right-hand side condition. The sufficiency of the right-hand side condition follows easily in a similar manner and is left to the reader.

Consider the case $\varrho_u = +\infty$. We then have

$$\varlimsup_{k\to+\infty} \frac{\ln^+ \ln^+ \overline{A}_k}{\ln k} = 1.$$

Otherwise, assume that $\varlimsup\limits_{k\to+\infty} \ln^+ \ln^+ \overline{A}_k / \ln k < 1$. Then there exists $\varrho_u'' < +\infty$ such that

$$\varlimsup_{k\to+\infty} \frac{\ln^+ \ln^+ \overline{A}_k}{\ln k} = \frac{\varrho_u''}{\varrho_u''+1}.$$

Clearly, by the case $\varrho_u < +\infty$, this yields a contradiction. □

Lemma 3. *If* $\Delta = 0$, *then* $\sigma_c = \sigma_u = \sigma_a = \sigma_0$.

P r o o f. Since $\Delta = 0$, for any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for any $k > K$,

$$p_k \leqslant \mathrm{e}^{k^\varepsilon} - 1.$$

For any sufficiently large n satisfying $\lambda_n \geqslant K+1$,

$$n < n_{K+1} + \sum_{i=K+1}^{[\lambda_n]} p_i < n_{K+1} + \sum_{i=K+1}^{[\lambda_n]} (\mathrm{e}^{i^\varepsilon} - 1) \leqslant n_{K+1} + [\lambda_n](\mathrm{e}^{[\lambda_n]^\varepsilon} - 1),$$

where $[\lambda_n]$ denotes the integer part of λ_n. Then

$$\begin{aligned}\varlimsup_{n\to+\infty} \frac{\ln n}{\lambda_n} &\leqslant \varlimsup_{n\to+\infty} \frac{\ln(n_{K+1} + [\lambda_n](\mathrm{e}^{[\lambda_n]^\varepsilon} - 1))}{[\lambda_n]} \\ &\leqslant \varlimsup_{n\to+\infty} \frac{\ln n_{K+1}}{[\lambda_n]} + \varlimsup_{n\to+\infty} \frac{\ln [\lambda_n]}{[\lambda_n]} + \varlimsup_{n\to+\infty} \frac{[\lambda_n]^\varepsilon}{[\lambda_n]} = 0.\end{aligned}$$

By G. Valiron's formula [10], [11]

$$\varlimsup_{n\to+\infty} \frac{\ln |a_n|}{\lambda_n} \leqslant \sigma_c \leqslant \sigma_u \leqslant \sigma_a \leqslant \varlimsup_{n\to+\infty} \frac{\ln |a_n|}{\lambda_n} + \varlimsup_{n\to+\infty} \frac{\ln n}{\lambda_n}.$$

The conclusion now follows. □

3. The proof of theorem 1

Proof. Since $\Delta = 0, \sigma_0 = 0$, by Lemma 3 we have $\sigma_c = \sigma_u = \sigma_a = 0$.

Consider the case $\varrho < +\infty$. We first prove the necessity of the right-hand side condition. Since $\overline{M}_u(\sigma) \geqslant M(\sigma)$, we have $\varrho_u \geqslant \varrho$.

For any $\varepsilon > 0$, when $\sigma(>0)$ is sufficiently small,

$$M(\sigma) < \exp\{\sigma^{-(\varrho+\varepsilon)}\}.$$

Take account of Hadamard's theorem [12], $a_n e^{-\lambda_n \sigma} \leqslant M(\sigma)$ and

$$\min\{\sigma^{-(\varrho+\varepsilon)} + \lambda_n \sigma\colon\ \sigma > 0\} = (\varrho+\varepsilon+1)\Big(\frac{\lambda_n}{\varrho+\varepsilon}\Big)^{(\varrho+\varepsilon)/(\varrho+\varepsilon+1)}.$$

Therefore, for sufficiently large $n \in \mathbb{N}$,

$$|a_n| < \exp\Big\{(\varrho+\varepsilon+1)\Big(\frac{\lambda_n}{\varrho+\varepsilon}\Big)^{(\varrho+\varepsilon)/(\varrho+\varepsilon+1)}\Big\}.$$

So, as $\varepsilon \to 0$,

$$\varlimsup_{n\to+\infty} \frac{\ln^+ \ln^+ |a_n|}{\ln \lambda_n} \leqslant \frac{\varrho}{\varrho+1}.$$

Suppose $\varlimsup\limits_{n\to+\infty} \ln^+ \ln^+ |a_n| / \ln \lambda_n < \varrho/(\varrho+1)$. Then there exists $0 \leqslant \varrho' < \varrho$ such that for sufficiently large $n \in \mathbb{N}$,

$$|a_n| < \exp\{\lambda_n^{\varrho'/(\varrho'+1)}\}.$$

Then for sufficiently large $k \in \mathbb{N}$,

$$\overline{A}_k < \sum_{j=n_k}^{n_k+p_k} \exp\{\lambda_j^{\varrho'/(\varrho'+1)}\} < \exp\{(k+1)^{\varrho'/(\varrho'+1)} + \ln(p_k+1)\}.$$

Since $\Delta = 0$, we conclude that

$$\varlimsup_{k\to+\infty} \frac{\ln^+ \ln^+ \overline{A}_k}{\ln k} \leqslant \frac{\varrho'}{\varrho'+1} < \frac{\varrho}{\varrho+1}.$$

By Lemma 2, $\varrho_u < \varrho$, which contradicts $\varrho_u \geqslant \varrho$. Hence,

$$\varlimsup_{n\to+\infty} \frac{\ln^+ \ln^+ |a_n|}{\ln \lambda_n} = \frac{\varrho}{\varrho+1}.$$

Second, we prove the sufficiency of the right-hand side condition. For any $\varepsilon > 0$, when n is sufficiently large,

$$|a_n| < \exp\{\lambda_n^{(\varrho+\varepsilon)/(\varrho+\varepsilon+1)}\}.$$

Then for sufficiently large $k \in \mathbb{N}$,

$$\overline{A}_k < \sum_{j=n_k}^{n_k+p_k} \exp\{\lambda_j^{(\varrho+\varepsilon)/(\varrho+\varepsilon+1)}\} < \exp\{(k+1)^{(\varrho+\varepsilon)/(\varrho+\varepsilon+1)} + \ln(p_k+1)\}.$$

Since $\Delta = 0$, then as $\varepsilon \to 0$,

$$\varlimsup_{k\to+\infty} \frac{\ln^+ \ln^+ \overline{A}_k}{\ln k} \leqslant \frac{\varrho}{\varrho+1}.$$

By Lemma 2, $\varrho_u \leqslant \varrho$. Since $M(\sigma) \leqslant \overline{M}_u(\sigma)$, we have

$$\varlimsup_{\sigma\to0^+} \frac{\ln^+ \ln^+ M(\sigma)}{-\ln \sigma} \leqslant \varrho.$$

If the equality does not hold, then by the necessity of the right-hand side condition,

$$\varlimsup_{n\to+\infty} \frac{\ln^+ \ln^+ |a_n|}{\ln \lambda_n} < \frac{\varrho}{\varrho+1},$$

which contradicts the right-hand side condition. Thus the sufficiency of the right-hand side condition is proved. Therefore the case $\varrho < +\infty$ is proved.

By the case $\varrho < +\infty$, it is easy to prove the case $\varrho = +\infty$. Thus Theorem 1 is proved. □

4. Corollary and examples

By Theorem 1, we can deduce Yu Jia-Rong's result [15], Theorem 2.2.

Corollary 1 [15]**.** *Let $f(s)$ be a Dirichlet series with frequencies $\{\lambda_n\}$ as in the introduction. If $\sigma_0 = 0$ and $\varlimsup_{n\to+\infty} n/\lambda_n = D < +\infty$, then*

$$\varlimsup_{\sigma\to0^+} \frac{\ln^+ \ln^+ M(\sigma)}{-\ln \sigma} = \varrho \Leftrightarrow \varlimsup_{n\to+\infty} \frac{\ln^+ \ln^+ |a_n|}{\ln \lambda_n} = \begin{cases} \dfrac{\varrho}{\varrho+1}, & \varrho < +\infty; \\ 1, & \varrho = +\infty. \end{cases} \tag{2}$$

P r o o f. Since $\overline{\lim\limits_{n\to+\infty}}\, n/\lambda_n = D < +\infty$, hence for any $\varepsilon > 0$ there exists N such that for any $n > N$,

$$p_{[\lambda_n]-1} \leqslant n < \lambda_n(D+\varepsilon) < ([\lambda_n]+1)(D+\varepsilon).$$

Therefore,

$$\Delta = \overline{\lim_{n\to+\infty}} \frac{\ln^+ \ln(p_{[\lambda_n]-1}+1)}{\ln([\lambda_n]-1)} \leqslant \overline{\lim_{n\to+\infty}} \frac{\ln^+ \ln(([\lambda_n]+1)(D+\varepsilon)+1)}{\ln([\lambda_n]-1)} = 0.$$

Hence $\Delta = 0$. Since $\sigma_0 = 0$, (2) holds by Theorem 1. □

Now we give two examples. Example 1 shows that $\Delta = 0$ is weaker than $\overline{\lim\limits_{n\to+\infty}}\, n/\lambda_n < +\infty$. Example 2 shows that $\Delta = 0$ cannot be weakened to $\Delta < +\infty$.

Example 1. Consider a Dirichlet series $f(s)$ with frequencies $\{\lambda_n\}$ as in the introduction. Take $a_n = 1$, $n = 0, 1, 2, \ldots$. When $\frac{1}{2}k(k+1) < n \leqslant \frac{1}{2}(k+1)(k+2)$, take $\lambda_{\frac{1}{2}k(k+1)+1+p} = k + p/(k+1)$, where $0 \leqslant p < k+1$. It is evident that $\sigma_0 = 0$, $\Delta = 0$ (but $\overline{\lim\limits_{n\to+\infty}}\, n/\lambda_n = +\infty$). Since $\overline{\lim\limits_{n\to+\infty}} \ln^+ \ln^+ |a_n| / \ln \lambda_n = 0$, by Theorem 1 we infer $\varrho = 0$.

Example 2. Consider a Dirichlet series $f(s)$ with frequencies $\{\lambda_n\}$ as in the introduction. Take $a_n = (-1)^n, n = 0, 1, 2, \ldots$. When $2^k \leqslant n < 2^{k+1}$, take $\lambda_n = \lambda_{2^k+p} = k + p/2^k$, where $0 \leqslant p < 2^k$. It is easily seen from the formulas for the abscissas $\sigma_c, \sigma_u, \sigma_a$ in terms of $A_k, \overline{A}_k, A_k^*$ in Section 2 that $\sigma_c = 0$ and $\sigma_a = \ln 2$. Since

$$\overline{A}_k \geqslant \left| \sum_{j=0}^{2^k-1} (-1)^j \mathrm{e}^{-\mathrm{i}(2^k k\pi + j\pi)} \right| = \left| \sum_{j=0}^{2^k-1} (-1)^j \mathrm{e}^{-\mathrm{i}j\pi} \right|$$
$$= \left| \sum_{j=0}^{2^k-1} (-1)^j (\cos j\pi + \mathrm{i} \sin j\pi) \right| = 2^k,$$

hence

$$\sigma_u = \overline{\lim_{k\to+\infty}} \frac{\ln \overline{A}_k}{k} = \ln 2.$$

We can see from this example that $\sigma_u = \sigma_a = \ln 2$ and $\sigma_c = 0$, while $\Delta = 1$ and $\sigma_0 = 0$. The conclusion of Theorem 1 does not hold for this Dirichlet series, as $M(\sigma)$ is infinite for $\sigma < \ln 2$, while $\ln^+ |a_n| \equiv 0$.

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Authors' addresses: Zhendong Gu, School of Mathematical Sciences, South China Normal University, Guangzhou, People's Republic of China. Postal Code: 510631, e-mail: `guzhd@qq.com`; Daochun Sun, School of Mathematical Sciences, South China Normal University, Guangzhou, People's Republic of China. Postal Code: 510631, e-mail: `sundch@scnu.edu.cn`.