Sohrab Sohrabi Laleh; Mir Yousef Sadeghi; Mahdi Hanifi Mostaghim
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Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 1, 105–110

Persistent URL: http://dml.cz/dmlcz/142043

SOME RESULTS ON THE COFINITENESS OF LOCAL COHOMOLOGY MODULES

Sohrab Sohrabi Laleh, Shabestar, Mir Yousef Sadeghi,
Mahdi Hanifi Mostaghim, Tehran

(Received September 4, 2010)

Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, M an R-module and t a non-negative integer. In this paper we show that the class of minimax modules includes the class of $\mathcal{AF}$ modules. The main result is that if the R-module $\operatorname{Ext}^t_R(R/\mathfrak{a}, M)$ is finite (finitely generated), $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < t$ and $H^t_{\mathfrak{a}}(M)$ is minimax then $H^t_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. As a consequence we show that if M and N are finite R-modules and $H^i_{\mathfrak{a}}(N)$ is minimax for all $i < t$ then the set of associated prime ideals of the generalized local cohomology module $H^t_{\mathfrak{a}}(M, N)$ is finite.

Keywords: local cohomology, cofinite modules, mimimax modules, AF modules, associated primes

MSC 2010: 13D45, 14B15, 13E05

1. Introduction

Throughout this note we assume that R is a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, M is an R-module, and t is a non-negative integer. For each $i \geqslant 0$, the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(M) = \varinjlim_n \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

For the basic properties of local cohomology the reader can refer to [2] of Brodmann and Sharp. An important problem in commutative algebra is to determine when the set of the associated primes of the i-th local cohomology module, $H^i_{\mathfrak{a}}(M)$ with respect to $\mathfrak{a}$, is finite.

It is well known that the local cohomology modules $H^i_{\mathfrak{a}}(M)$ are not always finitely generated. Taking this fact, Hartshorne [4] conjectured the following: If R is a Noetherian ring, then for any ideal $\mathfrak{a}$ of R and any finitely generated R-module M, the module $\operatorname{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is finitely generated for all $i, j \geqslant 0$.

In the same paper Hartshorne defined that an R-module M is $\mathfrak{a}$-cofinite whenever $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i \geqslant 0$. Hartshorne also gave a counterexample to his conjecture, which is essentially as follows. Let k be a field, $R = k[[X, Y, Z, U]]$, and $\mathfrak{a} = (X, U)R$. If we take $M = R/(XY - ZU)$, then $H^2_{\mathfrak{a}}(M)$ is not $\mathfrak{a}$-cofinite. Nonetheless, by using derived category theory, he proved that if R is a compelet regular local ring, then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite in two cases:

(i) $\mathfrak{a}$ is a non-zero principal ideal.
(ii) $\mathfrak{a}$ is a prime ideal with $\dim(R/\mathfrak{a}) = 1$.

In particular, using spectral sequence Mafi [7] showed that, if for a finite R-module M and an integer t, the local cohomology module $H^t_{\mathfrak{a}}(M)$ is Artinian and $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < t$, then $H^t_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. In this paper, in Theorem 4, we obtain this result with the minimax condition on $H^t_{\mathfrak{a}}(M)$ instead of the Artinian condition without using the spectral sequence theory. At the end, in Theorem 7, we show that if M and N are finite R-modules and $H^i_{\mathfrak{a}}(N)$ is minimax for all $i < t$, then the set of associated prime ideals of the generalized local cohomology $H^t_{\mathfrak{a}}(M, N)$ is finite.

2. The results

In [14] H. Zöschinger introduced the interesting class of minimax modules. He also has given many equivalent conditions for a module to be minimax in [14] and [15].

Definition 1. An R-module N is said to be a minimax module, if there is a finite submodule L of N such that N/L is Artinian.

Example 1. It was shown by T. Zink [13] and E. Enochs [3] that a module over a complete local ring is minimax if and only if it is Matlis reflexive.

S. Yassemi [12] introduced the following definition of the class of $\mathcal{AF}$ modules.

Definition 2. The R-module N is said to be an $\mathcal{AF}$ module, if there is an Artinian submodule L of N such that N/L is a finite.

Example 2. All finite modules and all Artinian modules are $\mathcal{AF}$ modules.

In the following Lemma we prove that every $\mathcal{AF}$ module is a minimax module.

Lemma 3. *Every $\mathcal{AF}$ module is minimax.*

Proof. Let N be an $\mathcal{AF}$ module, then there exists an Artinian submodule L of N such that N/L is finite. Since N/L is finite, there exists a finite submodule K of N such that $N = K + L$. Since $N/K \cong L/K \cap L$ and $L/K \cap L$ is Artinian, so N/K is Artinian as required. □

Example 3. By Lemma 3, the class of minimax modules includes the class of $\mathcal{AF}$ modules.

Now we prove the main theorem.

Theorem 4. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Let t be a non-negative integer, and M an R-module such that $\operatorname{Ext}^t_R(R/\mathfrak{a}, M)$ is a finite R-module. If $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < t$ and $H^t_{\mathfrak{a}}(M)$ is minimax, then $H^t_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite.*

Proof. In view of [9, Proposition 4.3], it is enough to show that $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is finite. To prove this, we use induction on t. If $t = 0$, since $\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is equal to the finite R-module $\operatorname{Hom}_R(R/\mathfrak{a}, M)$ the assertion is obvious. Now let $t > 0$ and suppose the result has been proved for smaller values of t. Since $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite, $\operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is finite for all i. Now from the long exact sequence induced by the exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0,$$

we can get that $\operatorname{Ext}^t_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is finite. Since $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ for all $i > 0$, we can assume that M is an $\mathfrak{a}$-torsion-free R-module. Let E be an injective envelope of M and put $L := E/M$. Then $\Gamma_{\mathfrak{a}}(E) = 0$ and so $\operatorname{Hom}_R(R/\mathfrak{a}, E) = 0$. Now, by using the exact sequence

$$0 \to M \to E \to L \to 0,$$

we get that $\operatorname{Ext}^i_R(R/\mathfrak{a}, L) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ and $H^i_{\mathfrak{a}}(L) \cong H^{i+1}_{\mathfrak{a}}(M)$ for all $i \geqslant 0$. Consequently, by the inductive hypothesis $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(L))$ is finite and hence $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is finite too. □

Melkersson in [10, Example 1.3] showed that in a local ring $(R, \mathfrak{m})$ a module M is $\mathfrak{m}$-cofinite if and only if it is Artinian. So we conclude the following result.

Corollary 5. *Let $(R, \mathfrak{m})$ be a local ring. Assume that the assumptions of Theorem 4 hold. Then $H^t_{\mathfrak{a}}(M)$ is an Artinian R-module.*

Proof. By [12, Theorem 1.2.v], $H^t_{\mathfrak{a}}(M)$ is $\mathfrak{m}$-cofinite, so that $H^t_{\mathfrak{a}}(M)$ is an Artinian R-module. □

Corollary 6. *Let the situation be as in Theorem 4. Moreover, assume that $H^i_{\mathfrak{a}}(M)$ is minimax for all $i \geqslant t$. Then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i. In this case if R is local with maximal ideal $\mathfrak{m}$ then $H^i_{\mathfrak{a}}(M)$ is an Artinian R-module for all i.*

Proof. The claim follows by Theorem 4 and the second part follows by Corollary 5. □

Now, we are ready to prove our final result about finiteness of the set of associated prime ideals of generalized local cohomology modules. Let M and N be R-modules, and let $\mathfrak{a}$ be an ideal of R. Then the generalized local cohomology module $H^i_{\mathfrak{a}}(M,N)$ which was introduced by Herzog in [5], is defined as

$$H^i_{\mathfrak{a}}(M,N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

If $M = R$, then $H^i_{\mathfrak{a}}(M,N)$ is equal to $H^i_{\mathfrak{a}}(N)$, the usual local cohomology module.

In [8] Mafi shows that if $\mathfrak{a}$ is an ideal of R, and M is a finite R-module, then for every R-module N and any positive integer t we have

$$\operatorname{Ass}_R(H^t_{\mathfrak{a}}(M,N)) \subseteq \bigcup_{i=0}^{t} \operatorname{Ass}_R(\operatorname{Ext}^i_R(M, H^{t-i}_{\mathfrak{a}}(N))).$$

By virtue of this result we prove the following theorem.

Theorem 7. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R, t a non-negative integer, and M and N finite R-modules. If $H^i_{\mathfrak{a}}(N)$ is a minimax R-module for all $i < t$ and* $\operatorname{supp}(M) \subseteq V(\mathfrak{a})$, *then the set* $\operatorname{Ass}_R(H^t_{\mathfrak{a}}(M,N))$ *is finite.*

In order to prove Theorem 7, we need to generalize [6, Lemma 4.2] as follows.

Lemma 8. *Let $\mathfrak{a}$ be an ideal of R and N an $\mathfrak{a}$-cofinite R-module. Then for any finite R-module M with* $\operatorname{supp}(M) \subseteq V(\mathfrak{a})$ *the R-module* $\operatorname{Ext}^i_R(M,N)$ *is finite for all i.*

Proof. Since $\operatorname{supp}(M) \subseteq V(\mathfrak{a})$, according to Gruson's Theorem [11, Theorem 4.1], there exists a chain of submodules of M,

$$0 = M_0 \subset M_1 \subset \ldots \subset M_k = M$$

such that the factors M_j/M_{j-1} are homomorphic images of a direct sum of finitely many $R/\mathfrak{a}$ ($1 \leqslant j \leqslant k$). Now consider the exact sequences

$$0 \to K \to (R/\mathfrak{a})^n \to M_1 \to 0$$
$$0 \to M_1 \to M_2 \to M_2/M_1 \to 0$$
$$\vdots$$
$$0 \to M_{k-1} \to M_k \to M_k/M_{k-1} \to 0$$

for some positive integer n. From the long exact sequence

$$\ldots \to \operatorname{Ext}_R^{i-1}(M_{j-1}, N) \to \operatorname{Ext}_R^i(M_j/M_{j-1}, N) \to \operatorname{Ext}_R^i(M_j, N) \to \operatorname{Ext}_R^i(M_{j-1}, N) \to \ldots$$

and by an easy induction on k, the assertion follows. So, it suffices to prove the case $k = 1$. From the exact sequence

$$0 \to K \to (R/\mathfrak{a})^n \to M \to 0$$

where $n \in \mathbb{N}$ and K is a finite R-module, and the induced long exact sequence, by using induction on i we show that $\operatorname{Ext}_R^i(M, N)$ is finite for all i. For $i = 0$, we have an exact sequence

$$0 \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R((R/\mathfrak{a})^n, N) \to \operatorname{Hom}_R(K, N) \to \ldots$$

Since $\operatorname{Hom}_R((R/\mathfrak{a})^n, N) \cong \bigoplus^n \operatorname{Hom}_R(R/\mathfrak{a}, N)$ and N is $\mathfrak{a}$-cofinite, $\operatorname{Hom}_R((R/\mathfrak{a})^n, N)$ is finite and then $\operatorname{Hom}_R(M, N)$ is finite. Now let $i > 0$. For any R-module M with $\operatorname{supp}(M) \subseteq V(\mathfrak{a})$ we have that the R-module $\operatorname{Ext}_R^{i-1}(M, N)$ is finite, in particular for K. Then from the long exact sequence

$$\ldots \to \operatorname{Ext}_R^{i-1}(K, N) \to \operatorname{Ext}_R^i(M, N) \to \operatorname{Ext}_R^i((R/\mathfrak{a})^n, N) \to \ldots$$

we can conclude that $\operatorname{Ext}_R^i(M, N)$ is finite. □

Now we can prove Theorem 7 by using Lemma 8.

Proof of Theorem 7. It is enough to show that $H_{\mathfrak{a}}^i(N)$ is $\mathfrak{a}$-cofinite for all $i < t$ and $\operatorname{Hom}_R(M, H_{\mathfrak{a}}^t(N))$ is finite. To show that $H_{\mathfrak{a}}^i(N)$ is $\mathfrak{a}$-cofinite, we use induction on i. The case $i = 0$ is obvious as $H_{\mathfrak{a}}^0(N)$ is finite. So, let $i > 0$ and suppose the result has been proved for smaller values of i. By the inductive hypothesis, $H_{\mathfrak{a}}^j(N)$ is $\mathfrak{a}$-cofinite for $j = 0, 1, \ldots, i-1$ and since $H_{\mathfrak{a}}^j(N)$ is minimax, hence by [1, Lemma 2.2] we can conclude that $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^i(N))$ is finite. Therefore by [9, Proposition 4.3], $H_{\mathfrak{a}}^i(N)$ is $\mathfrak{a}$-cofinite. If we use again [1, Lemma 2.2] then $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(N))$ is finite. So by Lemma 8, $\operatorname{Ext}_R^i(M, H_{\mathfrak{a}}^{t-i}(N))$ is finite and so $\operatorname{Ass}_R(H_{\mathfrak{a}}^i(M, N))$ is finite. □

Acknowledgement. The authors would like to thank Professor Khadijeh Ahmadi-Amoli for her careful reading of the first draft and many helpful suggestions.

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Authors' addresses: Sohrab Sohrabi Laleh, Department of Mathematics, Islamic Azad University, Shabestar Branch, Shabestar, Iran, e-mail: sohrabil6@yahoo.com; Mir Yousef Sadeghi, Payam-e-Noor University, Tehran Graduate Center, Tehran, Iran, e-mail: sadeghi@phd.pnu.ac.ir; Mahdi Hanifi Mostaghim, Payam-e-Noor University, Tehran Graduate Center, Tehran, Iran, e-mail: hanifi@phd.pnu.ac.ir.