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SPECTRAL CHARACTERIZATION OF MULTICONE GRAPHS

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Abstract. A *multicone graph* is defined to be the join of a clique and a regular graph. Based on Zhou and Cho's result [*B. Zhou, H. H. Cho*, Remarks on spectral radius and Laplacian eigenvalues of a graph, Czech. Math. J. *55* (*130*) (2005), 781–790], the spectral characterization of multicone graphs is investigated. Particularly, we determine a necessary and sufficient condition for two multicone graphs to be cospectral graphs and investigate the structures of graphs cospectral to a multicone graph. Additionally, lower and upper bounds for the largest eigenvalue of a multicone graph are given.

Keywords: adjacency matrix, cospectral graph, spectral characteriztion, multicone graph

MSC 2010: 05C50

1. Introduction

In this paper, we are concerned only with undirected simple graphs (loops and multiple edges are not allowed). All notions on graphs that are not defined here can be found in [1]. For a graph $G = (V(G), E(G))$, let $n(G)$, $m(G)$, $l(G)$ and $A(G)$ be respectively the order, size, line graph and adjacency matrix of G. We denote $\det(\lambda I - A(G))$, the *characteristic polynomial* of G, by $\varphi(G, \lambda)$ or simply $\varphi(G)$. The *adjacency spectrum* of G, denoted by $\mathrm{Spec}_A(G)$, is the multiset of eigenvalues of $A(G)$. Since $A(G)$ is symmetric, its eigenvalues are real and set $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \lambda_{n(G)}(G)$. The maximum eigenvalue $\lambda_1(G)$ of G is called the *spectral radius* of G and is often denoted by $\varrho(G)$. For a graph G and $U \subseteq V(G)$, $G[U]$ stands for the induced graph by U, and $\delta(G)$ is the minimum degree of G. Let $G_1 \cup G_2$ denote the *disjoint union* of graphs G_1 and G_2, and kG_1 the disjoint union

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of k copies of G_1. The *join* (or *complete product*) $G_1 \triangledown G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2. A *multicone graph* is defined to be the join of a clique and a regular (not necessarily connected) graph. Let $CP(k)$ denote the *cocktail-party graph* obtained from K_{2k} by removing k disjoint edges.

Two graphs G and H are said to be *A-cospectral* if the corresponding adjacency spectra are the same. A graph G is said to be *determined by the A-spectrum* (or simply *G is a DAS*-graph) if there is no other non-isomorphic graph A-cospectral to it, i.e., $\mathrm{Spec}_A(H) = \mathrm{Spec}_A(G)$ implies $G \cong H$. The background of the question "which graphs are determined by their spectrum?" originates from Chemistry (in 1956, Günthadr and Primas [7] raised this question in the context of Hückel's theory). For additional remarks on the topic we refer the readers to [3], [4]. A remarkable fact is that there are many wonderful papers to studying cospectral graphs and introducing kinds of methods of constructing them (see [5], [8], [9], [11] for example). By contrast, until now, determining what kinds of graphs are DAS-graphs is far from resolved. For the multicone graphs, only few graphs are proved to be DAS-graphs such as the complete graph $K_n = K_a \triangledown K_{n-a}$. In this paper we focus on studying under what conditions the multicone graph is a DAS-graph and determine the lower and upper bounds of its spectral radius.

2. Basic lemmas and results

In this section, we cite some useful results which put way for obtaining our main result.

Lemma 2.1 ([3]). *Let G and H be two A-cospectral graphs. Then*

(i) *$n(G) = n(H)$, $m(G) = m(H)$ and $\varphi(G) = \varphi(H)$.*
(ii) *G is k-regular if and only if H is k-regular.*
(iii) *G and H have the same number of closed walks of any length.*

Lemma 2.2 ([1]). *Let G_i be an r_i-regular graph of order n_i $(i = 1, 2)$. Then*

$$\varphi(G_1 \triangledown G_2, \lambda) = \frac{\varphi(G_1, \lambda)\varphi(G_2, \lambda)}{(\lambda - r_1)(\lambda - r_2)}[(\lambda - r_1)(\lambda - r_2) - n_1 n_2].$$

Corollary 2.1. *Let G_i be an r_i-regular graph of order n_i $(i = 1, 2)$. If H_i is A-cospectral to G_i (i=1,2), then $H_1 \triangledown H_2$ is A-cospectral to $G_1 \triangledown G_2$.*

P r o o f. Since H_i is A-cospectral to G_i, hence H_i is also an r_i-regular graph of order n_i $(i = 1, 2)$. Then the result follows from Lemma 2.2. □

Corollary 2.2. *Let $G = G_1 \triangledown G_2$, where G_i is an r_i-regular graph of order n_i $(i = 1, 2)$. If G is a DAS-graph, then G_i $(i = 1, 2)$ is also a DAS-graph.*

Proof. Let H_1 be any graph such that $\varphi(H_1) = \varphi(G_1)$. Then H_1 is an r_1-regular graph of order n_1. Set $H = H_1 \triangledown G_2$. By Corollary 2.1 we have $\varphi(H) = \varphi(G)$. Since G is a DAS-graph, we have $H \cong G$ and so $H_1 \cong G_1$. Similarly, G_2 can be proved. □

Hong et al. [10] first proved the following theorem for the connected graph. Nikiforov [12] showed it independently by a quite different method for a (not necessarily connected) graph, and mentioned some extreme graphs. Zhou and Cho [13] proved it for the unconnected case and completely characterized the extreme graphs.

Theorem 2.1 ([13]). *Let G be a graph with order n and size m and let δ be the minimal degree of vertices of G. Then*

$$\varrho(G) \leqslant \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}},$$

and equality holds if and only if in one component of G each vertex is either of degree δ or adjacent to all other vertices, and all other components are regular with degree δ.

Furthermore, Zhou and Cho [13] deduced that the equality in Theorem 2.1 occurs if and only if G is a graph of one of the following four types:

(i) a regular graph with the vertex degree δ;

(ii) a bidegreed graph in which every vertex is either of a degree δ or $n-1$ $(\delta < n-1)$;

(iii) the disjoint union of a complete graph with order at least $\delta + 2$ and a regular graph with the vertex degree δ;

(iv) the disjoint union of a connected bidegreed graph in which every vertex is either of the degree δ or adjacent to all other vertices, and a regular with the vertex degree δ.

In what follows, mainly the graphs of the second type are investigated. Let $\mathscr{B}(n-1, \delta)$ be the family of all bidegreed graphs of order n, size m, maximum degree $n-1$ and minimum degree δ $(\delta < n-1)$. Obviously, for $G \in \mathscr{B}(n-1, \delta)$, G is a connected graph containing a clique K_t (if G has t vertices of degree $n-1$) as its proper subgraph, and $B = G[V(G) \setminus V(K_t)]$ is a $(\delta - t)$-regular graph (clearly, $\delta \geqslant t$). Thus, G can be viewed as the join of K_t and B, where they are respectively called the *kernel* denoted by $\ker(G)$ and the *branch* denoted by $\mathrm{bra}(G)$ of G. Thus, $G = \ker(G) \triangledown \mathrm{bra}(G)$. Clearly, such a graph G is the so called multicone graph.

Corollary 2.3. *Let G be a multicone graph of order n with* $\ker(G) = K_t$ *and* $\operatorname{bra}(G) = B$. *If $k = \lambda_1(B), \lambda_2(B), \ldots, \lambda_{n-t}(B)$ are the eigenvalues of B, then the eigenvalues of G are*

$$\lambda_2(B), \ldots, \lambda_{n-t}(B), \underbrace{-1, \ldots, -1}_{t-1}, \frac{t+k-1 \pm \sqrt{(t+k-1)^2 + 4t(n-k-t) - 4k}}{2}.$$

Proof. From Lemma 2.2 and $\varphi(K_t) = (\lambda+1)^{t-1}(\lambda - t + 1)$ it follows that

$$(1) \qquad \varphi(G) = \varphi(K_t \triangledown B) = \frac{\varphi(K_t)\varphi(B)}{(\lambda - t + 1)(\lambda - k)}[(\lambda - t + 1)(\lambda - k) - t(n-t)]$$
$$= \prod_{i=2}^{n-t}(\lambda - \lambda_i(B))(\lambda+1)^{t-1}[\lambda^2 - (t+k-1)\lambda - (t(n-t-k)+k)],$$

which shows that the result holds. □

3. Spectral characterization of multicone graphs

In this section, spectral characterization of multicone graphs is investigated.

Lemma 3.1. *Let $G \in \mathscr{B}(n-1, \delta)$, where $\delta < n-1$. Then $\delta \notin \operatorname{Spec}_A(G)$.*

Proof. Suppose, without loss of generality, that G contains the kernel K_{t_1} ($t_1 \geqslant 1$) and the k_1-regular branch B_1 of order n_1. So, $\delta = k_1 + t_1$ and $G = K_{t_1} \triangledown B_1$. From (1) it follows that

$$\varphi(G) = \prod_{i=2}^{n_1}(\lambda - \lambda_i(B_1))(\lambda+1)^{t_1-1}[\lambda^2 - (t_1+k_1-1)\lambda - (t_1(n_1-k_1)+k_1)].$$

Put $f_1(\lambda) = \lambda^2 - (t_1+k_1-1)\lambda - (t_1(n_1-k_1)+k_1)$. Thus, $f_1(k_1+t_1) = -t_1(n_1-k_1-1)$. If $k_1 = n_1 - 1$, then $B_1 = K_{n_1}$ and so $G = K_{t_1} \triangledown K_{n_1} = K_{t_1+n_1}$ which contradicts the fact that G is a bidegreed graph. Thus, $n_1 - k_1 - 1 > 0$, since the spectral radius of a connected graph is at most its maximum degree (see [3, Theorem 1.1]). Hence, δ is not a root of $f_1(\lambda)$. Note that $\delta > k_1 > \lambda_i(B_i)$ for $2 \leqslant i \leqslant n_1$. Consequently, $\delta \notin \operatorname{Spec}_A(G)$. □

We next determine the structure of a graph A-cospectral to a multicone graph.

Theorem 3.1. *Let $G \in \mathscr{B}(n-1, \delta)$ and let H be a graph with the minimum degree $\delta(H) = \delta$, where $\delta < n-1$. If H is A-cospectral to G, then*

(i) *$H \in \mathscr{B}(n-1, \delta)$;*
(ii) *$\ker(H) = \ker(G)$;*
(iii) *$\operatorname{bra}(H)$ and $\operatorname{bra}(G)$ have the same degree.*

P r o o f. If H is A-cospectral to G, then $n(H)=n(G)$, $m(H)=m(G)$ and

$$\begin{aligned}\varrho(H)=\varrho(G)&=\frac{\delta-1}{2}+\sqrt{2m(G)-n(G)\delta+\frac{(\delta+1)^2}{4}}\\&=\frac{\delta(H)-1}{2}+\sqrt{2m(H)-n(H)\delta(H)+\frac{(\delta(H)+1)^2}{4}},\end{aligned}$$

which implies by virtue of Theorem 2.1 that H is a graph of types (i)–(iv). By Lemma 3.1 we get $\delta\notin\operatorname{Spec}_A(G)=\operatorname{Spec}_A(H)$. Note that any graph of types (i), (iii) and (iv) has an eigenvalue δ (since a δ-regular graph always has the eigenvalue δ). So, $H\in\mathscr{B}(n-1,\delta)$. This proves (i).

In what follows, we proceed to adopt the notation used in Lemma 3.1, i.e., $G=K_{t_1}\triangledown B_1$. Assume that H contains the kernel K_{t_2}. From (i) of the theorem we get $H\in\mathscr{B}(n-1,\delta)$, where $\delta=k_1+t_1$. Then $B_2=\operatorname{bra}(H)$ is $(k_1+t_1-t_2)$-regular branch with order n_2, and $H=K_{t_2}\triangledown B_2$. By (1) and $\varphi(K_{t_2})=(\lambda+1)^{t_2-1}(\lambda-t_2+1)$ we obtain

$$\varphi(H)=\prod_{i=2}^{n_2}(\lambda-\lambda_i(B_2))(\lambda+1)^{t_2-1}[\lambda^2-(t_1+k_1-1)\lambda-(t_2(n_1-k_1-1)+k_1+t_1)].$$

Put $f_2(\lambda)=\lambda^2-(t_1+k_1-1)\lambda-(t_2(n_1-k_1-1)+k_1+t_1)$. Note that $\varrho(G)>\lambda_i(B_1)$ for $2\leqslant i\leqslant n_1$ and $\varrho(H)>\lambda_i(B_2)$ for $2\leqslant i\leqslant n_2$. Therefore, $f_1(\varrho(G))=0$ and $f_2(\varrho(G))=0$, which yields

$$t_1(n_1-k_1-1)=t_2(n_1-k_1-1). \tag{2}$$

Recall that $n_1-k_1-1>0$. From (2) we have $t_1=t_2$, and so (ii) and (iii) holds. □

The theorem below determines a necessary and sufficient condition for two multicone graphs to be A-cospectral. For $\lambda\in\operatorname{Spec}_A(G)$, let $m_G(\lambda)$ denote the multiplicity of the eigenvalue λ.

Theorem 3.2. *Let $G,H\in\mathscr{B}(n-1,\delta)$, where $\delta<n-1$. Then H and G are A-cospectral graphs if and only if $m_G(-1)=m_H(-1)$ and* $\operatorname{bra}(H)$ *is A-cospectral to* $\operatorname{bra}(G)$.

P r o o f. Suppose that H and G are A-cospectral graphs. Then $m_G(-1)=m_H(-1)$. By Theorem 3.1 we obtain that $\ker(H)=\ker(G)$, and both $\operatorname{bra}(H)$ and $\operatorname{bra}(G)$ are regular graphs having the same degree (say, k). Without loss of generality, set $\ker(G)=\ker(H)=K_t$ and $\operatorname{bra}(G)=B_1$, $\operatorname{bra}(H)=B_2$ with orders $n(B_i)=n_i$

$(i = 1, 2)$. From $t + n_1 = n(G) = n(H) = t + n_2$ we get $n_1 = n_2$. From (1) it follows that

$$\varphi(G) = \varphi(K_t \triangledown B_1) = \frac{\varphi(K_t)\varphi(B_1)}{(\lambda - t + 1)(\lambda - k)}[(\lambda - t + 1)(\lambda - k) - tn_1] \tag{3}$$

and

$$\varphi(H) = \varphi(K_t \triangledown B_2) = \frac{\varphi(K_t)\varphi(B_2)}{(\lambda - t + 1)(\lambda - k)}[(\lambda - t + 1)(\lambda - k) - tn_2]. \tag{4}$$

By $\varphi(G) = \varphi(H)$ we get $\varphi(B_1) = \varphi(B_2)$. This completes the proof of necessity.

Next we show the sufficiency. Set $G = K_{t_1} \triangledown B_1$ and $H = K_{t_2} \triangledown B_2$. Recall that B_1 and B_2 are regular graphs. If B_1 is A-cospectral to B_2, by Lemma 2.1 they have the same degree (say, k) and $n_1 = n_2$. By equalities (3) and (4) we have

$$\varphi(G) = \prod_{i=2}^{n_1}(\lambda - \lambda_i(B_1))(\lambda + 1)^{t_1 - 1}[\lambda^2 - (t_1 + k - 1)\lambda - (t_1(n_1 - k) + k)]$$

and

$$\varphi(H) = \prod_{i=2}^{n_1}(\lambda - \lambda_i(B_2))(\lambda + 1)^{t_2 - 1}[\lambda^2 - (t_2 + k - 1)\lambda - (t_2(n_1 - k) + k)].$$

Since $\varphi(B_1) = \varphi(B_2)$, we have $\prod_{i=1}^{n_1}(\lambda - \lambda_i(B_1)) = \prod_{i=1}^{n_1}(\lambda - \lambda_i(B_2))$ and so $m_{B_1}(-1) = m_{B_2}(-1)$. Put $g_i(\lambda) = \lambda^2 - (t_i + k - 1)\lambda - (t_i(n_1 - k) + k)$. Thus, $g_i(-1) = -t_i(n_1 - k - 1)$ $(i = 1, 2)$. As proved in Theorem 3.1, $n_1 - k - 1 > 0$ and hence $g_i(-1) < 0$, i.e., -1 is not a root of $g_i(\lambda)$ $(i = 1, 2)$. Consequently, $m_G(-1) = t_1 - 1 + m_{B_1}(-1)$ and $m_H(-1) = t_2 - 1 + m_{B_2}(-1)$ which indicates by virtue of $m_G(-1) = m_H(-1)$ that $t_1 = t_2$. Therefore, $\varphi(G) = \varphi(H)$. □

The following theorem establishes a sufficient condition for the DAS-multicone graphs.

Theorem 3.3. *Let $G \in \mathscr{B}(n - 1, \delta)$, where $\delta < n - 1$. Then*

(i) *If G is a DAS-graph, so is* $\mathrm{bra}(G)$.
(ii) *Let H be any graph A-cospectral to G with $\delta(H) = \delta$. If* $\mathrm{bra}(G)$ *is a DAS-graph, so is G.*

P r o o f. Since $G = \ker(G) \triangledown \mathrm{bra}(G)$, (i) follows from Corollary 2.2. Now we show (ii). Let H be any graph A-cospectral to G. Since $\delta(H) = \delta$, from Theorem 3.1 it follows that $\ker(G) = \ker(H) = K_t$ and that $\mathrm{bra}(G) = B_1$ and $\mathrm{bra}(H) = B_2$ are $(\delta - t)$-regular graphs. By $t + n(B_1) = n(G) = n(H) = t + n(B_2)$ we have $n(B_1) = n(B_2)$. By (1) we get that

$$\varphi(G) = \varphi(K_t \triangledown B_1) = \frac{\varphi(K_t)\varphi(B_1)}{(\lambda - t + 1)(\lambda - \delta + t)}[(\lambda - t + 1)(\lambda - \delta + t) - tn(B_1)]$$

and

$$\varphi(H) = \varphi(K_t \triangledown B_2) = \frac{\varphi(K_t)\varphi(B_2)}{(\lambda - t + 1)(\lambda - \delta + t)}[(\lambda - t + 1)(\lambda - \delta + t) - tn(B_1)].$$

From $\varphi(G) = \varphi(H)$ we obtain that $\varphi(B_1) = \varphi(B_2)$. Since B_1 is a DAS-graph, then $B_2 \cong B_1$ and thus $H \cong G$. □

Theorem 3.3 provides a possible method how to construct the DAS-multicone graphs. The crucial point is that we need to show that any graph H A-cospectral to a multicone graph G must satisfy $\delta(H) = \delta(G)$. Unfortunately, it is not a fact in general. Here is a counterexample that $G = K_1 \triangledown 9K_1$ with $\delta(G) = 1$ and its A-cospectral graph is $H = 4K_1 \cup K_{3,3}$ with $\delta(H) = 0$. In fact, we have $\delta(G) \geqslant \delta(H)$, since the bound in Theorem 2.1 is a decreasing function of δ (see [10]). However, some computer calculations shows that the following statement may be true:

Conjecture 1. Let $G \in \mathscr{B}(n-1, \delta)$, where $1 < \delta < n - 1$. Then any graph A-cospectral to G has the minimum degree δ.

When $\delta = 2$, there is a famous graph $F_{n,t}$ with order n named *friendship graph* which consists of t triangles intersecting in a single vertex and is well-known for the *friendship theorem* (see [6], [15]). As a matter of fact, $F_{n,k} = K_1 \triangledown tK_2$. The authors [14] have proved it to be determined by the signless Laplacian spectrum and proposed

Conjecture 2. The friendship graph is a DAS-graph.

D. Cvetković et al. [2] introduced the *generalized cocktail party graph*, denoted by GCP, which is isomorphic to a clique with independent edges removed. Note that any vertex in GCP is of degree $n-1$ or $n-2$. The following lemma indicates that the GCP is a special multicone graph.

Lemma 4.1. *A graph G with order n is a GCP iff $G = K_{n-2k} \triangledown CP(k)$, where $k \geqslant 1$ and n is even.*

P r o o f. The sufficiency follows from the fact that $CP(k)$ is a $(2k-2)$-regular graph. For the necessity, by the definition of GCP we know that G is a bidegreed graph with vertex degree $n-1$ or $n-2$. Suppose that G has $n-t$ vertices of degree $n-1$. Clearly, $t \geqslant 2$. Thus, $G = K_{n-t} \triangledown H$, where H is a $(t-2)$-regular graph. Since the size $m(H) = t(t-2)/2$ is an integer, t is even. Set $t = 2k$. Note that the order $n(H) \neq t$ and H is $(t-2)$-regular. Therefore, H is obtained from K_t by removing $t/2$ mutually disjoint edges. So, $H = CP(k)$. □

Theorem 4.1. *Let G be a graph with order n. If G has t vertices of degree $n-1$, then*

$$\frac{t-1+\sqrt{(t-1)^2+4t(n-t)}}{2} \leqslant \varrho(G) \leqslant \frac{n-3+\sqrt{(n-1)^2+4t}}{2},$$

where the left and right equalities hold if and only if G is respectively $K_t \triangledown (n-t)K_1$ and $K_t \triangledown CP((n-t)/2)$.

P r o o f. We first show the right inequality. Partition $V(G)$ into $V_1 \cup V_2$ such that $V_1 = \{v\colon\ d(v) = n-1\}$ and $V_2 = \{u\colon\ d(u) < n-1\}$. Then $A(G)$ can be written as

$$A(G) = \begin{pmatrix} J-I & J \\ J & A(H) \end{pmatrix},$$

where $H = G[V_2]$, and I and J denote respectively the identity matrix and the all ones matrix. Since $d_H(u) \leqslant n-t-2$, we have $A(H)\mathbf{1} \leqslant (n-t-2)\mathbf{1}$, where $\mathbf{1}$ denotes the all one vector. Thus, $\varrho(G)$ does not exceed the Perron root of the 2×2 matrix

$$\begin{pmatrix} t-1 & n-t \\ t & n-t-2 \end{pmatrix}.$$

A direct calculation gives the Perron root $\frac{1}{2}\bigl(n-3+\sqrt{(n-1)^2+4t}\bigr)$.

It is easy to observe that the equality holds if and only if $d_H(u) = n-t-2$ for $u \in V_2$ and the vertex in V_1 (or V_2) has exactly $n-t$ (or t) neighbors in V_2 (or V_1). Thus, the degree of each vertex is at least $n-2$. Hence G is a GCP and the result follows from Lemma 4.1.

For the left inequality, consider $d_H(u) \geqslant 0$ and note that $\varrho(G)$ is no less than the Perron root of the 2×2 matrix

$$\begin{pmatrix} t-1 & n-t \\ t & 0 \end{pmatrix}.$$

Observe that the equality holds if and only if $d_H(u) = 0$ for $u \in V_2$ and the vertex in V_1 (or V_2) has exactly $n-t$ (or t) neighbors in V_2 (or V_1). Thus, $H = (n-t)K_1$ and so $G = K_t \triangledown (n-t)K_1$. $\square$

In what follows, let G be a multicone graph from Theorem 4.1. Then $1 \leqslant t < n$. Let $h(t)$ denote the right bound in the above theorem. The graph achieving $h(t)$ is $G = K_t \triangledown CP((n-t)/2)$. Differentiating $h(t)$ with respect to t we obtain $h'(t) > 0$. So, $h(t)$ is a strictly increasing function of t. Note that $(n-t)/2$ is an integer. Hence, the maximum of t is $n-2$ and so the maximum of $h(t)$ is $h(n-2)$ and $G = K_{n-2} \triangledown CP(1)$. On the other hand, the graph achieving the left bound in Theorem 4.1 is $G = K_t \triangledown (n-t)K_1$. Note that $\varrho(G)$ increases strictly if we increase any element of $A(G)$. Thus, $\varrho(K_1 \triangledown (n-1)K_1)$ is the minimum. So we have shown

Corollary 4.1. *For any multicone graph $G \in \mathscr{B}(n-1, \delta)$,*

$$\sqrt{n-1} \leqslant \varrho(G) \leqslant \frac{n-3+\sqrt{(n+1)^2-8}}{2},$$

where the left and right equalities hold if and only if G is respectively $K_1 \triangledown (n-1)K_1$ and $K_{n-2} \triangledown CP(1)$.

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References

[1] *D . Cvetković, M. Doob, H. Sachs*: Spectra of Graphs. Theory and Applications. 3rd revised a. enl. ed.. J. A. Barth Verlag, Leipzig, 1995.
[2] *D. Cvetković, M. Doob, S. Simić*: Generalized line graphs. J. Graph Theory *5* (1981), 385–399.
[3] *E. R. van Dam, W. H. Haemers*: Which graphs are determined by their spectrum? Linear Algebra Appl. *373* (2003), 241–272.
[4] *E. R. van Dam, W. H. Haemers*: Developments on spectral characterizations of graphs. Discrete Math. *309* (2009), 576–586.
[5] *C. D. Godsil, B. D. McKay*: Constructing cospectral graphs. Aequationes Math. *25* (1982), 257–268.

[6] *P. Erdős, A. Rényi, V. T. Sós*: On a problem of graph theory. Stud. Sci. Math. Hung. *1* (1966), 215–235.
[7] *Hs. H. Günthard, H. Primas*: Zusammenhang von Graphtheorie und Mo-Theorie von Molekeln mit Systemen konjugierter Bindungen. Helv. Chim. Acta *39* (1956), 1645–1653.
[8] *W. H. Haemers, E. Spence*: Enumeration of cospectral graphs. Eur. J. Comb. *25* (2004), 199–211.
[9] *F. Harary, C. King, A. Mowshowitz, R. Read*: Cospectral graphs and digraphs. Bull. Lond. Math. Soc. *3* (1971), 321–328.
[10] *Y. Hong, J.-L. Shu, K. Fang*: A sharp upper bound of the spectral radius of graphs. J. Comb. Theory, Ser. B *81* (2001), 177–183.
[11] *C. R. Johnson, M. Newman*: A note on cospectral graphs. J. Comb. Theory, Ser. B *28* (1980), 96–103.
[12] *V. Nikiforov*: Some inequalities for the largest eigenvalue of a graph. Comb. Probab. Comput. *11* (2002), 179–189.
[13] *B. Zhou, H. H. Cho*: Remarks on spectral radius and Laplacian eigenvalues of a graph. Czech. Math. J. *55 (130)* (2005), 781–790.
[14] *J. F. Wang, F. Belardo, Q. X. Huang, B. Borovićanin*: On the two largest Q-eigenvalues of graphs. Discrete Math. *310* (2010), 2858–2866.
[15] *H. S. Wilf*: The friendship theorem. Combinatorial Mathematics and Its Applications. Proc. Conf. Math. Inst., Oxford (D. J. A. Welsh, ed.). Academic Press, New York-Lodon, 1971, pp. 307–309.

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