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INTEGRALS AND BANACH SPACES FOR FINITE ORDER DISTRIBUTIONS

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Abstract. Let $\mathcal{B}_c$ denote the real-valued functions continuous on the extended real line and vanishing at $-\infty$. Let $\mathcal{B}_r$ denote the functions that are left continuous, have a right limit at each point and vanish at $-\infty$. Define $\mathcal{A}_c^n$ to be the space of tempered distributions that are the nth distributional derivative of a unique function in $\mathcal{B}_c$. Similarly with $\mathcal{A}_r^n$ from $\mathcal{B}_r$. A type of integral is defined on distributions in $\mathcal{A}_c^n$ and $\mathcal{A}_r^n$. The multipliers are iterated integrals of functions of bounded variation. For each $n \in \mathbb{N}$, the spaces $\mathcal{A}_c^n$ and $\mathcal{A}_r^n$ are Banach spaces, Banach lattices and Banach algebras isometrically isomorphic to $\mathcal{B}_c$ and $\mathcal{B}_r$, respectively. Under the ordering in this lattice, if a distribution is integrable then its absolute value is integrable. The dual space is isometrically isomorphic to the functions of bounded variation. The space $\mathcal{A}_c^1$ is the completion of the L^1 functions in the Alexiewicz norm. The space $\mathcal{A}_r^1$ contains all finite signed Borel measures. Many of the usual properties of integrals hold: Hölder inequality, second mean value theorem, continuity in norm, linear change of variables, a convergence theorem.

Keywords: regulated function, regulated primitive integral, Banach space, Banach lattice, Banach algebra, Schwartz distribution, generalized function, distributional Denjoy integral, continuous primitive integral, Henstock-Kurzweil integral, primitive

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1. Introduction

An integral means different things to different people: a Riemann sum, approximation by simple functions, inversion of the derivative of an absolutely continuous function, a type of linear functional. In this paper we will define an integration process on Schwartz distributions of finite order by inverting the nth order distributional derivative of regulated or continuous functions. An important part of the definition is that we will obtain a linear functional that acts on iterated integrals of functions of bounded variation.

We denote by $\mathcal{B}_c$ the continuous functions on the extended real line that vanish at $-\infty$. Then $\mathcal{A}_c^n$ is the set of distributions that are the nth distributional derivative of a unique function in $\mathcal{B}_c$. The case $n = 1$ was studied in [26]. The space $\mathcal{A}_c^1$ is the completion of the space of L^1 functions and the completion of the space of Henstock-Kurzweil integrable functions in the Alexiewicz norm. If $f\colon \mathbb{R} \to \mathbb{R}$ is a Henstock-Kurzweil integrable function then the Alexiewicz norm is $\|f\| = \sup_{x\in\mathbb{R}} |\int_{-\infty}^x f(t)\,\mathrm{d}t|$. If $F \in \mathcal{B}_c$ and $f = F'$ is its distributional derivative then the *continuous primitive integral* of f is $\int_a^b f = F(b) - F(a)$. In this paper we define $\int_{-\infty}^{\infty} fh$ for $f \in \mathcal{A}_c^n$ and h an n-fold iterated integral of a function of bounded variation (Definition 2.6). A function is called *regulated* if it has a left limit and a right limit at each point. We denote by $\mathcal{B}_r$ the left continuous regulated functions that vanish at $-\infty$. Then $\mathcal{A}_r^n$ is the set of distributions that are the nth distributional derivative of a function in $\mathcal{B}_r$. The case $n = 1$ was studied in [28]. The space $\mathcal{A}_r^1$ contains $\mathcal{A}_c^1$ as well as all finite signed Borel measures. An integral in $\mathcal{A}_r^n$ is defined as for $\mathcal{A}_c^n$. Under the uniform norm, $\mathcal{B}_c$ and $\mathcal{B}_r$ are Banach spaces. The nth order distributional derivative provides a linear isometric bijection between $\mathcal{B}_c$ and $\mathcal{A}_c^n$ and between $\mathcal{B}_r$ and $\mathcal{A}_r^n$. For each $n \in \mathbb{N}$, the spaces of distributions $\mathcal{A}_c^n$ and $\mathcal{A}_r^n$ are then Banach spaces that are isometrically isomorphic to $\mathcal{B}_c$ and $\mathcal{B}_r$, respectively. If $f \in \mathcal{A}_r^n$ with primitive $F \in \mathcal{B}_r$ then its norm is $\|f\|_{a,n} = \|F\|_\infty$. The spaces $\mathcal{B}_c$ and $\mathcal{A}_c^n$ are separable while the spaces $\mathcal{B}_r$ and $\mathcal{A}_r^n$ are not separable.

Below we define our notation for distributions.

The main definitions are given in Section 2. For $f \in \mathcal{A}_r^n$, the integral $\int_{-\infty}^{\infty} fh$ is defined by reducing to $\int_{-\infty}^{\infty} F'h^{(n-1)}$. With $h^{(n-1)}$ a function of bounded variation, this reduces to an integral in $\mathcal{A}_r^1$. This is evaluated using Henstock-Stieltjes integrals. See Definition 2.6. It is shown that distributions in $\mathcal{A}_r^n$ are tempered and of order at most n. There is translation invariance and for $\mathcal{A}_c^n$ there is continuity in norm. The multipliers for these integrals are iterated integrals of functions of bounded variation. Various properties of these functions are proved here.

Examples and further properties of the integral are given in Section 3. If δ is the Dirac distribution then $\delta^{(m)} \in \mathcal{A}_r^{m+1}$ for each $m \geqslant 0$. It is shown that each distributional derivative of a finite signed Borel measure is in some $\mathcal{A}_r^n$ space. A version of the second mean value theorem is established and a linear change of variables theorem is proved. Relationships amongst $\mathcal{A}_c^n$ and $\mathcal{A}_r^m$ are investigated.

In Section 4 a type of Hölder inequality is established. For $f \in \mathcal{A}_r^n$ with primitive $F \in \mathcal{B}_r$ we have $|\int_{-\infty}^{\infty} fh| \leqslant \|F\|_\infty \|g\|_{\mathcal{B}V}$, where g is of bounded variation and $g = h^{(n-1)}$. This leads to a convergence theorem when a sequence $(f_n) \subset \mathcal{A}_r^n$ converges in norm. Using the isometries $\mathcal{A}_c^n \leftrightarrow \mathcal{B}_c$ and $\mathcal{A}_r^n \leftrightarrow \mathcal{B}_r$ it is shown that the dual space of $\mathcal{A}_c^n$ is the space of functions of normalized bounded variation and the dual space of $\mathcal{A}_r^n$ is the space of functions of bounded variation.

Under pointwise operations $\mathcal{B}_r$ is a Banach lattice. If $F_1, F_2 \in \mathcal{B}_r$ then $F_1 \leqslant F_2$ means $F_1(x) \leqslant F_2(x)$ for all $x \in \mathbb{R}$. This Banach lattice structure is inherited by $\mathcal{A}_c^n$ and $\mathcal{A}_r^n$. If $f_1 = F_1^{(n)}, f_2 = F_2^{(n)} \in \mathcal{A}_r^n$ then $f_1 \preceq f_2$ if and only if $F_1 \leqslant F_2$ in $\mathcal{B}_r$. Elementary lattice properties are proved in Section 5, including the fact that $\mathcal{A}_c^n$ and $\mathcal{A}_r^n$ are abstract M spaces. Under this lattice ordering, the integrals introduced are absolute in the sense that if f is integrable then the absolute value of f in this ordering is integrable. This is the case even though functions in $\mathcal{A}_r^n$ may have conditionally convergent Henstock-Kurzweil integrals.

Banach algebras are considered in Section 6. Under pointwise operations $\mathcal{B}_r$ is a Banach algebra. If $F_1, F_2 \in \mathcal{B}_r$ then $(F_1F_2)(x) = F_1(x)F_2(x)$. By the isomorphism, $\mathcal{A}_c^n$ and $\mathcal{A}_r^n$ are also Banach algebras. If $f_1 = F_1^{(n)}, f_2 = F_2^{(n)} \in \mathcal{A}_r^n$ then $f_1f_2 = D^n(F_1F_2)$. For complex-valued distributions they are C^*-algebras. Under this multiplication $\delta^{(n)}\delta^{(n)} = \delta^{(n)}$ for each $n \geqslant 0$. J. F. Colombeau, E. E. Rosinger and others have embedded spaces of distributions in various algebras. For example, see [21]. The Banach algebra we construct here seems to be unrelated.

The starting point in this paper are the spaces $\mathcal{B}_c$ and $\mathcal{B}_r$. Basic properties are established in Section 2. The following lemma is used repeatedly to carry over Banach space, Banach lattice and Banach algebra properties to $\mathcal{A}_c^n$ and $\mathcal{A}_r^n$.

Lemma 1.1. *Let A be a set. Let B be a vector space over the field $\mathbb{R}$. Let $x, y \in A$; $a \in \mathbb{R}$.*

(a) *Suppose there is a bijection $\Phi\colon B \to A$. Define $x + y = \Phi(\Phi^{-1}(x) + \Phi^{-1}(y))$ and $ax = \Phi(a\Phi^{-1}(x))$. Then A is a vector space isomorphic to B and Φ is linear such that $\Phi \circ \Phi^{-1} = i_A$ and $\Phi^{-1} \circ \Phi = i_B$.*

(b) *Suppose B is a Banach space. Define $\|x\|_A = \|\Phi^{-1}(x)\|_B$. Then A is a Banach space isometrically isomorphic to B. If B is separable so is A.*

(c) *Suppose B is a Banach lattice. Define $x \preceq y$ in A if and only if $\Phi^{-1}(x) \preceq \Phi^{-1}(y)$ in B. Then A and B are isometrically isomorphic Banach lattices.*

(d) *Suppose B is a Banach algebra. Define $xy = \Phi(\Phi^{-1}(x)\Phi^{-1}(y))$. Then A and B are isomorphic Banach algebras.*

The proof is elementary. A related result is that if Φ is a surjective isometry between two normed linear spaces then Φ must be linear. This is the Mazur-Ulam theorem. For example, see [9].

The *test functions* are $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$, i.e., the smooth functions with compact support. The *support* of a function φ is the closure of the set on which φ does not vanish. Denote this as $\mathrm{supp}(\varphi)$. There is a notion of continuity in $\mathcal{D}(\mathbb{R})$. If

$\{\varphi_n\} \subset \mathcal{D}(\mathbb{R})$ then $\varphi_n \to \varphi \in \mathcal{D}(\mathbb{R})$ if there is a compact set $K \subset \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\operatorname{supp}(\varphi_n) \subset K$, and for each integer $m \geqslant 0$, $\varphi_n^{(m)} \to \varphi^{(m)}$ uniformly on K as $n \to \infty$. The *distributions* are the continuous linear functionals on $\mathcal{D}(\mathbb{R})$, denoted $\mathcal{D}'(\mathbb{R})$. If $T \in \mathcal{D}'(\mathbb{R})$ then $T\colon \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ and we write $\langle T, \varphi\rangle \in \mathbb{R}$ for $\varphi \in \mathcal{D}(\mathbb{R})$. If $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R})$ then $\langle T, \varphi_n\rangle \to \langle T, \varphi\rangle$ in $\mathbb{R}$. And, for all $a_1, a_2 \in \mathbb{R}$ and all $\varphi, \psi \in \mathcal{D}(\mathbb{R})$, $\langle T, a_1\varphi + a_2\psi\rangle = a_1\langle T, \varphi\rangle + a_2\langle T, \psi\rangle$. If $f \in L^p_{\mathrm{loc}}$ for some $1 \leqslant p \leqslant \infty$ then $\langle T_f, \varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\,\mathrm{d}x$ defines a distribution. For a locally integrable function we will often drop the distinction between f and T_f. The differentiation formula $\langle D^n T, \varphi\rangle = \langle T^{(n)}, \varphi\rangle = (-1)^n\langle T, \varphi^{(n)}\rangle$ ensures that all distributions have derivatives of all orders which are themselves distributions. This is known as the *distributional derivative* or *weak derivative*. We will usually denote distributional derivatives by $D^n F$, $F^{(n)}$ or F' and pointwise derivatives by $F^{(n)}(t)$ or $F'(t)$. For $T \in \mathcal{D}'(\mathbb{R})$ and $t \in \mathbb{R}$ the *translation* τ_t is defined by $\langle \tau_t T, \varphi\rangle = \langle T, \tau_{-t}\varphi\rangle$ where $\tau_t\varphi(x) = \varphi(x - t)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. If there is an integer $N \geqslant 0$ such that for each compact set $K \subset \mathbb{R}$ there is a real number $C \geqslant 0$ such that $|\langle T, \varphi\rangle| \leqslant C \sum\limits_0^N \|\varphi^{(n)}\|_\infty$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ with support in K, then the distribution T is said to be of *finite order*. The least such N is the *order* of T. If μ is a finite signed Borel measure then $\langle T_\mu, \varphi\rangle = \int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}\mu(x)$ defines $T_\mu \in \mathcal{D}'(\mathbb{R})$ as a distribution of order 0. Most of the results on distributions we use can be found in [10], [12] or [30].

Several authors have proposed various schemes for integrating distributions. L. Schwartz [23] considered the integral of $T \in \mathcal{D}'(\mathbb{R})$ as the linear functional $\langle T, 1\rangle$, whenever this exists. As will be seen in the next section, we generalize Schwartz's definition so that the integrable distributions are continuous linear functionals on iterated integrals of functions of bounded variation. J. C. Burkill [5] has sketched out a method of integrating distributions using higher order Stieltjes integrals. A. M. Russell [22] and A. G. Das [7] with coauthors have also used higher Stieltjes integrals. J. Mikusiński, J. A. Musielak and R. Sikorski have used convolutions to define a type of integral for distributions. See [19], [20], [24].

The extended real line is denoted $\overline{\mathbb{R}} = [-\infty, \infty]$. The space $C(\overline{\mathbb{R}})$ consists of the continuous functions $F\colon \overline{\mathbb{R}} \to \mathbb{R}$. A function is in $C(\overline{\mathbb{R}})$ if it is continuous at each point in $\mathbb{R}$ and if $F(\infty) = \lim\limits_{x\to\infty} F(x) \in \mathbb{R}$ and $F(-\infty) = \lim\limits_{x\to-\infty} F(x) \in \mathbb{R}$. This two-point compactification makes $\overline{\mathbb{R}}$ into a compact Hausdorff space. A topological base for $\overline{\mathbb{R}}$ consists of the usual open intervals (a, b) with $-\infty \leqslant a < b \leqslant \infty$, as well as $[-\infty, a)$ with $-\infty < a \leqslant \infty$, and $(a, \infty]$ with $-\infty \leqslant a < \infty$. In this paper the word *compact* will always refer to the usual topology on $\mathbb{R}$.

A function $F\colon \overline{\mathbb{R}} \to \mathbb{R}$ is *regulated* on $\overline{\mathbb{R}}$ if it has left and right limits at each point of $\mathbb{R}$ and real limits at $\pm\infty$, i.e., for each $x \in \mathbb{R}$ the limits $F(x-) = \lim\limits_{y\to x^-} F(y)$ and

$F(x+) = \lim_{y\to x^+} F(y)$ exist as real numbers and $\lim_{y\to-\infty} F(y)$ and $\lim_{y\to\infty} F(y)$ exist as real numbers. We will use the following normalizations for regulated functions. If F is regulated and $0 \leqslant \lambda \leqslant 1$ then $F_\lambda(x) = (1-\lambda)F(x-) + \lambda F(x+)$ for all $x \in \mathbb{R}$. The functions F and F_λ will then differ on a countable set. Note that F_0 is left continuous and F_1 is right continuous. If F is continuous then all normalizations are equal to F. The *Heaviside step function* has the left and right continuous normalizations $H_0 = \chi_{(0,\infty]}$ and $H_1 = \chi_{[0,\infty]}$. Unless otherwise stated, all regulated functions will satisfy $F(-\infty) = \lim_{y\to-\infty} F(y)$ and $F(\infty) = \lim_{y\to\infty} F(y)$. For more on regulated functions, see [11].

2. Banach spaces and integrals

A space of primitives is the space of regulated functions that vanish at $-\infty$. Each such regulated function is differentiated n times with the distributional derivative to yield a sequence of Banach spaces of integrable distributions, each being isometrically isomorphic to the space of primitives. A second space of primitives is the set of functions in $C(\overline{\mathbb{R}})$ that vanish at $-\infty$. These are also differentiated n times to give a sequence of Banach spaces. By integrating functions of bounded variation n times we find the corresponding set of multipliers. Distinction is made between functions of bounded variation, normalized bounded variation and essential bounded variation.

We take as our set of primitives $\mathcal{B}_r$. This consists of the functions $F\colon \overline{\mathbb{R}} \to \mathbb{R}$ that are regulated on $\overline{\mathbb{R}}$ such that $F(-\infty) = \lim_{x\to-\infty} F(x) = 0$, $F(x) = F(x-)$ for all $x \in \mathbb{R}$ and $F(\infty) = \lim_{y\to\infty} F(y)$. Hence, they are left continuous on $(-\infty,\infty]$, vanish at $-\infty$ and equal their limits at infinity. Under pointwise operations and the uniform norm, $\|F\|_\infty = \sup_{x\in\mathbb{R}} |F(x)|$, $\mathcal{B}_r$ is a Banach space. It is not separable. See [28, Theorem 2] where various properties of $\mathcal{B}_r$ are proved. A second set of primitives is the subspace $\mathcal{B}_c = \{F \in C(\overline{\mathbb{R}}) \mid F(-\infty) = 0\}$. This is then a separable Banach space with norm $\|F\|_\infty = \sup_{x\in\mathbb{R}} |F(x)| = \max_{x\in\overline{\mathbb{R}}} |F(x)|$. The separability of $\mathcal{B}_c$ follows from the compactness of $\overline{\mathbb{R}}$. See [8, Exercise V.7.12].

For each $n \in \mathbb{N}$ define $\mathcal{A}_r^n = \{f \in \mathcal{D}'(\mathbb{R}) \mid f = F^{(n)}$ for some $F \in \mathcal{B}_r\}$, i.e., $\langle f, \varphi\rangle = \langle F^{(n)}, \varphi\rangle = (-1)^n \langle F, \varphi^{(n)}\rangle = (-1)^n \int_{-\infty}^{\infty} F(x)\varphi^{(n)}(x)\,\mathrm{d}x$ for each $\varphi \in \mathcal{D}(\mathbb{R})$. This last integral is a Riemann integral with a compactly supported integrand. From this definition we see that elements of $\mathcal{A}_r^n$ are distributions. And, define $\mathcal{A}_c^n = \{f \in \mathcal{D}'(\mathbb{R}) \mid f = F^{(n)}$ for some $F \in \mathcal{B}_c\}$.

Theorem 2.1 (Uniqueness)**.** *For each $f \in \mathcal{A}_r^n$ there is a unique function $F \in \mathcal{B}_r$ such that $F^{(n)} = f$. For each $f \in \mathcal{A}_c^n$ there is a unique function $F \in \mathcal{B}_c$ such that $F^{(n)} = f$.*

Proof. Suppose $F^{(n)} = G^{(n)}$ for some $F, G \in \mathcal{B}_r$ then let $P = F - G$. Thus, $P \in \mathcal{B}_r$, $P^{(n)} = 0$ and P is a polynomial of degree at most $n-1$. This follows from the fact that the polynomials are Fourier transforms of linear combinations of the Dirac distribution and its derivatives [10, Exercise 9.25]. The only polynomial in $\mathcal{B}_r$ is 0. Similarly when $F, G \in \mathcal{B}_c$. $\square$

If $f \in \mathcal{A}_r^n$ we can then speak of the unique element $F \in \mathcal{B}_r$ such that $F^{(n)} = f$ as the *primitive* of f. Here it is essential that F be left continuous rather than just regulated. The mapping $\Phi\colon \mathcal{B}_r \to \mathcal{A}_r^n$ given by $\Phi(F) = F^{(n)}$ is a linear bijection. It is surjective by the definition of $\mathcal{A}_r^n$. It is injective by Theorem 2.1. It follows from Lemma 1.1 that $\mathcal{A}_r^n$ is a linear space. The norm inherited from $\mathcal{B}_r$ makes $\mathcal{A}_r^n$ into a Banach space. Similarly for $\mathcal{A}_c^n$. We call this the Alexiewicz norm and denote it $\|\cdot\|_{a,n}$. (See [1].) The Alexiewicz norm is translation invariant and we have continuity in norm in $\mathcal{A}_c^n$ but not in $\mathcal{A}_r^n$. And, $C^\infty(\mathbb{R})$ is dense in $\mathcal{A}_c^n$ but not in $\mathcal{A}_r^n$.

Theorem 2.2. *Let $n \in \mathbb{N}$ and let $f, f_1, f_2 \in \mathcal{A}_r^n$ with the respective primitives $F, F_1, F_2 \in \mathcal{B}_r$. Let $a_1, a_2 \in \mathbb{R}$. Let $\varphi \in \mathcal{D}(\mathbb{R})$.* (a) *With operations given by*

$$\begin{aligned}\langle a_1 f_1 + a_2 f_2, \varphi\rangle &= a_1\langle f_1, \varphi\rangle + a_2\langle f_2, \varphi\rangle \\ &= (-1)^n (a_1\langle F_1, \varphi^{(n)}\rangle + a_2\langle F_2, \varphi^{(n)}\rangle) \\ &= (-1)^n \Big(a_1 \int_{-\infty}^{\infty} F_1(x)\varphi^{(n)}(x)\,\mathrm{d}x + a_2 \int_{-\infty}^{\infty} F_2(x)\varphi^{(n)}(x)\,\mathrm{d}x\Big)\end{aligned}$$

$\mathcal{A}_r^n$ is a vector space. And, $\mathcal{A}_c^n$ is a subspace of $\mathcal{A}_r^n$. (b) *A norm on $\mathcal{A}_r^n$ is defined by $\|f\|_{a,n} = \|F\|_\infty$. This makes $\mathcal{A}_r^n$ into a Banach space that is not separable. Each of the spaces $\mathcal{A}_r^n$ is isometrically isomorphic to $\mathcal{B}_r$. Each of the spaces $\mathcal{A}_c^n$ is a separable Banach space isometrically isomorphic to $\mathcal{B}_c$.* (c) *Each distribution in $\mathcal{A}_r^n$ is tempered, of order at most n.* (d) *Let $\mathcal{O}\colon \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ be an operator that commutes with the derivative, $(\mathcal{O}T)' = \mathcal{O}T'$ for all $T \in \mathcal{D}'(\mathbb{R})$. Then $\langle \mathcal{O}f, \varphi\rangle = (-1)^n\langle \mathcal{O}F, \varphi\rangle$.* (e) *Let $t \in \mathbb{R}$ and let $T \in \mathcal{D}'(\mathbb{R})$. Then $T \in \mathcal{A}_r^n$ if and only if $\tau_t T \in \mathcal{A}_r^n$. Similarly for $\mathcal{A}_c^n$. If $f \in \mathcal{A}_r^n$ then $\|\tau_t f\|_{a,n} = \|f\|_{a,n}$.* (f) *For each $f \in \mathcal{A}_c^n$ it follows that $\lim\limits_{t\to 0}\|f - \tau_t f\|_{a,n} = 0$.* (g) *$C^\infty(\mathbb{R})$ is dense in $\mathcal{A}_c^n$ but not in $\mathcal{A}_r^n$.*

Proof. (a) and (b) These follow from Lemma 1.1.

(c) Let $K \subset \mathbb{R}$ be compact. Suppose $\varphi \in \mathcal{D}(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subset K$. Denote the Lebesgue measure of K by $|K|$. Let $f \in \mathcal{A}_r^n$ with primitive $F \in \mathcal{B}_r$. Then

$$|\langle f, \varphi\rangle| = \Big|\int_{-\infty}^{\infty} F(x)\varphi^{(n)}(x)\,\mathrm{d}x\Big| \leqslant \|F\|_\infty |K| \|\varphi^{(n)}\|_\infty.$$

This shows f is tempered and of order at most n. See [12] for the definition of tempered.

(d) It follows from associativity that $\mathcal{O}T^{(n)} = (\mathcal{O}T)^{(n)}$. Then $\langle \mathcal{O}f, \varphi\rangle = \langle \mathcal{O}F^{(n)}, \varphi\rangle = \langle (\mathcal{O}F)^{(n)}, \varphi\rangle = (-1)^n \langle \mathcal{O}F, \varphi^{(n)}\rangle$.

(e) Let $T \in \mathcal{D}'(\mathbb{R})$. Then $\langle \tau_t T', \varphi\rangle = \langle T', \tau_{-t}\varphi\rangle = -\langle T, (\tau_{-t}\varphi)'\rangle$. And, $\langle (\tau_t T)', \varphi\rangle = -\langle \tau_t T, \varphi'\rangle = -\langle T, \tau_{-t}\varphi'\rangle$. For $x \in \mathbb{R}$, $\tau_{-t}\varphi'(x) = (\tau_{-t}\varphi)'(x) = \partial\varphi(x+t)/\partial x$. It follows that τ_t commutes with derivatives. From the proof of (d), if $f \in \mathcal{A}_r^n$ then $\langle \tau_t f, \varphi\rangle = \langle \tau_t F^{(n)}, \varphi\rangle = \langle (\tau_t F)^{(n)}, \varphi\rangle$. Therefore, $\tau_t f$ is the nth derivative of $(\tau_t F) \in \mathcal{B}_r$, so $\tau_t f \in \mathcal{A}_r^n$. If $T \in \mathcal{D}'(\mathbb{R})$ such that $\tau_t T \in \mathcal{A}_r^n$ then write $T = \tau_{-t}(\tau_t T)$ to show $T \in \mathcal{A}_r^n$. Similarly for $\mathcal{A}_c^n$. For $f \in \mathcal{A}_r^n$, $\|\tau_t f\|_{a,n} = \|\tau_t F\|_\infty = \|F\|_\infty = \|f\|_{a,n}$.

(f) We have $\|f - \tau_t f\|_{a,n} = \|F^{(n)} - (\tau_t F)^{(n)}\|_{a,n} = \|F - \tau_t F\|_\infty \to 0$ as $t \to 0$.

(g) Let $\Phi_y(x) = (y/\pi)(x^2+y^2)^{-1}$ be the half plane Poisson kernel. Define the convolution $G_y = F * \Phi_y$. Since F is continuous on $\overline{\mathbb{R}}$, it is known that $\|G_y - F\|_\infty \to 0$ as $y \to 0^+$. See, for example, [4]. Note that $G_y \in C^\infty(\mathbb{R})$. By dominated convergence and the fact that $\int_{-\infty}^{\infty} \Phi_y(x)\,\mathrm{d}x = 1$ we see that $\lim_{x\to\infty} G_y(x) = F(\infty)$ and $\lim_{x\to-\infty} G_y(x) = F(-\infty) = 0$. Hence, $G_y \in \mathcal{B}_c$ for each $y > 0$. The density of $C^\infty(\mathbb{R})$ in $\mathcal{A}_c^n$ now follows. $\square$

Besides translations, other examples of operators commuting with the derivative are linear combinations of differential operators with coefficients independent of the differentiation variable.

It was shown in [27, Proposition 3.3] that L^1, and hence the space of Henstock-Kurzweil integrable functions, is dense in $\mathcal{A}_c^1$. Note that $C^\infty(\mathbb{R})$ is not dense in $\mathcal{A}_r^n$. The Heaviside step function H_0 is in $\mathcal{B}_r$. For each $\psi \in C(\mathbb{R})$ we have $\|H_0 - \psi\|_\infty \geqslant 1/2$. Therefore, $C^\infty(\mathbb{R})$ is not dense in $\mathcal{A}_r^n$. And, $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$ is not dense in $\mathcal{A}_c^n$. Define $F \in \mathcal{B}_c$ by $F(x) = \pi/2 + \arctan(x)$. Then for each $\varphi \in \mathcal{D}(\mathbb{R})$ we have $\|F - \varphi\|_\infty \geqslant \pi$.

We do not have continuity in norm in $\mathcal{A}_r^n$. For example, consider $\|H_0^{(n)} - \tau_t H_0^{(n)}\|_{a,n} = \|H_0 - \tau_t H_0\|_\infty = 1$ for all $t \neq 0$.

The *variation* of a function $g\colon \overline{\mathbb{R}} \to \mathbb{R}$ is the supremum of $\sum |g(x_i) - g(y_i)|$, taken over all disjoint intervals $(x_i, y_i) \subset \mathbb{R}$. This is denoted Vg. The functions of *bounded variation* are $\mathcal{B}V = \{g\colon \overline{\mathbb{R}} \to \mathbb{R} \mid Vg < \infty\}$. Functions of bounded variation are the difference of two increasing functions and thus are regulated on $\overline{\mathbb{R}}$. Under usual pointwise operations $\mathcal{B}V$ is a Banach space under the norm $\|g\|_{\mathcal{B}V} = \|g\|_\infty + Vg$. For each $-\infty \leqslant a \leqslant \infty$, an equivalent norm is $|g(a)| + Vg$. See, for example, [8], [10] and [15] for properties of $\mathcal{B}V$ functions.

The following spaces will serve as multipliers for $\mathcal{A}_r^n$. Each space $\mathcal{I}BV^n$ is defined inductively.

Definition 2.3. Define $\mathcal{IBV}^0 = \mathcal{BV}$. Suppose $\mathcal{IBV}^{n-1}$ is known for $n \in \mathbb{N}$. Define $\mathcal{IBV}^n = \{h\colon \mathbb{R} \to \mathbb{R} \mid h(x) = \int_0^x q(t)\,\mathrm{d}t \text{ for some } q \in \mathcal{IBV}^{n-1}\}$.

Hence, a function $h \in \mathcal{IBV}^n$ is an n-fold iterated integral.

Definition 2.4. Let $q \in L^1_{\mathrm{loc}}$. Define $I^0[q](x) = q(x)$. For $n \in \mathbb{N}$ define

$$I^n[q](x) = \int_{x_n=0}^{x} \cdots \int_{x_i=0}^{x_{i+1}} \cdots \int_{x_1=0}^{x_2} q(x_1)\,\mathrm{d}x_1 \ldots \mathrm{d}x_i \ldots \mathrm{d}x_n.$$

Proposition 2.5. *If $h \in \mathcal{IBV}^n$ for $n \in \mathbb{N}$ then there is a function $g \in \mathcal{BV}$ such that for all $x \in \mathbb{R}$,*

$$h(x) = I^n[g](x) = \frac{1}{(n-1)!}\int_0^x (x-s)^{n-1} g(s)\,\mathrm{d}s.$$

The proof follows from induction and the Fubini-Tonelli theorem.

The function g is not unique since there are functions of bounded variation that differ only on a countable set. Imposing a normalization on $\mathcal{BV}$ makes g unique. Fix $0 \leqslant \lambda \leqslant 1$. Functions g and g_λ differ on a set that is countable and $Vg_\lambda = \inf Vh$ where the infimum is taken over all $h \in \mathcal{BV}$ such that $g_\lambda = h$ almost everywhere. The value of $0 \leqslant \lambda \leqslant 1$ does not affect the value of Vg_λ. The functions of *normalized bounded variation* are then $\mathcal{NBV}_\lambda = \{g_\lambda \mid g \in \mathcal{BV}\}$. It is easy to see that for each $0 \leqslant \lambda \leqslant 1$ there is a unique $g \in \mathcal{NBV}_\lambda$ such that if $h \in \mathcal{IBV}^n$ then $h = I^n[g]$. Clearly, $\mathcal{NBV}_\lambda \subsetneq \mathcal{BV}$.

Note that if $g \in \mathcal{BV}$ then it is bounded, so the function $x \mapsto \int_0^x g(t)\,\mathrm{d}t$ is Lipschitz continuous and $I^n[g] \in C^{n-1}(\mathbb{R})$. The same applies if $g \in \mathcal{EBV}$ (see below). If $h \in \mathcal{IBV}^n$ then $h^{(m)}(0) = 0$ for all $0 \leqslant m \leqslant n-1$ and $h(x) = O(x^n)$ as $|x| \to \infty$.

Now we can define integrals on $\mathcal{A}_r^n$. A distribution $f \in \mathcal{A}_r^1$ is the distributional derivative of a unique function $F \in \mathcal{B}_r$. Its *regulated primitive integrals* are

$$\int_{(a,b)} f = \int_{a+}^{b-} f = F(b-) - F(a+) = F(b) - F(a+), \tag{2.1}$$

$$\int_{(a,b]} f = \int_{a+}^{b+} f = F(b+) - F(a+), \tag{2.2}$$

$$\int_{[a,b)} f = \int_{a-}^{b-} f = F(b-) - F(a-) = F(b) - F(a), \tag{2.3}$$

$$\int_{[a,b]} f = \int_{a-}^{b+} f = F(b+) - F(a-) = F(b+) - F(a) \tag{2.4}$$

for all $-\infty < a < b < \infty$. We also have $\int_{-\infty}^{\infty} f = F(\infty)$ with similar definitions for semi-infinite intervals. And, $\int_{\{a\}} f = F(a+) - F(a-)$. This integral was described in detail in [28]. The multipliers are the functions of bounded variation. If $g \in \mathcal{BV}$ then

$$\int_{-\infty}^{\infty} fg = \int_{-\infty}^{\infty} g(x)\,\mathrm{d}F(x) = F(\infty)g(\infty) - \int_{-\infty}^{\infty} F(x)\,\mathrm{d}g(x)$$
$$- \sum_{n\in\mathbb{N}} [F(c_n) - F(c_n+)]\,[g(c_n) - g(c_n+)]\,.$$

The sum is over all $c_n \in \mathbb{R}$ at which F and g are not both right continuous. The integrals $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}F(x)$ and $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}g(x)$ are Henstock-Stieltjes integrals. They are known to exist when one of F and g is regulated and one is of bounded variation. They are defined by using tagged partitions of $\overline{\mathbb{R}}$. If F is regulated but not required to be left continuous then an additional term containing coincident jump discontinuities of F and g from the left must be added. See [18, p. 199].

If $f \in \mathcal{A}_c^1$ then the four integrals (2.1)–(2.4) all give $\int_a^b f = F(b)-F(a)$ and the sum in (2.5) vanishes. In this case, $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}g(x)$ is a Riemann-Stieltjes integral over an unbounded domain. It can also be defined by taking limits of Riemann-Stieltjes integrals over finite subintervals. See [18, p. 187] and [25] for details.

Definition 2.6. Let $n \in \mathbb{N}$ and let $0 \leqslant \lambda \leqslant 1$. For $f \in \mathcal{A}_r^n$ let F be its primitive in $\mathcal{B}_r$. For $h \in \mathcal{IBV}^{n-1}$ such that $h = I^{n-1}[g]$ for $g \in \mathcal{N}BV_\lambda$, define the regulated primitive integral of f with respect to h as

$$\int_{-\infty}^{\infty} fh = \int_{-\infty}^{\infty} F^{(n)}h = (-1)^{n-1}\int_{-\infty}^{\infty} F'h^{(n-1)} \tag{2.6}$$

$$= (-1)^{n-1}F(\infty)g(\infty) - (-1)^{n-1}\int_{-\infty}^{\infty} F(x)\,\mathrm{d}g(x) \tag{2.7}$$

$$- (-1)^{n-1}\sum_{n\in\mathbb{N}} [F(c_n) - F(c_n+)]\,[g(c_n) - g(c_n+)] \tag{2.8}$$

$$= (-1)^{n-1}\int_{-\infty}^{\infty} h^{(n-1)}(x)\,\mathrm{d}F(x). \tag{2.8}$$

If $f \in \mathcal{A}_c^n$, or if $f \in \mathcal{A}_r^n$ and $g \in \mathcal{N}\mathrm{BV}_1$ (i.e. right continuous), then the sum in (2.8) vanishes.

To distinguish them from the regulated primitive integral, we will always explicitly show the integration variable and differential in Lebesgue, Henstock-Stieltjes and Riemann integrals. It is shown in [26] (following Definition 6) that if $g_1, g_2 \in \mathcal{BV}$ differ on a countable set and $F \in C(\overline{\mathbb{R}})$ then $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}g_1(x) = \int_{-\infty}^{\infty} F(x)\,\mathrm{d}g_2(x)$. Hence, if $f \in \mathcal{A}_c^n$ it makes no difference in Definition 2.6 if we use $g \in \mathcal{N}BV_\lambda$ or any function of bounded variation that differs from g on a countable set.

Proposition 2.7. *Let $n \in \mathbb{N}$. For $f \in \mathcal{A}_c^n$ let F be its primitive in $\mathcal{B}_c$. Let $h \in \mathcal{IBV}^{n-1}$ and let $g \in \mathcal{B}V$ such that $h = I^{n-1}[g]$. Then $\int_{-\infty}^{\infty} fh = (-1)^{n-1}F(\infty)g(\infty) - (-1)^{n-1}\int_{-\infty}^{\infty} F(x)\,\mathrm{d}g(x)$.*

The integral in (2.7) takes different forms when g is the primitive for different types of integrals. The set of primitives for L^1 functions is $AC(\overline{\mathbb{R}}) := AC(\mathbb{R}) \cap \mathcal{B}V$, where $AC(\mathbb{R})$ are the functions that are absolutely continuous on each compact set in $\mathbb{R}$. If f is a measurable function on the real line then $f \in L^1$ if and only if there is a function $F \in AC(\overline{\mathbb{R}})$ such that $f(x) = F'(x)$ for almost all $x \in \mathbb{R}$.

Proposition 2.8. *Let $F \in \mathcal{B}_r$.* (a) *If $g \in AC(\overline{\mathbb{R}})$ then $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}g(x) = \int_{-\infty}^{\infty} F(x)g'(x)\,\mathrm{d}x$. This last is a Lebesgue integral.* (b) *If $g \in C^1(\mathbb{R}) \cap C(\overline{\mathbb{R}})$ then $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}g(x) = \int_{-\infty}^{\infty} F(x)g'(x)\,\mathrm{d}x$. This last is an improper Riemann integral.*

This follows from the form the fundamental theorem of calculus takes for each integral. For example, see [16, p. 74]. The primitives for the Riemann integral are the functions of bounded slope variation. See [29]. This set of primitives properly contains the Lipschitz functions and is a proper subset of $C^1(\mathbb{R})$.

Let $g\colon \mathbb{R} \to \mathbb{R}$. Then g is of *essential bounded variation* if its distributional derivative is a signed Radon measure, i.e., there is a unique signed Radon measure μ_g such that $\langle g', \varphi\rangle = -\langle g, \varphi'\rangle = -\int_{-\infty}^{\infty} g(x)\varphi'(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}\mu_g(x)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. Denote the functions of essential bounded variation by $\mathcal{E}BV$. If $g \in \mathcal{E}BV$ then its *essential variation* is $EVg = \sup\limits_{\varphi} \int_{-\infty}^{\infty} g(x)\varphi'(x)\,\mathrm{d}x$, the supremum being taken over all functions $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\|\varphi\|_\infty \leqslant 1$. Changing a function on a set of measure zero does not affect its essential variation. The elements of $\mathcal{E}BV$ are equivalence classes of functions that are equal almost everywhere. For each $0 \leqslant \lambda \leqslant 1$ there is exactly one function from $\mathcal{N}BV_\lambda$ in each equivalence class. If $g \in \mathcal{E}BV$ then there is exactly one function $h \in \mathcal{N}BV_\lambda$ such that $EVg = Vh$. Hence, $\mathcal{E}BV$ and $\mathcal{N}BV_\lambda$ are isometrically isomorphic. The space $\mathcal{E}BV$ is a Banach space under the norm $\|g\|_{\mathcal{E}BV} = \operatorname{ess\,sup}|g| + EVg = \|h\|_{\mathcal{B}V}$. Also, $EVg = |\mu_g|(\mathbb{R}) = \inf\limits_h Vh$ where the infimum is taken over all $h \in \mathcal{B}V$ such that $g = h$ almost everywhere. It is shown in [27, Corollary 15] that $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}g(x) = \int_{-\infty}^{\infty} F(x)\,\mathrm{d}\mu_g(x)$ for all $F \in C(\overline{\mathbb{R}})$. A limiting process is used in [26, Theorem 8] to define $\int_{-\infty}^{\infty} F'g$ for $F \in \mathcal{B}_c$ and $g \in \mathcal{E}BV$. Hence, in Definition 2.6 we can use $g \in \mathcal{E}BV$ when $f \in \mathcal{A}_c^n$. All of the results in this paper can be rewritten under this assumption. If $h \in \mathcal{N}BV_\lambda$ is the unique function such that $EVg = Vh$ then we can define $g(\pm\infty) = h(\pm\infty)$. The limit does not depend on the choice of $0 \leqslant \lambda \leqslant 1$. For more on functions of essential bounded variation, see [3] and [31].

For each $n \in \mathbb{N}$, the operator $I^n\colon \mathcal{B}V \to \mathcal{IBV}^n$ is linear. By the definition of $\mathcal{IBV}^n$ it is surjective. Similarly if the domain is $\mathcal{N}BV_\lambda$ or $\mathcal{E}BV$. If $g \in \mathcal{B}V$ such that $I^n[g]=0$ then by the fundamental theorem of calculus, $\int_0^x g(t)\,\mathrm{d}t=0$ for each $x\in\mathbb{R}$. This does not imply $g=0$. For example, $g=\chi_{\{0\}}$. Hence, $I^n\colon \mathcal{B}V\to\mathcal{I}BV^n$ is not injective. However, I^n is a bijection when we use $\mathcal{N}BV_\lambda$ or $\mathcal{E}BV$.

Theorem 2.9. *Let $n\in\mathbb{N}$. Let $0\leqslant\lambda\leqslant 1$.* (a) *The sets $\mathcal{I}BV^n$ are equal if $\mathcal{I}BV^0$ is taken to be $\mathcal{B}V$, $\mathcal{N}BV_\lambda$ or $\mathcal{E}BV$.* (b) *The linear operator $I^n\colon \mathcal{N}BV_\lambda\to\mathcal{I}BV^n$ is a bijection and $\mathcal{I}BV^n$ is a Banach space with the norm $\|h\|_{\mathcal{I}BV^n}=\|(I^n)^{-1}h\|_\infty+V[(I^n)^{-1}h]$.* (c) *The linear operator $I^n\colon \mathcal{E}BV\to\mathcal{I}BV^n$ is a bijection and $\mathcal{I}BV^n$ is a Banach space with the norm $\|h\|_{\mathcal{I}BV^n}=\|h^{(n)}\|_{\mathcal{E}BV}$.*

Proof. (a) Note that $\mathcal{N}BV_\lambda\subsetneq\mathcal{B}V$. For each element $g\in\mathcal{B}V$ the function $g_\lambda\in\mathcal{N}BV_\lambda$ differs from g on a countable set. Hence, $I^n[g]=I^n[g_\lambda]$. For each $g\in\mathcal{E}BV$ there is exactly one function $h\in\mathcal{N}BV_\lambda$ such that $g=h$ almost everywhere. Then $I^n[g]=I^n[h]$.

(b) From (a) and Definition 2.4 the operator I^n is linear and surjective. If $I^n[g]=0$ for $g\in\mathcal{N}BV_\lambda$ then $\int_0^x g(t)\,\mathrm{d}t=0$ for all $x\in\mathbb{R}$. By the fundamental theorem of calculus, $g(x)=0$ at all points of continuity of g. Hence, $g=0$ except on a countable set. But g has a left limit and a right limit at each point. Suppose there is $a\in\mathbb{R}$ such that $g(a+)=\alpha>0$. Then there is $\delta>0$ such that $g(x)\geqslant\alpha/2$ for all $x\in(a,a+\delta)$. This contradicts the fact that g vanishes except perhaps on a countable set. Similarly if $\alpha<0$ and similarly with the left limit. Hence, $g(x+)=g(x-)=0$ at all $x\in\mathbb{R}$. It follows that $g=0$ on $\mathbb{R}$. Hence, $I^n\colon \mathcal{N}BV_\lambda\to\mathcal{I}BV^n$ is a bijection. If $h\in\mathcal{I}BV^n$ then $h=I[g]$ for a unique function $g\in\mathcal{N}BV_\lambda$. The pointwise derivative $h^{(n)}(x)=g(x)$ at all points of continuity of g. In general, we cannot recover $g(x)$ for all $x\in\mathbb{R}$ with the nth order pointwise derivative. To compute the inverse of the operator I^n, let S be the set of points in $\mathbb{R}$ at which g is not continuous. Then S is countable and $h^{(n)}(x)=g(x)$ for all $x\notin S$. Suppose $a\notin S$. The limits

$$\lim_{\substack{x\to a^+\\ x\notin S}} g(x)=g(a+)\quad\text{and}\quad\lim_{\substack{x\to a^-\\ x\notin S}} g(x)=g(a-)$$

both exist. Since $g\in\mathcal{N}BV_\lambda$ we have $g(a)=(1-\lambda)g(a-)+\lambda g(a+)$. This then defines $(I^n)^{-1}$. By Lemma 1.1, $\mathcal{I}BV^n$ is a Banach space with the norm $\|h\|_{\mathcal{I}BV^n}=\|(I^n)^{-1}h\|_\infty+V[(I^n)^{-1}h]$.

(c) If $g\in\mathcal{E}BV$ such that $I^n[g]=0$ then by the fundamental theorem of calculus, $\int_0^x g(t)\,\mathrm{d}t=0$ for each $x\in\mathbb{R}$. Hence, $g=0$ almost everywhere. But then $\langle g,\varphi'\rangle=0$ for all $\varphi\in\mathcal{D}(\mathbb{R})$. Hence, $\mu_g=0$ and $g=0$ as an element of $\mathcal{E}BV$. Then $I^n\colon \mathcal{E}BV\to\mathcal{I}BV^n$ is a bijection. If $h\in\mathcal{I}BV^n$ then the pointwise derivative $h^{(n)}(x)$ exists almost everywhere and defines a function in $\mathcal{E}BV$. By Lemma 1.1, $\mathcal{I}BV^n$ is a Banach space with the norm $\|h\|_{\mathcal{I}BV^n}=\|h^{(n)}\|_{\mathcal{E}BV}$. $\square$

If $h \in \mathcal{IBV}^n$ and $h = I^n[g]$ for $g \in \mathcal{BV}$ then $h = I^n[g_\lambda]$ for each $\lambda \in \mathbb{R}$. Thus the normalization on functions of bounded variation does not affect the multiplier h. If $g \in C(\overline{\mathbb{R}}) \cap \mathcal{BV}$ then all normalizations g_λ equal g and the integral $\int_{-\infty}^{\infty} fg$ is independent of λ. However, if g is not continuous, different values of λ may give different values for this integral. An example is given in the next section.

In the definition of $F \in \mathcal{B}_r$ the condition $\lim_{x\to-\infty} F(x) = 0$ is imposed. This is arbitrary but convenient because it makes primitives unique. If we merely require $\lim_{x\to-\infty} F(x)$ to exist in $\mathbb{R}$ then formula (2.7) must be modified by the addition of the term $(-1)^n F(-\infty)g(-\infty)$. Since an element of $\mathcal{A}_r^n$ is the nth order derivative of a function in $\mathcal{B}_r$, adding a polynomial of degree at most $n-1$ does not affect $\int_{-\infty}^{\infty} F^{(n)}h$ or $\mathcal{A}_r^n$. The norm on $\mathcal{B}_r$ could then be modified to a difference formula. Define $J_n[F](h;x) = \sum_{i=0}^{n} (-1)^n \binom{n}{i} F(x+ih)$. Then $J_n[P] = 0$ if P is a polynomial of degree at most $n-1$. The norm on $\mathcal{B}_r$ could then be replaced by $\sup_{x,h\in\mathbb{R}} |J_n[F](h;x)|$. For example, if $C(\overline{\mathbb{R}})$ is used instead of $\mathcal{B}_c$ then use the norm $\sup_{x,h\in\mathbb{R}} |F(x) - F(x+h)|$.

In the definition of $h \in \mathcal{IBV}^n$ we have arbitrarily imposed the condition $h^{(m)}(0) = 0$ for all $0 \leqslant m \leqslant n-1$. This does not affect the integral $\int_{-\infty}^{\infty} fh$ if we use (2.7) to define the integral. Different lower limits of integration in Definition 2.4 would change h by the addition of a polynomial of degree at most $n-1$. Addition of such a polynomial also does not affect the norm $\|\cdot\|_{\mathcal{IBV}^n}$.

Proposition 2.10. (a) *Let $n \geqslant 2$. Let $f \in \mathcal{A}_r^n$ such that $f = F^{(n)}$ for $F \in \mathcal{B}_r$. For each polynomial P of degree at most $n-2$, define $\int_{-\infty}^{\infty} fP = (-1)^{n-1} \int_{-\infty}^{\infty} F' P^{(n-1)}$. Then $\int_{-\infty}^{\infty} fP = 0$. In particular, if $P_k(x) = x^k$ then each of the moments $\int_{-\infty}^{\infty} fP_k = 0$ for all integers $0 \leqslant k \leqslant n-2$.* (b) *Let $g \in \mathcal{BV}$. Let $a_1, a_2, \ldots, a_{n-1}$ be real numbers. Define $h(x) = \int_{x_n=a_{n-1}}^{x} \ldots \int_{x_i=a_i}^{x_{i+1}} \ldots \int_{x_1=a_1}^{x_2} g(x_1)\,\mathrm{d}x_1 \ldots \mathrm{d}x_i \ldots \mathrm{d}x_n$. With $f \in \mathcal{A}_r^n$ and $F \in \mathcal{B}_r$ such that $f = F^{(n)}$, define $\int_{-\infty}^{\infty} fh = (-1)^{n-1} \int_{-\infty}^{\infty} F' h^{(n-1)}$. Then $\int_{-\infty}^{\infty} fh = \int_{-\infty}^{\infty} f I^{n-1}[g]$.* (c) *Let $n \geqslant 1$. Let $h \in \mathcal{IBV}^n$ and let P be a polynomial of degree at most $n-1$. Then $\|h+P\|_{\mathcal{IBV}^n} = \|h\|_{\mathcal{IBV}^n}$.*

Proof. (a) Let P be a polynomial of degree not exceeding $n-2$. Then $\int_{-\infty}^{\infty} fP = (-1)^{n-1} \int_{-\infty}^{\infty} F' P^{(n-1)} = 0$ since $P^{(n-1)}(x) = 0$ for all $x \in \mathbb{R}$. (b) The difference between h and $I^{n-1}[g]$ is a polynomial of degree at most $n-2$. (c) It follows from the proof of Theorem 2.9 that $(I^n)^{-1}P = 0$. □

If φ is a test function then $\varphi \in \mathcal{IBV}^n$ for each $n \geqslant 0$. For each $f \in \mathcal{A}_r^n$ with primitive $F \in \mathcal{B}_r$ the distributional derivative formula $\langle f, \varphi \rangle = \langle F^{(n)}, \varphi \rangle = (-1)^n \int_{-\infty}^{\infty} F(x)\varphi^{(n)}(x)\,\mathrm{d}x$ agrees with the definition of the integral in (2.6). If $F, h \in C^{n-1}(\mathbb{R})$ such that $F^{(n-1)}, h^{(n-1)} \in AC(\overline{\mathbb{R}})$ then the integration by parts formula is

$$\int_{-\infty}^{\infty} F^{(n)}(x)h(x)\,\mathrm{d}x = (-1)^n \int_{-\infty}^{\infty} F(x)h^{(n)}(x)\,\mathrm{d}x$$
$$+\sum_{k=0}^{n-1}(-1)^{n-k-1}[F^{(k)}(\infty)h^{(n-k-1)}(\infty) - F^{(k)}(-\infty)h^{(n-k-1)}(-\infty)].$$

When $h \in C_c^\infty(\mathbb{R})$ the limits $F^{(k)}(\pm\infty)h^{(n-k-1)}(\pm\infty)$ vanish for each $0 \leqslant k \leqslant n-1$ and each F. In the case of $F \in \mathcal{D}'(\mathbb{R})$ these limits are ignored in the formula for the distributional derivative D^nF, even though $F^{(k)}$ need not have any pointwise meaning. Similarly in the definition of the integral (2.7). We now show that when F and h have pointwise derivatives as above, these limits $F^{(k)}(\pm\infty)h^{(n-k-1)}(\pm\infty)$ vanish, provided $F^{(k)}$ has a monotonicity property. Suppose $h \in \mathcal{I}BV^{n-1}$. Then there is $g \in \mathcal{B}V$ such that $h = I^{n-1}[g]$. Hence,

$$|h^{(n-k-1)}(x)| = \left|\int_{x_k=0}^{x} \cdots \int_{x_1=0}^{x_2} g(x_1)\,\mathrm{d}x_1 \ldots \mathrm{d}x_k\right| \leqslant \frac{\|g\|_\infty |x|^k}{k!}$$

so $h^{(n-k-1)}(x) = O(x^k)$ as $x \to \infty$. This growth condition is sharp; take g to be constant. Suppose $F \in \mathcal{B}_c$ such that $F \in C^{k-1}(\mathbb{R})$, $F^{(k-1)} \in AC(\overline{\mathbb{R}})$ and there is $M > 0$ such that $F^{(k)}(x) > 0$ and $F^{(k)}(x)$ is decreasing for almost all $x > M$. Then F is given by the iterated integral

$$F(x) = \int_{x_k=-\infty}^{x} \cdots \int_{x_1=-\infty}^{x_2} F^{(k)}(x_1)\,\mathrm{d}x_1 \ldots \mathrm{d}x_k.$$

Now consider $I(x) := \int_{x_k=x/2}^{x} \cdots \int_{x_1=x/2}^{x_2} F^{(k)}(x_1)\,\mathrm{d}x_1 \ldots \mathrm{d}x_k$. For x large enough we have $I(x) \geqslant F^{(k)}(x)x^k/(2^k k!)$. Since $\lim_{x\to\infty} I(x) = 0$ it follows that $F^{(k)}(x) = o(x^{-k})$ as $x \to \infty$. But then $\lim_{x\to\infty} F^{(k)}(x)h^{(n-k-1)}(x) = 0$. If $F^{(k)}(x)$ is increasing, instead of $I(x)$ integrate over the interval $[x, 2x]^k \subset \mathbb{R}^k$. Similarly with limits as $x \to -\infty$.

3. Examples and properties of the integral

The space $\mathcal{A}_c^1$ consists of the derivatives of functions in $\mathcal{B}_c$. Hence, it contains all functions integrable in the Lebesgue, Henstock-Kurzweil and wide Denjoy sense over $\mathbb{R}$. For each interval $I \subset \overline{\mathbb{R}}$ the characteristic function χ_I is of bounded variation. So if $f \in \mathcal{A}_c^1$ with primitive $F \in \mathcal{B}_c$ then $\int_a^b f = \int_{-\infty}^{\infty} f\chi_I = F(b) - F(a)$. The same formula holds for integration over $I = [a,b], [a,b), (a,b]$ and (a,b). Similarly, we can integrate over all semi-infinite intervals. If F is continuous on $\overline{\mathbb{R}}$ but has a pointwise derivative nowhere then $F' \in \mathcal{A}_c^1$ and $\int_a^b F' = F(b) - F(a)$ for all $-\infty \leqslant a < b \leqslant \infty$. If $F \in \mathcal{B}_c$ is a continuous singular function, $F'(x) = 0$ for almost all $x \in \mathbb{R}$, then the Lebesgue integral $\int_a^b F'(x)\,\mathrm{d}x = 0$ but $\int_a^b F' = F(b) - F(a)$. Other examples of integration in $\mathcal{A}_c^1$ are given in [26].

The Schwartz space, $\mathcal{S}$, of rapidly decreasing test functions, consists of the functions $\psi \in C^\infty(\mathbb{R})$ such that for each $m, n \geqslant 0$ we have $x^m \psi^{(n)}(x) \to 0$ as $|x| \to \infty$. Let $\psi \in \mathcal{S}$. Then $\psi^{(m)} \in \mathcal{B}_c$ for each $m \geqslant 0$. Define $\Psi(x) = \int_{-\infty}^x \psi(t)\,\mathrm{d}t$. Then $\Psi^{(m)} \in \mathcal{B}_c$ for each $m \geqslant 0$ and $\Psi'(x) = \psi(x)$ for each $x \in \mathbb{R}$. For each $1 \leqslant n \leqslant m+1$, $\psi^{(m)} \in \mathcal{A}_c^n$. An example of a function in $\mathcal{S}$ is $\psi(x) = \exp(-x^2)$. If we take $F_1(x) = \exp(-x^2)$ and $F_2(x) = \int_{-\infty}^x \exp(-t^2)\,\mathrm{d}t$ then $F_1, F_2 \in \mathcal{B}_c$ and if $f(x) = F_1'(x) = F_2''(x) = -2x\exp(-x^2)$ then $f \in \mathcal{A}_c^1 \cap \mathcal{A}_c^2$. Note that $\|f\|_{a,1} = \|F_1\|_\infty = 1$ while $\|f\|_{a,2} = \int_{-\infty}^\infty \exp(-t^2)\,\mathrm{d}t = \sqrt{\pi}$, so a distribution can have different norms in different spaces $\mathcal{A}_c^n$.

The space $\mathcal{A}_r^1$ consists of the distributional derivative of regulated functions. Clearly, $\mathcal{A}_c^1 \subsetneq \mathcal{A}_r^1$. The Dirac distribution is $\delta = H_0'$. Hence, $\delta \in \mathcal{A}_r^1$. According to (2.5), for each $g \in \mathcal{B}V$ we have

$$\begin{aligned}\int_{-\infty}^\infty \delta g &= H_0(\infty)g(\infty) - \int_{-\infty}^\infty H_0(x)\,\mathrm{d}g(x) - [H_0(0) - H_0(0+)][g(0) - g(0+)]\\ &= g(\infty) - [H_0(\infty)g(\infty) - H_0(0+)g(0+)] - [0-1][g(0) - g(0+)]\\ &= g(0).\end{aligned}$$

The Henstock-Stieltjes integral $\int_{-\infty}^\infty H_0(x)\,\mathrm{d}g(x)$ can be evaluated using a tagged partition of $\overline{\mathbb{R}}$ that forces 0 and ∞ to be tags. This agrees with the action of δ as a tempered distribution, for which g must be in the Schwartz space $\mathcal{S}$. Notice that changing the value of $H_0(0)$ does not affect the value of $\int_{-\infty}^\infty \delta g$. When δ acts as a measure, this equation is written $\int_{-\infty}^\infty g(x)\,\mathrm{d}\delta(x) = g(0)$ and holds for all functions $g\colon \mathbb{R} \to \mathbb{R}$. Similarly, every signed Radon measure is in $\mathcal{A}_r^1$.

Note that changing a function of bounded variation at one point can affect the value of $\int_{-\infty}^\infty fg$. For example, let $F \in \mathcal{B}_r$ and $g = a\chi_{\{0\}}$. Then $\int_{-\infty}^\infty F'g = a[F(0+) - F(0-)]$. See [28] for more examples of integration in $\mathcal{A}_r^1$.

Using (2.9), an example in $\mathcal{A}_r^n$ is

$$\begin{aligned}\int_{-\infty}^\infty \delta^{(n-1)} I^{n-1}[H_\lambda] = \int_{-\infty}^\infty H_0^{(n)} I^{n-1}[H_\lambda] &= (-1)^{n-1}\int_{-\infty}^\infty H_\lambda(x)\,\mathrm{d}H_0(x)\\ &= (-1)^{n-1} H_\lambda(0)[H_0(0+) - H_0(0-)]\\ &= (-1)^{n-1}\lambda.\end{aligned}$$

Hence, the choice of λ affects the value of the integral.

Proposition 3.1. *Let $m \geqslant 0$.* (a) *Let $n \geqslant 1$. Then $\delta^{(m)} \in \mathcal{A}_r^n$ if and only if $n = m+1$.* (b) *For no $n \geqslant 1$ we have $\delta^{(m)} \in \mathcal{A}_c^n$.* (c) $\|\delta^{(m)}\|_{a,m+1} = 1$. (d) *Let μ be a finite signed Borel measure. Then $D^m\mu \in \mathcal{A}_r^{m+1}$.*

Proof. (a) We have $\delta^{(m)} = D^{m+1}H_0$, so $\delta^{(m)} \in \mathcal{A}_r^{m+1}$. Suppose $\delta^{(m)} = F^{(n)}$ for some $n \geqslant m+2$. Then $F^{(n-m)} = \delta + P$ where P is a polynomial of degree at most $m-1$ and the equality is in $\mathcal{A}_r^1$. If $m = 0$ then $P = 0$. Integrating $n-m$ times over the interval $[0,x]$ gives $F(x) = x^{n-m-1}\chi_{(0,\infty]}(x)/(n-m-1)! + Q(x)$ where Q is a polynomial of degree at most $n-1$. But then $F \notin \mathcal{B}_r$. If $n \leqslant m$ then $F = \delta^{(m-n)} + P$ where P is a polynomial of degree at most $n-1$. Comparing supports shows this impossible for all $F \in \mathcal{B}_r$. (b) Part (a) includes the proof. (c) Notice that $\|\delta^{(m)}\|_{a,m+1} = \|H_0\|_\infty = 1$ for each $m \geqslant 0$. (d) Define $F(x) = \int_{(-\infty,x)} \mathrm{d}\mu$. Then $F \in \mathcal{N}\mathrm{BV}_0 \subset \mathcal{B}_r$ and $F' = \mu$. $\square$

Observe that (2.6) gives $\int_{-\infty}^{\infty} \delta^{(m)} I^m[g] = (-1)^m g(0)$ for each $g \in \mathcal{N}BV_\lambda$ and $m \geqslant 0$. Let $K(x) = x\chi_{[0,\infty]}(x)$. Then $\delta^{(m)} = D^{m+2}K$ for each $m \geqslant 0$. Although K is continuous, it is not in $\mathcal{B}_c$. However, if we let $F_1(x) = 0$ for $x \leqslant 0$, $F_1(x) = x$ for $0 \leqslant x \leqslant 1$, $F_1(x) = 1$ for $x \geqslant 1$ then $F_1 \in \mathcal{B}_c$. Let $F_2(x) = H_0(x) - H_0(x-1)$. Then $F_2 \in \mathcal{B}_r \setminus \mathcal{B}_c$. Let $f = F_1'' \in \mathcal{A}_c^2$ then $f = F_2' \in \mathcal{A}_r^1$ and $f = \delta - \tau_1\delta$. So a linear combination of elements from $\mathcal{A}_r^1$ is in $\mathcal{A}_c^2$.

Proposition 3.2. (a) *For each $1 \leqslant m < n$, $\mathcal{A}_c^n$ is not a subset of $\mathcal{A}_c^m$ and $\mathcal{A}_c^m$ is not a subset of $\mathcal{A}_c^n$.* (b) *$\mathcal{A}_c^m \subset \mathcal{A}_r^n$ if and only if $m = n$.* (c) *For each $1 \leqslant m < n$, $\mathcal{A}_r^n$ is not a subset of $\mathcal{A}_r^m$ and $\mathcal{A}_r^m$ is not a subset of $\mathcal{A}_r^n$.* (d) *For no $m, n \in \mathbb{N}$ we have $\mathcal{A}_r^m \subset \mathcal{A}_c^n$.* (e) *For each $m, n \in \mathbb{N}$, $\mathcal{A}_c^m \cap \mathcal{A}_c^n \neq \emptyset$. Hence, $\mathcal{A}_r^m \cap \mathcal{A}_r^n \neq \emptyset$.*

Proof. (a) There is an increasing function $F \in \mathcal{B}_c \cap C^\infty(\mathbb{R})$ such that $F = 0$ on $(-\infty, 0]$ and $F = 1$ on $[1,\infty)$. Let $f = F^{(m)} \in \mathcal{A}_c^m$. Suppose $f \in \mathcal{A}_c^n$. The only function $G \in \mathcal{B}_c$ that satisfies $G^{(n)} = f$ is given by the iterated improper Riemann integrals

$$\begin{aligned}
G(x) &= \int_{x_n=-\infty}^{x} \dots \int_{x_i=-\infty}^{x_{i+1}} \dots \int_{x_1=-\infty}^{x_2} F^{(m)}(x_1)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \\
&= \int_{x_n=-\infty}^{x} \dots \int_{x_i=-\infty}^{x_{i+1}} \dots \int_{x_{m+1}=-\infty}^{x_{m+2}} F(x_{m+1})\,\mathrm{d}x_{m+1} \dots \mathrm{d}x_n \\
&\geqslant \int_{x_n=1}^{x} \dots \int_{x_i=1}^{x_{i+1}} \dots \int_{x_{m+1}=1}^{x_{m+2}} \mathrm{d}x_{m+1} \dots \mathrm{d}x_n \qquad \text{if } x \geqslant 1 \\
&= \frac{(x-1)^{n-m}}{(n-m)!}.
\end{aligned}$$

Hence, $G \notin \mathcal{B}_c$ so $f \notin \mathcal{A}_c^n$.

Let $F(x)=0$ for $x\leqslant 0$, $F(x)=x$ for $0\leqslant x\leqslant 1$ and $F(x)=1$ for $x\geqslant 1$. Then $F\in\mathcal{B}_c$. Define $f=F^{(n)}\in\mathcal{A}_c^n$. The Heaviside step function is $H_0=\chi_{(0,\infty]}$. The Dirac distribution is $\delta=H_0'$. If $a\in\mathbb{R}$ we write $\delta_a=\tau_a\delta$ for the Dirac distribution supported at a. For $n\geqslant 2$ we have $f=F^{(n)}=\delta^{(n-2)}-\delta_1^{(n-2)}$. Suppose $f\in\mathcal{A}_c^m$ is given by $f=G^{(m)}$ for $G\in\mathcal{B}_c$. If $n\geqslant m+2$ then $G=\delta^{(n-m-2)}-\delta_1^{(n-m-2)}+P$ where P is a polynomial of degree at most $m-1$. If $n=m+1$ then $G(x)=H_0(x)-H_0(x-1)$. It follows that $G\notin\mathcal{B}_c$. Hence, $f\notin\mathcal{A}_c^m$.

(b) Since $\mathcal{B}_c\subset\mathcal{B}_r$ we have $\mathcal{A}_c^n\subset\mathcal{A}_r^n$. For the other part of the proof use examples as in part (a). Replace the second example in (a) by $F(x)=H_0(x)-H_0(x-1)$.

(c) By Proposition 3.1, $\delta^{(m-1)}\in\mathcal{A}_r^n$ if and only if $n=m$.

(d) By Proposition 3.1, $\delta^{(m-1)}\in\mathcal{A}_r^m$ but is not in any of the $\mathcal{A}_c^n$ spaces.

(e) See the example in the second paragraph of this section. □

Let $F(x)=H_0(x-1)(x-1)^\alpha e^{-(x-1)}$ where $\alpha>0$. Then $F\in\mathcal{B}_c$. Define $f\in\mathcal{A}_c^n$ by $f=F^{(n)}$. For all $x\neq 1$ the pointwise derivative gives $f(x)\sim\alpha(\alpha-1)\ldots(\alpha-n+1)(x-1)^{\alpha-n}$ as $x\to 1+$. Then f is singular at 1 such that $f\notin L^1_{\mathrm{loc}}$ if $\alpha\leqslant n-1$ and yet $\int_{-\infty}^{\infty} fh$ exists for each $h\in\mathcal{I}BV^n$.

In $\mathcal{A}_c^n$ there is a version of the second mean value theorem for integrals.

Theorem 3.3. *Let $F\in\mathcal{B}_c$. Let $h\in\mathcal{I}BV^{n-1}$ such that $h^{(n-1)}$ is a monotonic function. Then $\int_{-\infty}^{\infty}F^{(n)}h=(-1)^{n-1}[h^{(n-1)}(-\infty)\int_{-\infty}^{\xi}F'+h^{(n-1)}(\infty)\int_{\xi}^{\infty}F']$ for some $\xi\in\overline{\mathbb{R}}$.*

Proof. Integrate and use the mean value theorem for Riemann-Stieltjes integrals [18, §7.10]:

$$\begin{aligned}\int_{-\infty}^{\infty}F^{(n)}h&=(-1)^{n-1}\left[F(\infty)h^{(n-1)}(\infty)-\int_{-\infty}^{\infty}F(x)\,\mathrm{d}h^{(n-1)}(x)\right]\\&=(-1)^{n-1}\left[F(\infty)h^{(n-1)}(\infty)-F(\xi)\int_{-\infty}^{\infty}\mathrm{d}h^{(n-1)}(x)\right]\\&=(-1)^{n-1}\{F(\infty)h^{(n-1)}(\infty)-F(\xi)[h^{(n-1)}(\infty)-h^{(n-1)}(-\infty)]\}\\&=(-1)^{n-1}\{h^{(n-1)}(-\infty)F(\xi)+h^{(n-1)}(\infty)[F(\infty)-F(\xi)]\}.\end{aligned}$$

□

This proof is adapted from a similar theorem for the wide Denjoy integral in [6], where a proof of the Bonnet form of the second mean value theorem can also be found.

If $\psi\in C^\infty(\mathbb{R})$ is a bijection such that $\psi'>0$ on $\mathbb{R}$ then for any distribution $T\in\mathcal{D}'(\mathbb{R})$ the composition $T\circ\psi$ is defined by $\langle T\circ\psi,\varphi\rangle=\langle T,\varphi\circ\psi^{-1}/\psi'\circ\psi^{-1}\rangle$ for all $\varphi\in\mathcal{D}(\mathbb{R})$. In [26] a change of variables formula was proved in $\mathcal{A}_c^1$ when $\psi\in C(\mathbb{R})$, i.e., no monotonicity or pointwise differentiability is assumed. In [28] a change of variables formula was proved in $\mathcal{A}_r^1$ when ψ was piecewise monotonic. For $\mathcal{A}_r^n$ we have the simple case of composition with a linear function.

Theorem 3.4. *Let $\psi(x) = ax + b$ for $a, b \in \mathbb{R}$, $a \neq 0$. Let $F \in \mathcal{B}_r$. Then $(F \circ \psi)^{(n)} = a^n(F^{(n)} \circ \psi)$. Let $h \in \mathcal{IBV}^{n-1}$. Then $\int_{-\infty}^{\infty} F^{(n)}h = |a| \int_{-\infty}^{\infty} (F^{(n)} \circ \psi)(h \circ \psi)$.*

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then

$$\begin{aligned}\langle F^{(n)} \circ \psi, \varphi\rangle &= \operatorname{sgn}(a)\Big\langle F^{(n)}, \frac{\varphi \circ \psi^{-1}}{\psi' \circ \psi^{-1}}\Big\rangle \\ &= \frac{\operatorname{sgn}(a)(-1)^n}{a^{n+1}} \int_{-\infty}^{\infty} F(y)\varphi^{(n)} \circ \psi^{-1}(y)\,\mathrm{d}y \\ &= \frac{(-1)^n}{a^n} \int_{-\infty}^{\infty} F(ax+b)\varphi^{(n)}(x)\,\mathrm{d}x \\ &= a^{-n}\langle (F \circ \psi)^{(n)}, \varphi\rangle.\end{aligned}$$

This shows that $(F \circ \psi)^{(n)} = a^n(F^{(n)} \circ \psi)$. Note that $F \circ \psi \in \mathcal{B}_r$. Suppose $a > 0$. Then

$$\begin{aligned}a^n \int_{-\infty}^{\infty} (F^{(n)} \circ \psi)h \circ \psi &= \int_{-\infty}^{\infty} (F \circ \psi)^{(n)} h \circ \psi \\ &= (-1)^{n-1}a^n \int_{-\infty}^{\infty} h^{(n-1)}(ax+b)\,\mathrm{d}F(ax+b) \\ &= (-1)^{n-1}a^{n-1} \int_{-\infty}^{\infty} h^{(n-1)}(y)\,\mathrm{d}F(y) \\ &= a^{n-1} \int_{-\infty}^{\infty} F^{(n)}h.\end{aligned} \tag{3.1}$$

If $a < 0$ then there is a sign change in (3.1) upon change of variables. □

Define $r_x(y) = x - y$. Then for $F \in \mathcal{B}_r$ and $h \in \mathcal{IBV}^{n-1}$, Theorem 3.4 shows the equality of the two convolution integrals $\int_{-\infty}^{\infty} (F^{(n)} \circ r_x)h = \int_{-\infty}^{\infty} F^{(n)}(h \circ r_x)$.

If $f \in \mathcal{A}_c^n$ for $n \geqslant 2$ then in general $\int_a^b f$ does not exist. However, if $h \in \mathcal{IBV}^{n-1}$ has compact support and is in $C^{n-1}(\mathbb{R})$ then fh can be integrated over a subinterval. For example, let $a < b$. Define $h(x) = (x-a)^p(x-b)^q\chi_{[a,b]}(x)$ for $p, q \geqslant n-1$. There is a polynomial, P, of degree at most $n-2$ such that $h + P \in \mathcal{IBV}^{n-1} \cap C^{n-2}(\mathbb{R})$. It follows that $h + P = I^{n-1}[g]$ where

$$g(x) = \chi_{(a,b)}(x) \sum_{i=0}^{n-1} \binom{n-1}{i} (p-i+1)_i(q-n+i+2)_{n-i-1}(x-a)^{p-i}(x-b)^{q-n+i+1}.$$

The Pochhammer symbol is $(z)_m = z(z+1)\dots(z+m-1)$ for $m \in \mathbb{N}$ with $(z)_0 = 1$. The formula for g comes from the Leibniz rule for differentiating a product. Note that $g \in \mathcal{B}V$. The value of g at a and b is irrelevant. Let $f = F^{(n)}$ for $F \in \mathcal{B}_c$. Then since $g(a-) = g(b+) = 0$, Proposition 2.10 gives

$$\begin{aligned}\int_{-\infty}^{\infty} fh = (-1)^{n-1}\int_{-\infty}^{\infty} F'h^{(n-1)} &= (-1)^{n-1}\int_{-\infty}^{\infty} F'g\\ &= (-1)^n\biggl[F(a)g(a) - F(b)g(b) + \int_a^b F(x)\,\mathrm{d}g(x)\biggr].\end{aligned}$$

This defines $\int_a^b F^{(n)}g = (-1)^n[F(a)g(a) - F(b)g(b)] + (-1)^n\int_a^b F(x)\,\mathrm{d}g(x)$. If $p, q > n-1$ then $h \in C^{n-1}(\mathbb{R})$, g is continuous, $g(a) = g(b) = 0$ and $\int_{-\infty}^{\infty} fh = (-1)^n\int_a^b F(x)\,\mathrm{d}g(x)$. If $p, q > n$ then $g \in C^1(\mathbb{R})$ and from Proposition 2.8, $\int_{-\infty}^{\infty} fh = (-1)^n\int_a^b F(x)g'(x)\,\mathrm{d}x$. If $f \in \mathcal{A}_r^n$ then we must adopt a normalization on g. If $g \in \mathcal{N}BV_\lambda$ then

$$\begin{aligned}\int_{-\infty}^{\infty} fh = (-1)^n\{&F(a)[g_\lambda(a) - g_\lambda(a-)] + F(b)[g_\lambda(b+) - g_\lambda(b)]\\ &+ \int_a^b F(x)\,\mathrm{d}g_\lambda(x) + [F(a) - F(a+)][g_\lambda(a) - g_\lambda(a+)]\\ &+ [F(b) - F(b+)][g_\lambda(b) - g_\lambda(b+)]\}.\end{aligned}$$

There are functions in $\mathcal{I}BV^n$ that play the role of characteristic functions of intervals. These lead to a version of the fundamental theorem of calculus that is built in to the definition of the integral. This also gives an explicit formula for the inverse of the nth derivative operator.

Theorem 3.5. *Let $x \in \mathbb{R}$. Let $\lambda = 1$. Define $h(x,t) = (x-t)^{n-1}H_0(x-t) \times \bigl((n-1)!\bigr)^{-1}$. (a) $h(0,\cdot) = I^{n-1}[(-1)^{n-1}\chi_{[-\infty,0)}] \in \mathcal{I}BV^{n-1}$. (b) Let $F \in \mathcal{B}_r$. Then $\int_{-\infty}^{\infty} F^{(n)}h(x,\cdot) = F(x)$. (c) Let $f \in \mathcal{A}_r^n$ and define $G(x) = \int_{-\infty}^{\infty} fh(x,\cdot)$. Then $G^{(n)} = f$. (d) The operator $\Phi\colon \mathcal{A}_r^n \to \mathcal{B}_r$ defined by $\Phi[f](x) = \int_{-\infty}^{\infty} fh(x,\cdot)$ is a linear isometry and is the inverse of $D^n\colon \mathcal{B}_r \to \mathcal{A}_r^n$ given by $D^n[F] = F^{(n)}$. Similarly with $\Phi\colon \mathcal{A}_c^n \to \mathcal{B}_c$.*

Proof. (a) Let $g(x,t) = (-1)^{n-1}\chi_{[-\infty,x)}(t)$. Note that $g(x,\cdot) \in \mathcal{N}\mathrm{BV}_1$, $h(x,\cdot) \in C^{n-2}(\mathbb{R})$ and that $\partial^{n-1}h(x,t)/\partial t^{n-1} = g(x,t)$ for all $t \neq x$. Therefore, $h(0,\cdot) = I^{n-1}[g(0,\cdot)] \in \mathcal{I}BV^{n-1}$. (b) From (2.6) and Proposition 2.10, $\int_{-\infty}^{\infty} F^{(n)}h(x,\cdot) = (-1)^{n-1}\int_{-\infty}^{\infty} F'(-1)^{n-1}\chi_{[-\infty,x)} = \int_{[-\infty,x)} F' = F(x-) = F(x)$. (c) This follows from (b). $\square$

Comparing the result of Proposition 2.5 and Theorem 3.5(b), it is clear how to define the iterated integral of the nth derivative of functions in $\mathcal{B}_c$.

Definition 3.6. Let $F \in \mathcal{B}_c$. Define $\int_{x_n=-\infty}^{x} \cdots \int_{x_i=-\infty}^{x_{i+1}} \cdots \int_{x_1=\infty}^{x_2} F^{(n)} = F(x)$ for each $x \in \mathbb{R}$.

The definition can be justified as follows. The set of functions $\mathcal{B}_c^\infty$ is defined to be those functions $\psi \in C^\infty(\mathbb{R})$ for which there are real numbers $a < b$ and c such that $\psi = 0$ on $(-\infty, a]$ and $\psi = c$ on $[b, \infty)$. It is clear that $\mathcal{B}_c^\infty$ is dense in $\mathcal{B}_c$. If (ψ_k) is a Cauchy sequence in $\mathcal{B}_c^\infty$ then

$$\Big| \int_{x_n=-\infty}^{x} \cdots \int_{x_1=-\infty}^{x_2} \psi_k^{(n)}(x_1)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n - \int_{x_n=-\infty}^{x} \cdots \int_{x_1=-\infty}^{x_2} \psi_l^{(n)}(x_1)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n \Big| \\ = |\psi_k(x) - \psi_l(x)| \leqslant \|\psi_k - \psi_l\|_\infty.$$

Hence, the sequence of functions $\int_{x_n=-\infty}^{x} \cdots \int_{x_1=-\infty}^{x_2} \psi_k^{(n)}(x_1)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n$ has a limit in $\mathcal{B}_c$. We can define the limit to be $F(x)$ in the case when $F \in \mathcal{B}_c$ and $\lim\limits_{k\to\infty} \|F - \psi_k\|_\infty = 0$. It is easy to see that the value of the limit is independent of the choice of the sequence (ψ_k).

It is a classical result that the initial value problem: given $f \in C(\mathbb{R})$ such that $\int_{-\infty}^{0} |f(t)||t|^{n-1}\,\mathrm{d}t < \infty$, find $F \in C^n(\mathbb{R})$ such $F^{(n)}(x) = f(x)$ for all $x \in \mathbb{R}$, with initial condition $\lim\limits_{x\to-\infty} F^{(k)}(x) = 0$ for each $0 \leqslant k \leqslant n-1$, has the unique solution $F(x) = [1/(n-1)!] \int_{-\infty}^{x} f(t)(x-t)^{n-1}\,\mathrm{d}t$. By the Fubini-Tonelli theorem the solution can also be written as $F(x) = \int_{-\infty}^{x} \cdots \int_{-\infty}^{x_2} f(x_1)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n$. Hence, an alternative approach to the integral is to use Theorem 3.5(b) and Definition 3.6, rather than Definition 2.6.

4. Hölder inequality and dual space

One of the many useful properties of functions in an L^p space is the Hölder inequality. Distributions in $\mathcal{A}_r^n$ also satisfy a type of Hölder inequality. For each $f \in \mathcal{A}_r^n$ and each $h \in \mathcal{I}BV^{n-1}$ the integral Definition 2.6 provides a type of product.

Theorem 4.1 (Hölder inequality). *Let $f \in \mathcal{A}_r^n$ with primitive $F \in \mathcal{B}_r$. Let $h \in \mathcal{I}BV^{n-1}$ such that $h = I^{n-1}[g]$ for $g \in \mathcal{B}V$. Then $|\int_{-\infty}^{\infty} fh| \leqslant \|F\|_\infty \|g\|_{\mathcal{B}V} = \|f\|_{a,n} \|h\|_{\mathcal{I}BV^{n-1}}$.*

The proof follows from Definition 2.6. The case $n = 1$ was considered in [26] for $\mathcal{A}_c^1$ and in [28] for $\mathcal{A}_r^1$ where various other forms of this inequality can be found. The estimates do not depend on the choice of $0 \leqslant \lambda \leqslant 1$ since $\|g_\lambda\|_{\mathcal{B}V} \leqslant \|g\|_{\mathcal{B}V}$ for all $g \in \mathcal{B}V$. For $g \in \mathcal{E}BV$ and $f \in \mathcal{A}_c^n$ use $\|g\|_{\mathcal{E}BV}$.

An application of the Hölder inequality is the following convergence theorem.

Theorem 4.2. *Fix $n \in \mathbb{N}$. Fix $0 \leqslant \lambda \leqslant 1$. Let $f \in \mathcal{A}_r^n$ and for each $k \in \mathbb{N}$ let $f_k \in \mathcal{A}_r^n$ such that $\|f_k - f\|_{a,n} \to 0$. Let $h, h_k \in \mathcal{I}BV^{n-1}$ such that $\|h_k - h\|_{\mathcal{I}BV^{n-1}} \to 0$. Then $\lim_{k\to\infty} \int_{-\infty}^{\infty} f_k h_k = \int_{-\infty}^{\infty} fh$.*

Proof. The Hölder inequality gives

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f_k h_k - \int_{-\infty}^{\infty} fh\right| &\leqslant \left|\int_{-\infty}^{\infty} (f_k - f)h_k\right| + \left|\int_{-\infty}^{\infty} f(h_k - h)\right| \\ &\leqslant \|f_k - f\|_{a,n}\|h_k\|_{\mathcal{I}BV^{n-1}} + \|f\|_{a,n}\|h_k - h\|_{\mathcal{I}BV^{n-1}}.\end{aligned}$$

Since $\|h_k\|_{\mathcal{I}BV^{n-1}}$ is bounded, the result follows. □

The Hölder inequality shows that for each $f \in \mathcal{A}_c^n$ the integral is a continuous linear functional on $\mathcal{I}BV^{n-1}$ and that for each $h \in \mathcal{I}BV^{n-1}$ the integral is a continuous linear functional on $\mathcal{A}_c^n$. There is also an equivalent norm in terms of these functionals. Similarly for $\mathcal{A}_r^n$.

Theorem 4.3. (a) *Let $f \in \mathcal{A}_c^n$. Define $\Phi_f\colon \mathcal{I}BV^{n-1} \to \mathbb{R}$ by $\Phi_f(h) = \int_{-\infty}^{\infty} fh$. Then Φ_f is a continuous linear functional.* (b) *Let $h \in \mathcal{I}BV^{n-1}$. Define $\Psi_h\colon \mathcal{I}BV^{n-1} \to \mathbb{R}$ by $\Psi_h(f) = \int_{-\infty}^{\infty} fh$. Then Ψ_h is a continuous linear functional.* (c) *Let $f \in \mathcal{A}_c^n$. Define $\|f\|'_{a,n} = \sup_h \int_{-\infty}^{\infty} fh$ where the supremum is taken over all $h \in \mathcal{I}BV^{n-1}$ with $h = I^{n-1}[g]$ for $g \in \mathcal{B}V$ such that $\|g\|_\infty \leqslant 1$ and $Vg \leqslant 1$. Then $\|\cdot\|_{a,n}$ and $\|\cdot\|'_{a,n}$ are equivalent norms on $\mathcal{A}_c^n$.* (d) *Fix $0 \leqslant \lambda \leqslant 1$. Results analogous to* (a), (b) *and* (c) *hold for $\mathcal{A}_r^n$.*

Proof. Linearity in each argument follows from linearity of the derivatives defining f and the integrals defining h. If $f \in \mathcal{A}_c^n$ and $(h_k) \subset \mathcal{I}BV^{n-1}$ such that $\|h_k\|_{\mathcal{I}BV^{n-1}} \to 0$ as $k \to \infty$ then by Theorem 4.2, $\Phi_f(h_k) \to 0$. If $h \in \mathcal{I}BV^{n-1}$ and $(f_k) \subset \mathcal{A}_c^n$ such that $\|f_k\|_{a,n} \to 0$ as $k \to \infty$ then by Theorem 4.2, $\Psi_h(f_k) \to 0$. Part (c) follows from Theorem 29 in [26], which proves equivalence on $\mathcal{A}_c^1$. (d) See [28, Theorem 15] for equivalent norms in $\mathcal{A}_r^n$. □

It is a classical result that if $[a,b]$ is a compact interval then $C([a,b])^* = \mathcal{N}BV_\lambda$ and if L is an element of the dual space then there is a function $g \in \mathcal{N}BV_\lambda$ such that $L(F) = \int_a^b F(x)\,\mathrm{d}g(x)$ for all $F \in C([a,b])$. By the compactification, the same holds for $C(\overline{\mathbb{R}})$. The choice of $0 \leqslant \lambda \leqslant 1$ is immaterial. The distinction between $\mathcal{B}V$ and $\mathcal{N}BV_\lambda$ is sometimes ignored (including in [26] in the paragraphs preceding Theorem 8). The distinction is important since if F is continuous and $g = \chi_{\{0\}}$ then $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}g(x) = 0$. This function g is not 0 as an element of $\mathcal{B}V$ but its normalized version in $\mathcal{N}BV_\lambda$ is 0. It was shown in [26] that the dual space of $\mathcal{A}_c^1$ is $\mathcal{E}BV$. It then follows from Lemma 1.1 that the dual space of $\mathcal{A}_c^n$ is also $\mathcal{E}BV$. For each $0 \leqslant \lambda \leqslant 1$

we can choose to represent each element of $\mathcal{E}BV$ by a unique function in $\mathcal{N}BV_\lambda$. Hence, the dual space of $\mathcal{A}_c^n$ is $\mathcal{N}BV_\lambda$. Meanwhile, the dual space of $\mathcal{B}_r$ is $\mathcal{B}V$. See [28] for a discussion of this point.

Theorem 4.4. *For each $n \in \mathbb{N}$ the dual space of $\mathcal{A}_c^n$ is isometrically isomorphic to $\mathcal{E}BV$ and $\mathcal{N}BV_\lambda$. The dual space of $\mathcal{A}_r^n$ is isometrically isomorphic to $\mathcal{B}V$.*

By Theorem 2.9 and Lemma 1.1, the spaces $\mathcal{N}BV_\lambda$ and $\mathcal{I}BV^n$ are isometrically isomorphic. Hence, $\mathcal{A}_c^n \subset \mathcal{N}BV_\lambda^*$. Spaces of distributions are often defined as the duals of some topological vector spaces. No explicit description of the dual of $\mathcal{B}V$ or $\mathcal{N}BV_\lambda$ seems to be known. We are thus reluctant to define our space of integrable distributions as $\mathcal{N}BV_\lambda^*$ and instead have chosen the concrete description in terms of derivatives of functions in $\mathcal{B}_c$ and $\mathcal{B}_r$.

5. Banach lattice

The usual pointwise ordering makes $\mathcal{B}_r$ into a Banach lattice. Each of the spaces $\mathcal{A}_r^n$ inherits this Banach lattice structure. We will point out a few of the most basic lattice properties of $\mathcal{A}_r^n$ but leave a detailed study for later. A reference for this section is [2]. To keep the paper reasonably self contained we prove all results in this section ab initio, although some of them follow from more general lattice theorems.

If $\preceq$ is a binary operation on a set S then it is a *partial order* if for all $x, y, z \in S$ it is *reflexive* ($x \preceq x$), *antisymmetric* ($x \preceq y$ and $y \preceq x$ imply $x = y$) and *transitive* ($x \preceq y$ and $y \preceq z$ imply $x \preceq z$). If S is a Banach space with a norm $\|\cdot\|_S$ and $\preceq$ is a partial order on S then S is a *Banach lattice* if for all $x, y, z \in S$

(1) $x \vee y$ and $x \wedge y$ are in S. The *join* is $x \vee y = \sup\{x, y\} = w$ such that $x \preceq w$, $y \preceq w$ and if $x \preceq \tilde{w}$ and $y \preceq \tilde{w}$ then $w \preceq \tilde{w}$. The *meet* is $x \wedge y = \inf\{x, y\} = w$ such that $w \preceq x$, $w \preceq y$ and if $\tilde{w} \preceq x$ and $\tilde{w} \preceq y$ then $\tilde{w} \preceq w$.
(2) $x \preceq y$ implies $x + z \preceq y + z$.
(3) $x \preceq y$ implies $kx \preceq ky$ for all $k \in \mathbb{R}$ with $k \geqslant 0$.
(4) $|x| \preceq |y|$ implies $\|x\|_S \leqslant \|y\|_S$.

If $x \preceq y$ we write $y \succeq x$. We also define $|x| = x \vee (-x)$, $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. Then $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

The usual pointwise ordering, $F_1 \leqslant F_2$ if and only if $F_1(x) \leqslant F_2(x)$ for all $x \in \mathbb{R}$, is a partial order on $\mathcal{B}_r$. Since $\mathcal{B}_r$ is closed under the operations $(F_1 \vee F_2)(x) = \sup(F_1, F_2)(x) = \max(F_1(x), F_2(x))$ and $(F_1 \wedge F_2)(x) = \inf(F_1, F_2)(x) = \min(F_1(x), F_2(x))$, it is then a vector lattice (or a Riesz space). The inequality $\|F_1 F_2\|_\infty \leqslant \|F_1\|_\infty \|F_2\|_\infty$ shows $\mathcal{B}_r$ is also a Banach lattice. See [28]. Clearly, $\mathcal{B}_c$ is a sublattice. Notice that the ordering in $\mathcal{B}_r$ depends on our choice of using left continuous primitives.

A partial ordering in $\mathcal{A}_r^n$ is inherited from $\mathcal{B}_r$. If $f_1, f_2 \in \mathcal{A}_c^n$ with the respective primitives $F_1, F_2 \in \mathcal{B}_r$ then $f_1 \preceq f_2$ if and only if $F_1 \leqslant F_2$ in $\mathcal{B}_r$. By Lemma 1.1, $\mathcal{A}_r^n$ is a Banach lattice and $\mathcal{A}_c^n$ is a sublattice.

An element $e \geqslant 0$ such that for each $x \in S$ there is $\lambda > 0$ such that $|x| \leqslant \lambda e$ is an *order unit* for the lattice S. In the theorem below we show $\mathcal{B}_r$ and hence $\mathcal{A}_r^n$ do not have an order unit.

We have absolute integrability: if $f \in \mathcal{A}_r^n$ so is $|f|$. The nth derivative operator D^n commutes with $\vee$ and $\wedge$ and hence with $|\cdot|$.

Theorem 5.1 (Banach lattice). (a) *$\mathcal{B}_r$ is a Banach lattice and $\mathcal{B}_c$ is a Banach sublattice.* (b) *For $f_1, f_2 \in \mathcal{A}_r^n$ with the respective primitives $F_1, F_2 \in \mathcal{B}_r$, define $f_1 \preceq f_2$ if $F_1 \leqslant F_2$ in $\mathcal{B}_r$. Then $\mathcal{A}_r^n$ is a Banach lattice isomorphic to $\mathcal{B}_r$.* (c) *$\mathcal{B}_r$ and $\mathcal{A}_r^n$ do not have an order unit.* (d) *Let $F_1, F_2 \in \mathcal{B}_r$. Then $D^n(F_1 \vee F_2) = (F_1)^{(n)} \vee F_2^{(n)}$, $D^n(F_1 \wedge F_2) = F_1^{(n)} \wedge F_2^{(n)}$, $|F^{(n)}| = D^n|F|$, $D^n(F^+) = (D^nF)^+$, and $D^n(F^-) = (D^nF)^-$.* (e) *If $f \in \mathcal{A}_r^n$ with primitive $F \in \mathcal{B}_r$ then $|f| \in \mathcal{A}_r^n$ with primitive $|F| \in \mathcal{B}_r$. Let $h = I^{n-1}[(-1)^{n-1}\chi_{(a,b)}]$ for $(a,b) \subset \mathbb{R}$. Then $|\int_{-\infty}^{\infty} fh| \geqslant |\int_{-\infty}^{\infty} |f|h|$ for all $f \in \mathcal{A}_c^n$. Now let $h(x,t) = (x-t)^{n-1}H_0(x-t)/(n-1)!$. Then $|\int_{-\infty}^{\infty} fh(x,\cdot)| = \int_{-\infty}^{\infty} |f|h(x,\cdot) = |F(x)|$ for all $f \in \mathcal{A}_c^n$. This formula also holds for $f \in \mathcal{A}_r^n$ if $\lambda = 1$. And, $|||f|||_{a,n} = \|f\|_{a,n}$, $\|f^\pm\|_{a,n} \leqslant \|f\|_{a,n}$.* (f) *If $f \in \mathcal{A}_r^n$ then $f^\pm \in \mathcal{A}_r^n$ with the the respective primitives $F^\pm \in \mathcal{B}_r$. Jordan decomposition: $f = f^+ - f^-$. And, $\int_{-\infty}^{\infty} fh = \int_{-\infty}^{\infty} f^+h - \int_{-\infty}^{\infty} f^-h$ for every $h \in \mathcal{IBV}^{n-1}$.* (g) *$\mathcal{A}_r^n$ is distributive: $f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h)$ and $f \vee (g \wedge h) = (f \vee g) \wedge (f \vee h)$ for all $f, g, h \in \mathcal{A}_r^n$.* (h) *$\mathcal{A}_r^n$ is modular: For all $f, g \in \mathcal{A}_r^n$, if $f \preceq g$ then $f \vee (g \wedge h) = g \wedge (f \vee h)$ for all $h \in \mathcal{A}_r^n$.* (i) *Let F_1 and F_2 be continuous functions on $\overline{\mathbb{R}}$. Then*

$$F_1^{(n)} \preceq F_2^{(n)} \iff F_1(x) - F_1(-\infty) \leqslant F_2(x) - F_2(-\infty) \quad \forall x \in \mathbb{R}. \tag{5.1}$$

Let F_1 and F_2 be regulated functions on $\overline{\mathbb{R}}$. Then

$$\begin{aligned} F_1^{(n)} \preceq F_2^{(n)} &\iff F(x-) - F(-\infty) \leqslant G(x-) - G(-\infty) \quad \forall x \in \mathbb{R} \\ &\iff F(x+) - F(-\infty) \leqslant G(x+) - G(-\infty) \quad \forall x \in \mathbb{R}. \end{aligned}$$

Proof. (a) It is clear that $\mathcal{B}_r$ is closed under supremum and infimum. See [28]. Hence, it is a Banach sublattice of the bounded functions on $\overline{\mathbb{R}}$. (b) This follows from Lemma 1.1. (c) Suppose $e \in \mathcal{B}_r$ is an order unit. Then F defined by $F(x) = \sqrt{e(x)}$ is in $\mathcal{B}_r$. And, $\lambda \geqslant \lim_{x \to -\infty} |F(x)|/e(x) = \lim_{x \to -\infty} 1/\sqrt{e(x)} = \infty$. Hence, $\mathcal{B}_r$ has no order unit. This shows $\mathcal{A}_r^n$ has no order unit. (d) Suppose $f \in \mathcal{A}_r^n$ with primitive $F \in \mathcal{B}_r$ such that $D^nF_1 \vee D^nF_2 = f$. Then $D^nF_1 \preceq f$, $D^nF_2 \preceq f$ and if $D^nF_1 \preceq \tilde{f}$, $D^nF_2 \preceq \tilde{f}$ for some $\tilde{f} \in \mathcal{A}_r^n$ then $f \preceq \tilde{f}$. These statements are equivalent to $F_1 \leqslant F$,

$F_2 \leqslant F$ and if $F_1 \leqslant \tilde{F}$, $F_2 \leqslant \tilde{F}$ then $F \leqslant \tilde{F}$, where $\tilde{F} \in \mathcal{B}_r$ is the primitive of $\tilde{f}$. Therefore, $F = F_1 \vee F_2$ so $D^n(F_1 \vee F_2) = D^n F = f = D^n F_1 \vee D^n F_2$. The other parts are similar. (e) For $f \in \mathcal{A}_c^n$, note that $|\int_{-\infty}^{\infty} fh| = |\int_{-\infty}^{\infty} F' h^{(n-1)}| = |F(b) - F(a)|$ and $\int_{-\infty}^{\infty} |f|h = |F(b)| - |F(a)|$. The other parts of (e) and (f) follow from (d) and the definitions, together with Proposition 2.10 and Theorem 3.5. (g) The real-valued functions on any set form a distributed lattice due to inheritance from $\leqslant$ in $\mathbb{R}$. Therefore, $\mathcal{B}_r$ is a distributed lattice and so is $\mathcal{A}_r^n$. See [17, p. 484] for an elementary proof and for another property of distributed lattices. (h) Modularity is also inherited from $\leqslant$ in $\mathbb{R}$ via $\mathcal{B}_r$. (i) We have $F_1^{(n)}, F_2^{(n)} \in \mathcal{A}_c^n$ with the respective primitives $\Phi_1, \Phi_2 \in \mathcal{B}_c$ given by $\Phi_1(x) = F_1(x) - F_1(-\infty)$ and $\Phi_2(x) = F_2(x) - F_2(-\infty)$. The definition of order then gives (5.1). The relations $F(x\pm) = \lim_{y \to x^\pm} F(y) = \lim_{y \to x^\pm} F(y-)$ give (5.2) and (5.3). $\square$

Let $f_1, f_2 \in \mathcal{A}_r^n$ with the respective primitives $F_1, F_2 \in \mathcal{B}_r$. Note that if $F_1 \leqslant F_2$ in $\mathcal{B}_r$ then we can differentiate both sides of this inequality with D^n to get $f_1 \preceq f_2$ in $\mathcal{A}_r^n$. And, if $f_1 \preceq f_2$ in $\mathcal{A}_r^n$ we can integrate both sides against $h(x, \cdot)$ to get $F_1 \leqslant F_2$ in $\mathcal{B}_r$. See Theorem 3.5. This also shows the derivative D^n is a positive operator on $\mathcal{B}_r$ and its inverse is a positive operator on $\mathcal{A}_r^n$.

Define $H_{-\infty}\colon \overline{\mathbb{R}} \to \mathbb{R}$ by $H_{-\infty} = \chi_{(-\infty,\infty]}$. Then $H_{-\infty}$ behaves like an order unit for $\mathcal{B}_r$. However, as a distribution it is equal to the constant distribution 1. Hence, all of its distributional derivatives are 0. To include an order unit we have to use a more general type of differentiation with respect to test functions that are not necessarily 0 at $-\infty$. Two possibilities are $\mathcal{B}V$ or $C^\infty(\mathbb{R}) \cap C(\overline{\mathbb{R}})$. We can take as a space of primitives $\mathcal{B}$, which consists of the functions $F\colon \overline{\mathbb{R}} \to \mathbb{R}$ that are regulated on $\overline{\mathbb{R}}$ such that $F(-\infty) = 0$, $F(x) = F(x-)$ for all $x \in \mathbb{R}$ and $F(\infty) = \lim_{y \to \infty} F(y)$. Hence, they are left continuous on $(-\infty, \infty]$, vanish at $-\infty$ and can have a jump discontinuity at $-\infty$ but not at ∞. Then $H_{-\infty} \in \mathcal{B}$. For $F \in \mathcal{B}$ and $g \in \mathcal{B}V$ define $\langle F', g\rangle = \int_{-\infty}^{\infty} g(x)\,\mathrm{d}F(x)$. Then $\langle H'_{-\infty}, g\rangle = g(-\infty)$. Hence, for this type of differentiation with respect to functions of bounded variation, $H'_{-\infty} \neq 0$. The Banach lattice $\mathcal{B}$ then has $H_{-\infty}$ as an order unit. We will explore this type of differentiation elsewhere.

The usual pointwise ordering makes L^1 into a Banach lattice. But the space of Henstock-Kurzweil integrable functions is not a vector lattice. It is not closed under supremum and infimum since there are functions integrable in this sense for which $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ converges but $\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x$ diverges. For example, $f(x) = x^2 \sin(\exp(x^2))$. Thus, even for functions, when we allow conditional convergence we must look elsewhere to find a lattice structure.

This order $\preceq$ is not compatible with the usual order on distributions: if $T, U \in \mathcal{D}'(\mathbb{R})$ then $T \geqslant U$ if and only if $\langle T - U, \varphi\rangle \geqslant 0$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi \geqslant 0$. If

$T \geqslant 0$ then it is known that T is a Borel measure. The usual ordering on distributions does not give a vector lattice on $\mathcal{A}_c^n$. For example, if $F(x) = H_0(x) \int_0^x \sin(t)\,\mathrm{d}t/t$ then $F(x) \geqslant 0$ for all $x \in \mathbb{R}$ so $F^{(n)} \succeq 0$ in $\mathcal{A}_c^n$. With the distributional ordering, $\sup(F', 0)$ is the function equal to $\sin(x)/x$ when $x \in (2n\pi, (2n+1)\pi]$ for some integer $n \geqslant 0$ and is equal to 0 otherwise. This function is not in $\mathcal{A}_c^1$ since the integral defining F converges conditionally. None of the derivatives $F^{(n)}(x)$ are positive in the pointwise or distributional sense. Note that in $\mathcal{A}_c^n$ we have $(D^n F)^+ = |F^{(n)}| = F^{(n)}$ and $(D^n F)^- = 0$.

If two distributions are in more than one of the $\mathcal{A}_c^n$ spaces they may have different order relations in each such space. For example, let $f(x) = -2(1-2x^2)\exp(-x^2)$, $F_1(x) = -2x\exp(-x^2)$ and $F_2(x) = \exp(-x^2)$. Then $f, F_1, F_2 \in \mathcal{B}_c$ and we have the pointwise derivatives $f(x) = F_1'(x) = F_2''(x)$ for each $x \in \mathbb{R}$. Hence, $f \in \mathcal{A}_c^1 \cap \mathcal{A}_c^2$. Since $f(0) = -2 < 0$ and $f(1) = 2/\mathrm{e} > 0$, it follows that f is neither positive nor negative in $\mathcal{B}_c$. Since $F_1(-1) = 2/\mathrm{e} > 0$ and $F_1(1) = -2/\mathrm{e} < 0$, it follows that f is neither positive nor negative in $\mathcal{A}_c^1$. But $F_2(x) > 0$ for all $x \in \mathbb{R}$ so $f \succeq 0$ in $\mathcal{A}_c^2$.

A vector lattice is *order complete* (or *Dedekind complete*) if every nonempty subset that is bounded above has a supremum. But $\mathcal{B}_r$ is not complete. Let $F_n(x) = H_0(x - 1/n)\sin(\pi/x)$ with $F_n(0) = 0$. Let $S = \{F_n \mid n \in \mathbb{N}\}$ then $S \subset \mathcal{B}_c$. An upper bound for S is the Heaviside step function H_0 but $\sup(S)(x) = H_0(x)\sin(\pi/x)$, which is not regulated. Hence, $\mathcal{A}_r^n$ is not complete, either.

A vector lattice is *Archimedean* if whenever $0 \leqslant x \leqslant ny$ for all $n \in \mathbb{N}$ and some $y \geqslant 0$ then $x = 0$. Applying the Archimedean property at each point of $\mathbb{R}$ shows $\mathcal{B}_r$ and hence $\mathcal{A}_r^n$ are Archimedean. All lattice inequalities that hold in $\mathbb{R}$ also hold in all Archimedean spaces and all lattice equalities that hold in $\mathbb{R}$ also hold in all vector lattices. See [2]. This expands the list of identities and inequalities proved in Theorem 5.1.

A Banach lattice is an *abstract L space* if $\|x+y\| = \|x\| + \|y\|$ for all $x, y \geqslant 0$. A Banach lattice is an *abstract M space* if $\|x \vee y\| = \max(\|x\|, \|y\|)$ for all $x, y \geqslant 0$. See, for example [2]. We next show that $\mathcal{B}_r$ and $\mathcal{A}_r^n$ are abstract M spaces but neither is an abstract L space.

Theorem 5.2. (a) *All of $\mathcal{B}_r$, $\mathcal{B}_c$, $\mathcal{A}_r^n$ and $\mathcal{A}_c^n$ are abstract M spaces.* (b) *None of $\mathcal{B}_r$, $\mathcal{B}_c$, $\mathcal{A}_r^n$ or $\mathcal{A}_c^n$ are abstract L spaces.*

Proof. (a) It suffices to prove $\mathcal{B}_r$ is an abstract M space. If $F_1, F_1 \geqslant 0$ in $\mathcal{B}_r$ then $\|F_1 \vee F_2\|_\infty = \sup\limits_{x\in\mathbb{R}} \max(F_1(x), F_2(x)) \geqslant \sup\limits_{x\in\mathbb{R}} F_1(x) = \|F_1\|_\infty$. Similarly, $\|F_1 \vee F_2\|_\infty \geqslant \|F_2\|_\infty$. So, $\|F_1 \vee F_2\|_\infty \geqslant \max(\|F_1\|_\infty, \|F_2\|_\infty)$. And, $\|F_1 \vee F_2\|_\infty = \sup\limits_{x\in\mathbb{R}} \max(F_1(x), F_2(x)) \leqslant \sup\limits_{x\in\mathbb{R}} \max(\|F_1\|_\infty, \|F_2\|_\infty) = \max(\|F_1\|_\infty, \|F_2\|_\infty)$. Hence, $\mathcal{B}_r$ is an abstract M space.

(b) It suffices to show $\mathcal{B}_c$ is not an abstract L space. Let $F_1(x) = 1 - |x|$ for $|x| \leqslant 1$ and $F_1(x) = 0$, otherwise. Let $F_2(x) = 1 - |x-2|$ for $|x-2| \leqslant 1$ and $F_2(x) = 0$, otherwise. Then $F_1(x), F_2(x) \geqslant 0$ for all $x \in \overline{\mathbb{R}}$. And, $\|F_1\|_\infty = \|F_2\|_\infty = \|F_1 + F_2\|_\infty = 1$. So $\mathcal{B}_c$ is not an abstract L space. $\square$

For every measure μ it is known that $L^1(\mu)$ is an abstract L space and that a Banach lattice is an abstract L space if and only if it is lattice isometric to $L^1(\nu)$ for some measure ν. A Banach lattice is an abstract M space with unit if and only if it is lattice isometric to $C(K)$ for some compact Hausdorff space K. The space $C(K)$ is the set of all real-valued continuous functions on K. These results are due to S. Kakutani, M. Krein and others. For references see [2]. The fact that $\mathcal{A}_c^n$ is an abstract M space but not an abstract L space indicates that what we have termed an integral here is fundamentally different from the Lebesgue integral.

6. Banach algebra

A *commutative algebra* is a vector space V over the scalar field $\mathbb{R}$ with a multiplication $V \times V \mapsto V$ such that for all $u, v, w \in V$ and all $a \in \mathbb{R}$, $u(vw) = (uv)w$ (associative), $uv = vu$ (commutative), $u(v+w) = uv + uw$ and $(u+v)w = uw + vw$ (distributive), $a(uv) = (au)v$. If $(V, \|\cdot\|_V)$ is a Banach space and $\|uv\|_V \leqslant \|u\|_V \|v\|_V$ then it is a Banach algebra. For any compact Hausdorff space, K, the set of continuous real-valued functions $C(K)$ is a commutative Banach algebra under pointwise multiplication and the uniform norm. Since $\overline{\mathbb{R}}$ is compact and $\mathcal{B}_r$ and $\mathcal{B}_c$ are closed under pointwise multiplication, $\mathcal{B}_r$ is a subalgebra of $C(\overline{\mathbb{R}})$ and $\mathcal{B}_c$ is a subalgebra of $\mathcal{B}_r$. The usual pointwise multiplication, $(FG)(x) = [F(x)][G(x)]$ for all $x \in \overline{\mathbb{R}}$, then makes $\mathcal{B}_r$ into a commutative algebra. The inequality $\|F_1 F_2\|_\infty \leqslant \|F_1\|_\infty \|F_2\|_\infty$ for all $F_1, F_2 \in \mathcal{B}_r$ shows $\mathcal{B}_r$ is a commutative Banach algebra.

There is no unit. For suppose $F(x) > 0$ for all $x \in \mathbb{R}$. If $eF = F$ then $e(x) = 1$ for all $x \in \mathbb{R}$ so $e \notin \mathcal{B}_r$. Consider the sequence $(u_n) \subset \mathcal{B}_c$ defined by $u_n(x) = 0$ for $x \leqslant -n$, $u_n(x) = x + n$ for $-n \leqslant x \leqslant 1 - n$ and $u_n(x) = 1$ for $x \geqslant 1 - n$. For each $F \in \mathcal{B}_c$ we have $\|F - u_n F\|_\infty \to 0$. Given $\varepsilon > 0$ there is $a \in \mathbb{R}$ such that $|F(x)| < \varepsilon$ for all $x \leqslant a$. We then have $|F(x) - u_n(x)F(x)| = |F(x)||1 - u_n(x)| < \varepsilon$ for all $x \leqslant a$. If $x \geqslant a$ take $n \geqslant 1 - a$. Then $u_n(x) = 1$. Hence, $\|F - u_n F\|_\infty \to 0$. $\mathcal{B}_c$ is then said to have an *approximate identity*.

By Lemma 1.1, $\mathcal{A}_r^n$ is a commutative Banach algebra, isomorphic to $\mathcal{B}_r$ for each $n \in \mathbb{N}$. If $f_1, f_2 \in \mathcal{A}_r^n$ with the respective primitives $F_1, F_2 \in \mathcal{B}_r$ then $f_1 f_2 = D^n(F_1 F_2)$.

Theorem 6.1. *For each $n \in \mathbb{N}$, $\mathcal{A}_r^n$ is a commutative Banach algebra without unit, with approximate identity, isomorphic to $\mathcal{B}_r$. Similarly with $\mathcal{A}_c^n$ and $\mathcal{B}_c$.*

There is no difficulty in allowing functions in $\mathcal{B}_r$ to be complex-valued and using $\mathbb{C}$ as the field of scalars. Complex conjugation is then an involution on $\mathcal{B}_r$. Then $\mathcal{B}_r$ is a C^*-algebra since for each $F \in \mathcal{B}_r$ we have $\|\overline{F}\|_\infty = \|F\|_\infty$ and $\|F\overline{F}\|_\infty = \|F\|_\infty^2$. Thus, each space $\mathcal{A}_r^n$ is also a C^*-algebra.

If $f_1, f_2 \in \mathcal{A}_r^n$ with the respective primitives $F_1, F_2 \in \mathcal{B}_r$ then for all $h \in \mathcal{IBV}^{n-1}$,

$$\begin{aligned}\int_{-\infty}^{\infty}(f_1 f_2)h &= (-1)^{n-1}\int_{-\infty}^{\infty} D^n(F_1F_2)D^{n-1}h\\ &= (-1)^{n-1}F_1(\infty)F_2(\infty)h^{(n-1)}(\infty) - (-1)^{n-1}\int_{-\infty}^{\infty} F_1(x)F_2(x)\,\mathrm{d}h^{(n-1)}(x).\end{aligned}$$

Let $a < \infty$, $h = I^{n-1}[\chi_{(-\infty,a)}]$ with $\lambda = 1$. Then $\int_{-\infty}^{\infty}(f_1f_2)h = (-1)^n F_1(a)F_2(a)$. In particular, in $\mathcal{A}_c^1$ we have $\int_{-\infty}^{a}(f_1f_2) = (\int_{-\infty}^{a} f_1)(\int_{-\infty}^{a} f_2)$.

There are zero divisors. Let $F_1, F_2 \in \mathcal{D}(\mathbb{R})$ with disjoint supports. Then $F_1F_2 = 0$ in $\mathcal{B}_r$ so $F_1^{(n)}F_2^{(n)} = 0$ in $\mathcal{A}_r^n$, yet neither $F_1^{(n)}$ nor $F_2^{(n)}$ need be zero. This example also shows the multiplication introduced in $\mathcal{A}_r^n$ is not compatible with pointwise multiplication in the case when elements of $\mathcal{A}_r^n$ are functions.

If two distributions are in more than one of the $\mathcal{A}_c^n$ spaces they may have a different product in each such space. For example, let $f(x) = \sin(x)$ for $|x| \leqslant \pi$ and $f(x) = 0$, otherwise. Then $f \in \mathcal{B}_c$ and its square in $\mathcal{B}_c$ is the function $f^2(x) = \sin^2(x)$ for $|x| \leqslant \pi$ and $f^2(x) = 0$, otherwise. Now let $F_1(x) = -1 - \cos(x)$ for $|x| \leqslant \pi$ and $F_1(x) = 0$, otherwise. Let $F_2(x) = 0$ for $x \leqslant -\pi$, $F_2(x) = -x - \sin(x) - \pi$ for $|x| \leqslant \pi$ and $F_2(x) = -2\pi$ for $x \geqslant \pi$. Then $F_1, F_2 \in \mathcal{B}_c$ and we have the pointwise derivatives $f(x) = F_1'(x) = F_2''(x)$ for each $x \in \mathbb{R}$. Hence, $f \in \mathcal{A}_c^1 \cap \mathcal{A}_c^2$. In $\mathcal{A}_c^1$, $f^2(x) = D(F_1^2)(x) = -2[1 + \cos(x)]\sin(x)$ for $|x| \leqslant \pi$ and $f^2(x) = 0$, otherwise. In $\mathcal{A}_c^2$, $f^2(x) = D^2(F_2^2)(x) = 2[(1 + \cos(x))^2 - (x + \sin(x) + \pi)\sin(x)]$ for $|x| \leqslant \pi$ and $f^2(x) = 0$, otherwise. Hence, $f \in \mathcal{B}_c \cap \mathcal{A}_c^1 \cap \mathcal{A}_c^2$ but has a different product in each of these three spaces.

By Proposition 3.1, for each integer $n \geqslant 0$, $\delta^{(n)} \in \mathcal{A}_r^{n+1}$. Let $\lambda \in \mathbb{R}$. Then $\delta^{(n)}\delta^{(n)} = D^{n+1}[H_\lambda^2] = D^{n+1}[H_{\lambda^2}] = \delta^{(n)}$.

It is easy to see that $\mathcal{B}_c$ is a maximal ideal of $C(\overline{\mathbb{R}})$. See [14] for the definition. It then follows that $\mathcal{A}_c^n$ is a maximal ideal of the space of nth derivatives of functions in $C(\overline{\mathbb{R}})$. Similarly, $\mathcal{B}_r$ is a maximal ideal of the space $\mathcal{B}$, introduced in Section 5, consisting of the left continuous regulated functions on $\overline{\mathbb{R}}$ such that $F(-\infty) = 0$. It then follows that $\mathcal{A}_r^n$ is a maximal ideal of the nth derivatives of such functions. Note that $\mathcal{B}$ has a unit, $H_{-\infty} = \chi_{(-\infty,\infty]}$. As pointed out in Section 5, the distributional derivative is too coarse to distinguish between $H_{-\infty}$ and the constant functions so this would entail using a finer notion of the derivative such as using functions of bounded variation for test functions. This is something we will pursue elsewhere.

Define $AC_{-\infty}(\overline{\mathbb{R}})$ to be the functions in $AC(\overline{\mathbb{R}})$ whose limit vanishes at $-\infty$. And, $AC^n_{-\infty}(\mathbb{R})$ consists of the distributions $f \in \mathcal{D}'(\mathbb{R})$ such that $f = D^nF$ for some $F \in AC_{-\infty}(\overline{\mathbb{R}})$. Note that $AC^1_{-\infty}(\overline{\mathbb{R}}) = L^1$. It is easy to show that $AC_{-\infty}(\overline{\mathbb{R}})$ is closed under pointwise multiplication. Hence, it is a Banach subalgebra of $\mathcal{B}_c$. Then $AC^n_{-\infty}(\mathbb{R})$ is a Banach subalgebra of $\mathcal{A}^n_c$. Of course, L^1 is not an algebra under the usual pointwise multiplication. Similarly for the spaces of nth derivatives of primitives of Henstock-Kurzweil and wide Denjoy integrable functions. See [13] for the definitions of the relevant spaces of primitives. Under convolution L^1 is a Banach algebra. Although convolution has been defined in $\mathcal{A}^1_c \times L^1$ in [27] it does not seem possible to define convolution in $\mathcal{A}^1_c \times \mathcal{A}^1_c$. Convolutions can be defined for distributions but restrictions on the supports are generally imposed. See [30].

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