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# A NOTE ON TOPOLOGICAL GROUPS AND THEIR REMAINDERS

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*Abstract.* In this note we first give a summary that on property of a remainder of a non-locally compact topological group $G$ in a compactification $bG$ makes the remainder and the topological group $G$ all separable and metrizable.

If a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG \setminus G$ of $G$ belongs to $\mathscr{P}$, then $G$ and $bG \setminus G$ are separable and metrizable, where $\mathscr{P}$ is a class of spaces which satisfies the following conditions:
(1) if $X \in \mathscr{P}$, then every compact subset of the space $X$ is a $G_\delta$-set of $X$;
(2) if $X \in \mathscr{P}$ and $X$ is not locally compact, then $X$ is not locally countably compact;
(3) if $X \in \mathscr{P}$ and $X$ is a Lindelöf $p$-space, then $X$ is metrizable.

Some known conclusions on topological groups and their remainders can be obtained from this conclusion. As a corollary, we have that if a non-locally compact topological group $G$ has a compactification $bG$ such that compact subsets of $bG \setminus G$ are $G_\delta$-sets in a uniform way (i.e., $bG \setminus G$ is CSS), then $G$ and $bG \setminus G$ are separable and metrizable spaces.

In the last part of this note, we prove that if a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG \setminus G$ has a point-countable weak base and has a dense subset $D$ such that every point of the set $D$ has countable pseudo-character in the remainder $bG \setminus G$ (or the subspace $D$ has countable $\pi$-character), then $G$ and $bG \setminus G$ are both separable and metrizable.

*Keywords*: topological group, remainder, compactification, metrizable space, weak base

*MSC 2010*: 54A25, 54B05

## 1. Introduction

All spaces in this note are Tychonoff spaces unless stated otherwise, a "compactification" is a "Hausdorff compactification". A *remainder* of a space $X$ is the subspace $bX \setminus X$ of a compactification $bX$ of $X$.

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In 1958, M. Henriksen and J. R. Isbell [15] showed that a space $X$ is of countable type if and only if the remainder in any (or in some) compactification of $X$ is Lindelöf. In recent years, there are many results on topological groups and their remainders. In 2005, A. V. Arhangel'skii [2] showed that if a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG \setminus G$ has a $G_\delta$-diagonal, then $G$ is metrizable. In 2007, A. V. Arhangel'skii [3] obtained that both $G$ and $bG \setminus G$ are separable and metrizable if $G$ is a non-locally compact topological group and has a compactification $bG$ such that the remainder $bG \setminus G$ has a $G_\delta$-diagonal. Some other results on a topological group and its remainder can be found in [4], [5], [6], [7], and [18].

Most of the known results on topological groups and their remainders study the relationship between properties of topological groups and their remainders. In this note, we give a summary on what property of a remainder of a non-locally compact topological group $G$ in a compactification $bG$ makes the remainder $bG \setminus G$ and $G$ all separable and metrizable. The following is a result on it.

If a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG \setminus G$ belongs to $\mathscr{P}$, then $G$ and $bG \setminus G$ are separable and metrizable, where $\mathscr{P}$ is a class of spaces which satisfies the following conditions:

(1) if $X \in \mathscr{P}$, then every compact subset of the space $X$ is a $G_\delta$-set of $X$;

(2) if $X \in \mathscr{P}$ and $X$ is not locally compact, then $X$ is not locally countably compact;

(3) if $X \in \mathscr{P}$ and $X$ is a Lindelöf $p$-space, then $X$ is metrizable.

Some known conclusions on topological groups and their remainders can be obtained from this conclusion. As a corollary, we have that if a non-locally compact topological group $G$ has a compactification $bG$ such that compact sets of $bG \setminus G$ are $G_\delta$-sets in a uniform way (i.e., $bG \setminus G$ is CSS), then $G$ and $bG \setminus G$ are separable and metrizable spaces.

In [7] Arhangel'skii showed that if $G$ is a non-locally compact topological group, and the remainder of $G$ in a compactification $bG$ is the union of a finite collection of hereditarily $D$-spaces each of which is first countable (of countable $\pi$-character) at a dense set of points, then $G$ is metrizable. In [21] Peng proved that a space with a point-countable weak base is a $D$-space. So we will study the property of a non-locally compact topological group $G$ which has a compactification $bG$ such that the remainder $bG \setminus G$ has countable tightness and is the union of a finite collection of spaces with point-countable weak bases. The following question appears in [19].

Let $G$ be a non-locally compact topological group, if the remainder $Y = bG \setminus G$ of $G$ in a compactification $bG$ of $G$ has a point-countable weak base, are $G$ and $bG$ separable and metrizable ([19, Question 5.2])?

In the last part of this note, we prove that if a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG \setminus G$ has a point-countable weak base and has a dense subset $D$ such that every point of the set $D$ has countable pseudo-character in the remainder $bG\setminus G$ (or the subspace $D$ has countable $\pi$-character), then $G$ and $bG\setminus G$ are both separable and metrizable; if a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG\setminus G$ has countable tightness and is the union of a finite collection $\{X_i\colon\ i\leqslant n\}$ of spaces such that $X_i$ has a point-countable weak base and has a dense subspace $D_i$ which has countable $\pi$-character for each $i\leqslant n$, then $G$ is metrizable.

The set of all positive integers is denoted by $\mathbb{N}$, and $\omega$ is $\mathbb{N}\cup\{0\}$. In notions and terminology we will follow [11], [13], and [26].

## 2. On remainders of metrizable spaces

Recall that a space $X$ is of *countable type* if every compact subset $P$ of $X$ is contained in a compact subset $F\subset X$ that has a countable base of open neighborhoods in $X$ [1]. All metrizable spaces, and all locally compact Hausdorff spaces, as well as all Čech-complete spaces are of countable type [1].

Recall that a space $X$ is a *$p$-space* [1], if in any (or in some) compactification $bX$ of $X$ there exists a countable family $\{\mathscr{U}_n\}_{n\in\mathbb{N}}$ of families $\mathscr{U}_n$ of open subsets of $bX$ such that $x\in\bigcap\limits_{n\in\mathbb{N}}\operatorname{st}(x,\mathscr{U}_n)\subset X$ for each $x\in X$. It was shown in [1] that every $p$-space is of countable type, and that every metrizable space is of countable type. A. V. Arhangel'skii [1] proved that a paracompact $p$-space is a preimage of a metrizable space under a perfect mapping. A *Lindelöf $p$-space* is a preimage of a separable and metrizable space under a perfect mapping. A mapping is said to be *perfect* if it is continuous, closed and all fibers are compact.

**Lemma 2.1** ([15]). *A space $X$ is of countable type if and only if the remainder in any (or in some) compactification of $X$ is Lindelöf.*

Recall that a space $X$ is said to have a *$G_\delta$-diagonal* if the diagonal $\Delta_X=\{(x,x)\colon x\in X\}$ is the intersection of countably many of open subsets of $X\times X$. A countably compact space $X$ with a $G_\delta$-diagonal is metrizable [9].

**Lemma 2.2** ([13]). *A Lindelöf $p$-space with a $G_\delta$-diagonal is separable and metrizable.*

**Proposition 2.3.** *Let $X$ be a locally separable meta-Lindelöf space, then $X=\bigoplus\limits_{\alpha\in\Lambda}X_\alpha$, where $\{X_\alpha\colon\ \alpha\in\Lambda\}$ is a discrete family of open separable subspaces of $X$.*

**Lemma 2.4** ([2, Theorem 2.1]). *If $X$ is a Lindelöf $p$-space, then any remainder of $X$ is a Lindelöf $p$-space.*

By the proof of the last part of Theorem 5 in [3], we can get Theorem 2.5. To assist the reader, we give a proof.

**Theorem 2.5.** *If a nowhere locally compact locally separable metrizable space $X$ has a compactification $bX$ such that every compact subset of the remainder $bX \setminus X$ is a $G_\delta$-set of $bX \setminus X$ and every Lindelöf $p$-subspace of the remainder $bX \setminus X$ is metrizable, then $X$ and $bX \setminus X$ are separable and metrizable.*

Proof. Since $X$ is a locally separable metrizable space, $X = \bigoplus_{\alpha\in\Lambda} X_\alpha$ by Proposition 2.3, where $X_\alpha$ is separable and metrizable for each $\alpha \in \Lambda$. If $F$ is the set of all accumulation points for the family $\{X_\alpha\colon \alpha \in \Lambda\}$ in $bX$, then the set $F$ is a closed subset of $bX$ and $F \subset bX \setminus X$. Thus $F$ is a compact subset of $bX$.

Since every compact subset of the remainder $bX \setminus X$ is a $G_\delta$-set of $bX \setminus X$ and every Lindelöf $p$-subspace of the remainder $bX \setminus X$ is metrizable, the subspace $F$ is a $G_\delta$-set of $bX \setminus X$ and is separable and metrizable.

Put $F = \bigcap\{O_n\colon n \in \mathbb{N}\}$, where $O_n$ is an open subset of $bX \setminus X$ for each $n \in \mathbb{N}$. Denote $M = (bX \setminus X) \setminus F = \bigcup\{A_n\colon n \in \mathbb{N}\}$, where $A_n = (bX \setminus X) \setminus O_n$ for each $n \in \mathbb{N}$. Thus the set $A_n$ is a closed subset of $bX \setminus X$. $X$ is a metrizable space, hence $X$ is of countable type. So $bX \setminus X$ is Lindelöf by Lemma 2.1. Thus the subspace $A_n$ is Lindelöf for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ and for each $y \in A_n$, there exists an open subset $U_y$ of $bX$ such that $y \in U_y$ and $\overline{U_y} \cap F = \emptyset$. So there exists $m_y \in \mathbb{N}$ such that $U_y \cap X = \bigcup\{U_y \cap X_{\alpha_i}\colon i \leqslant m_y, \alpha_i \in \Lambda\}$. If we let $P = \bigcup\{X_{\alpha_i}\colon i \leqslant m_y, \alpha_i \in \Lambda\}$, then $U_y \cap X \subset P$. Since $\overline{U_y \cap X} = \overline{U_y}$ and $\overline{U_y \cap X} \subset \overline{P}$, $\overline{U_y} \subset \overline{P}$.

The set $P$ is a separable and metrizable subspace of $X$, hence $\overline{P} \setminus P$ is a Lindelöf $p$-space by Lemma 2.4. Thus $\overline{P} \setminus P$ is a separable and metrizable space, and so is the set $U_y \cap (bX \setminus X)$. Thus the subspace $A_n$ is Lindelöf and every point of $A_n$ has a neighborhood which has a countable base, hence the subspace $A_n$ has a countable network for each $n \in \mathbb{N}$. The subspace $(bX \setminus X) \setminus F$ has a countable network and the subspace $F$ has a countable network, so $bX \setminus X$ has a countable network. Thus $bX \setminus X$ is separable, hence the Souslin number of $bX \setminus X$ is countable. So the Souslin numbers of $bX$ and $X$ are both countable. Thus $X$ is separable and metrizable. So $bX \setminus X$ is a Lindelöf $p$-space by Lemma 2.4. In addition, $bX \setminus X$ has a countable network, so it has a $G_\delta$-diagonal. Thus $bX \setminus X$ is separable and metrizable by Lemma 2.2. □

**Lemma 2.6.** *If $X$ is a regular space and $X$ has a $G_\delta$-diagonal, then every compact subset of $X$ is a $G_\delta$-set of $X$.*

P r o o f. $X$ has a $G_\delta$-diagonal, thus there is a sequence $\{\mathscr{U}_n\colon n \in \mathbb{N}\}$ of open covers of $X$ such that for any distinct points $x$ and $y$ of $X$ there is $n \in \mathbb{N}$ such that $x \notin \mathrm{st}(y, \mathscr{U}_n)$, where $\mathrm{st}(y, \mathscr{U}_n) = \bigcup\{U\colon y \in U \text{ and } U \in \mathscr{U}_n\}$. Let $C$ be any compact subset of $X$. In what follows we show that the set $C$ is a $G_\delta$-set of $X$.

For each $m \in \mathbb{N}$ there are $n_m \in \mathbb{N}$ and an open subset $V(m,i)$ of $X$ for each $i \leqslant n_m$ such that $C \subset \bigcup\{V(m,i)\colon i \leqslant n_m\}$, $V(m,i) \cap C \neq \emptyset$, and there are $U(m,i) \in \mathscr{U}_m$ and some $j \leqslant n_{m-1}$ such that $V(m,i) \subset U(m,i)$ and $\overline{V(m,i)} \subset V(m-1,j)$.

Suppose there is a point $y \in \bigcap\{\bigcup\{V(m,i)\colon i \leqslant n_m\}\colon m \in \mathbb{N}\} \setminus C$. For each $m \in \mathbb{N}$ and for each $j \leqslant m$ there is $i_j^m \leqslant n_j$ such that $y \in V(j, i_j^m) \subset \overline{V(j, i_j^m)} \subset V(j-1, i_{j-1}^m)$. Since $\{V(m,i)\colon i \leqslant n_m\}$ is a finite family for each $m \in \mathbb{N}$, there is $i_m \leqslant n_m$ such that $y \in V(m, i_m) \subset \overline{V(m, i_m)} \subset V(m-1, i_{m-1})$ for each $m \in \mathbb{N}$ by König's Lemma. Thus $\bigcap\{V(m,i_m)\colon m \in \mathbb{N}\} \cap C = \bigcap\{\overline{V(m,i_m)}\colon m \in \mathbb{N}\} \cap C \neq \emptyset$. Let $x \in \bigcap\{V(m,i_m)\colon m \in \mathbb{N}\} \cap C$. Since the point $y \in V(m,i_m)$ and $V(m,i_m) \subset U(m,i_m)$ for each $m \in \mathbb{N}$, $y \in \mathrm{st}(x, \mathscr{U}_m)$ for each $m \in \mathbb{N}$. This contradicts $y \notin \bigcap\{\mathrm{st}(x, \mathscr{U}_m)\colon m \in \mathbb{N}\}$.

So $\bigcap\{\bigcup\{V(m,i)\colon i \leqslant n_m\}, m \in \mathbb{N}\} = C$, hence $C$ is a $G_\delta$-set of $X$. $\square$

By Lemma 2.2, Theorem 2.5, and Lemma 2.6, we get a corollary.

**Corollary 2.7.** *Let $X$ be a nowhere locally compact locally separable metrizable space. If $X$ has a compactification $bX$ such that the remainder $bX \setminus X$ has a $G_\delta$-diagonal, then both $X$ and $bX \setminus X$ are separable and metrizable.*

**Lemma 2.8** ([3, Proposition 4]). *Let $X$ be a nowhere locally separable metrizable space and let $bX$ be a compactification of $X$. If $\mathscr{B} = \bigcup\{\mathscr{B}_n\colon n \in \omega\}$ is a base of $X$ such that each family $\mathscr{B}_n$ is discrete in $X$, then $Z = \bigcup\{F_n\colon n \in \omega\}$ is dense in $Y = bX \setminus X$ and $F_n$ is compact for each $n$, where $F_n$ is the set of all accumulation points for $\mathscr{B}_n$ in $bX$ for each $n$.*

Let us recall that a topological space $X$ is *homogeneous* if for any two points $a, b \in X$ there exists a homeomorphism $f\colon X \to X$ such that $f(a) = b$.

**Theorem 2.9.** *Let $X$ be a nowhere locally compact homogeneous metrizable space and let $bX$ be a compactification of $X$ such that every compact subset of the remainder $Y = bX \setminus X$ is metrizable, then $X$ is locally separable.*

P r o o f. Suppose $X$ is not locally separable. Since $X$ is homogeneous, the space $X$ is nowhere locally separable if $X$ is not locally separable. $X$ is a metrizable space, there exists a $\sigma$-discrete base $\mathscr{B} = \bigcup\{\mathscr{B}_n \colon n \in \mathbb{N}\}$ of $X$. For each $n \in \mathbb{N}$, denote by $F_n$ the set of all accumulation points for $\mathscr{B}_n$ in $bX$. The set $F_n$ is a closed subset of $bX$ and $F_n \subset Y$. The set $F_n$ is a compact subset of $bX \setminus X$, so $F_n$ is separable and metrizable. Thus $bX \setminus X$ is separable by Lemma 2.8. Thus the Souslin number of $bX$ and $X$ are all countable, and hence $X$ is separable. A contradiction. Thus $X$ is locally separable. □

## 3. A general result on topological groups and their remainders

By some known conclusions, we will give a more general result on the metrizable property of a non-locally compact topological group $G$ and its remainder.

By the proof of Case 2 of Theorem 4.19 in [2], we can get the following lemma.

**Lemma 3.1.** *Let $G$ be a non-locally compact topological group. If $G$ is a paracompact $p$-space and has a compatification $bG$ such that every compact subset of the remainder $Y = bG \setminus G$ is metrizable, then $G$ is a metrizable space.*

**Theorem 3.2.** *Let $G$ be a non-locally compact topological group. If $G$ is a paracompact $p$-space and has a compatification $bG$ such that every compact subset of the remainder $Y = bG \setminus G$ is metrizable, then $G$ is a locally separable and metrizable space.*

Proof. $G$ is a metrizable space by Lemma 3.1. By Proposition 1.1 in [26] every topological group $G$ is homogeneous. Thus $G$ is a locally separable by Theorem 2.9. □

Recall that a family $\mathscr{U}$ of non-empty open subsets of a space $X$ is called a $\pi$-*base* of a point $x \in X$, if for any non-empty open subset $V$ of $X$ there is $U \in \mathscr{U}$ such that $U \subset V$. The $\pi$-character of $x$ in $X$ is defined by $\pi_\chi(x, X) = \min\{|\mathscr{U}| \colon \mathscr{U}$ is a $\pi$-base of the point $x\}$. If $\sup\{\pi_\chi(x, X) \colon x \in X\}$ is countable, then $X$ is called to have *countable $\pi$-character*.

**Lemma 3.3.** *Let $Y$ be a dense subspace of a regular space $X$. If the subspace $Y$ is first countable (or has countable $\pi$-character), then every point of $Y$ has a countable open neighborhood base (or has a countable $\pi$-base) in $X$, and if $x$ is an accumulation point of a countable subset $C$ of $Y$ then the point $x$ has a countable $\pi$-base in $X$.*

Proof. We only prove the case of the space $Y$ being first countable. The proof of the case that $Y$ has countable $\pi$-character is similar.

For any $y \in Y$ we let $\{V_n(y)\colon n \in \mathbb{N}\}$ be a countable open neighborhood base of the point $y$ in $Y$. For each $n \in \mathbb{N}$ there is an open neighborhood $U_n(y)$ of $y$ in $X$ such that $U_n(y) \cap Y = V_n(y)$. If $O$ is an open neighborhood of the point $y$ in $X$, then there is an open subset $O_1$ of $X$ such that $y \in O_1 \subset \overline{O_1} \subset O$ by the regularity property of $X$. So there is $n \in \mathbb{N}$ such that $y \in V_n(y) \subset O_1$, hence $\overline{V_n(y)} \subset \overline{O_1} \subset O$. Since $\overline{V_n(y)} = \overline{U_n(y)}$, the set $\overline{U_n(y)} \subset \overline{O_1} \subset O$. Thus $\{U_n(y)\colon n \in \mathbb{N}\}$ is a countable open neighborhood base of the point $y$ in $X$.

Let $x$ be an accumulation point of a countable subset $C$ of $Y$. If $W$ is an open neighborhood of the point $x$ in $X$, then there are $y \in C$ and $n \in \mathbb{N}$ such that $y \in U_n(y) \subset W$. So $\{U_n(y)\colon n \in \mathbb{N},\ y \in C\}$ is a countable $\pi$-base of the point $x$ in $X$. □

Recall that a point $x$ of a space $X$ is said to have *countable pseudo-character in* $X$ if the set $\{x\}$ is the intersection of countably many open subsets of $X$. A space $X$ is said to have *countable pseudo-character*, if every point of $X$ has countable pseudo-character in $X$.

**Lemma 3.4** ([4, Theorem 5.1]). *Suppose that $G$ is a topological group with a remainder of countable pseudo-character. Then at least one of the following conditions is satisfied:*
(1) *$G$ is a paracompact $p$-space;*
(2) *the remainder $bG \setminus G$ is first countable.*

**Lemma 3.5** ([6, Proposition 1.3]). *Let $G$ be a topological group. If some point of $G$ has a countable $\pi$-base, then $G$ is metrizable.*

**Lemma 3.6.** *If a non-locally compact topological group $G$ has a compatification $bG$ such that the remainder $Y = bG \setminus G$ has countable pseudo-character, $Y$ is not locally countably compact, and every compact subset of $Y$ is metrizable, then $G$ is a locally separable and metrizable space.*

P r o o f. By Lemma 3.4 $G$ is a paracompact $p$-space or the remainder $bG \setminus G$ is first countable. If $G$ is a paracompact $p$-space, then $G$ is a locally separable and metrizable space by Theorem 3.2. Since $Y$ is not locally countably compact, the space $Y$ is not countably compact. There is a countable infinite subset $C \subset Y$ such that $\overline{C} \cap G \neq \emptyset$. If the remainder $bG \setminus G$ is first countable and $x \in \overline{C} \cap G$, then the point $x$ has a countable $\pi$-base in $bG$ by Lemma 3.3, hence the point $x$ has a countable $\pi$-base in $G$. Thus $G$ is metrizable by Lemma 3.5. So $G$ is a locally separable and metrizable space by Theorem 3.2 □

**Theorem 3.7.** *If a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG \setminus G$ belongs to $\mathscr{P}$, then $G$ and $bG \setminus G$ are separable and metrizable spaces, where $\mathscr{P}$ is a class of spaces which satisfies the following conditions:*

(1) *if $X \in \mathscr{P}$, then every compact subset of the space $X$ is a $G_\delta$-set of $X$;*

(2) *if $X \in \mathscr{P}$ and $X$ is not locally compact, then $X$ is not locally countably compact;*

(3) *if $X \in \mathscr{P}$ and $X$ is a Lindelöf $p$-space, then $X$ is metrizable.*

P r o o f. Since $bG \setminus G$ has property $\mathscr{P}$, every compact subset of $bG \setminus G$ is a $G_\delta$-set of $bG \setminus G$ by the condition (1) and is metrizable by the condition (3). The remainder $Y = bG \setminus G$ is not locally compact, thus it is not locally countably compact by the condition (2). By the condition (1) the remainder $Y = bG \setminus G$ has countable pseudo-character. So the conditions of Lemma 3.6 are satisfied, hence $G$ is locally separable and metrizable. Thus $G$ and $bG \setminus G$ are separable and metrizable spaces by Theorem 2.5. □

A space $X$ is said to have a *locally $G_\delta$-diagonal* if every point $x$ of $X$ has a neighborhood $V_x$ which has a $G_\delta$-diagonal.

**Lemma 3.8.** *If $X$ has a locally $G_\delta$-diagonal, then every compact subset of $X$ is a $G_\delta$-set of $X$.*

P r o o f. Let $C$ be any compact subset of $X$. For each $x \in C$ there is an open neighborhood $V_x$ of $x$ such that $V_x$ has a $G_\delta$-diagonal. The set $C$ is compact, there are some $n \in \mathbb{N}$ and a point $x_i$ for each $i \leqslant n$ such that $C \subset \bigcup\{V_{x_i} \colon i \leqslant n\} = Y$. Since $\mathscr{P} = \{V_{x_i} \colon i \leqslant n\}$ is a finite open cover of the subspace $Y$ and each element of $\mathscr{P}$ has a $G_\delta$-diagonal, the subspace $Y$ has a $G_\delta$-diagonal by Lemma 11 in [18].

Thus the set $C$ is a $G_\delta$-set of $Y$ by Lemma 2.6, hence it is a $G_\delta$-set of $X$. □

In what follows, we denote $\mathscr{P}$ by a class of spaces which satisfies the conditions appearing in Theorem 3.7. By Lemma 2.2 and 2.6, we know that if a space $X$ has a $G_\delta$-diagonal then $X \in \mathscr{P}$. By Lemma 11 in [18] we know that if $X$ is a regular Lindelöf space with a locally $G_\delta$-diagonal then $X$ has a $G_\delta$-diagonal. Thus by Lemma 2.2 in this note we know that every Lindelöf $p$-space with a locally $G_\delta$-diagonal is metrizable. If a space $X$ has a locally $G_\delta$-diagonal and $X$ is not locally compact then $X$ is not locally countably compact. By these conclusions and Lemma 3.8 we have that $X \in \mathscr{P}$ if $X$ has a locally $G_\delta$-diagonal.

**Corollary 3.9** ([3, Theorem 5])**.** *If a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG \setminus G$ has a $G_\delta$-diagonal, then $G$ and $bG \setminus G$ are separable and metrizable.*

**Corollary 3.10** ([6, Theorem 2.17; 18, Theorem 12]). *If a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG \setminus G$ has a locally $G_\delta$-diagonal, then $G$ and $bG \setminus G$ are separable and metrizable.*

In 1973, H. Martin introduced the class of CSS spaces [20]. Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{C}$ be the family of all non-empty compact subsets of $X$. If there exists a function $U\colon \mathbb{N} \times \mathscr{C} \longrightarrow \mathscr{T}$ such that:

(1) for every $C \in \mathscr{C}$, $C = \bigcap\{U(n, C)\colon n \in \mathbb{N}\}$ and $U(n+1, C) \subset U(n, C)$ for $n \in \mathbb{N}$;

(2) if $D \in \mathscr{C}$, $C \in \mathscr{C}$, and $C \subset D$, then $U(n, C) \subset U(n, D)$ for each $n \in \mathbb{N}$.

Then $X$ is called a *c-semi-stratifiable* (CSS) *space*.

It is obvious that every subspace of a CSS space is CSS.

**Lemma 3.11** ([8, Proposition 3.8]). *If $X$ is a* CSS *countably compact space, then $X$ is a compact metrizable space.*

**Lemma 3.12** ([8, Proposition 3.8]). *If $X$ is a* CSS *paracompact $p$-space, then $X$ is metrizable.*

**Lemma 3.13** ([23, Theorem 4]). *If $X = \bigcup\{X_n\colon n \in \mathbb{N}\}$ and $X_n$ is a closed* CSS *subspace of $X$ for each $n \in \mathbb{N}$, then $X$ is a* CSS *space.*

**Theorem 3.14.** *If a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG \setminus G$ is a locally* CSS *space, then $G$ and $bG \setminus G$ are separable and metrizable spaces.*

P r o o f. By Lemma 3.11 we know that if a locally CSS space $X$ is not a locally compact space then $X$ is not locally countably compact. Every regular Lindelöf locally CSS space is a CSS space by Lemma 3.13. Thus a Lindelöf locally CSS $p$-space is metrizable by Lemma 3.12.

Let $X$ be a locally CSS regular space and let $F$ be a non-empty compact subset of $X$. For each $x \in F$ there is an open neighborhood $V_x$ of $x$ such that $\overline{V_x}$ is CSS. There are $n \in \mathbb{N}$ and a point $x_i \in F$ for each $i \leqslant n$ such that $F \subset \bigcup\{V_{x_i}\colon i \leqslant n\} \subset \bigcup\{\overline{V_{x_i}}\colon i \leqslant n\}$. By Lemma 3.13 the subspace $Y = \bigcup\{\overline{V_{x_i}}\colon i \leqslant n\}$ is CSS. Thus the set $F$ is a $G_\delta$-set of $Y$, hence it is a $G_\delta$-set of $X$.

So a locally CSS space belongs to $\mathscr{P}$. Thus $G$ and $bG \setminus G$ are separable and metrizable spaces. □

Recall that a space $X$ has a *quasi-$G_\delta(2)$-diagonal* provided there is a sequence $\{\mathscr{U}_n\colon n \in \mathbb{N}\}$ of collections of open subsets of $X$ with the property that, given distinct points $x, y \in X$, there is $n \in \mathbb{N}$ with $x \in \operatorname{st}^2(x, \mathscr{U}_n) \subset X \setminus \{y\}$.

**Proposition 3.15** ([23, Theorem 9]). *If $X$ has a quasi-$G_\delta(2)$-diagonal, then $X$ is a* CSS *space.*

By Theorem 3.14 and Proposition 3.15, we can obtain:

**Corollary 3.16.** *If a non-locally compact topological group $G$ has a compactification $bG$ such that the remainder $bG \setminus G$ has a locally quasi-$G_\delta(2)$-diagonal, then $G$ and $bG \setminus G$ are separable and metrizable spaces.*

In [3, Theorem 10], it was proved that if $G$ is a non-locally compact topological group and has a compactification $bG$ such that the remainder $bG \setminus G$ has a point-countable base, then $G$ and $bG \setminus G$ are separable and metrizable. Every Lindelöf $p$-space with a point-countable base is metrizable [14]. Every countably compact space with a point-countable base is compact and metrizable [11]. We can get the following proposition.

**Proposition 3.17.** *If $X$ is a space such that every point of $X$ has an open neighborhood which has a point-countable base, then the following conclusions hold:*
(1) *$X$ has a point-countable base if $X$ is meta-Lindelöf;*
(2) *$X$ is metrizable if $X$ is a Lindelöf $p$-space;*
(3) *a subset $C$ of $X$ is a $G_\delta$-set of $X$ if the set $C$ is a compact subset of $X$;*
(4) *$X$ is not locally countably compact if $X$ is not locally compact.*

Proof. We just need to prove the item (3). Let $C$ be a compact subset of $X$. For each $x \in C$ there is an open neighborhood $V_x$ of $x$ such that the subspace $V_x$ has a point-countable base. There are $n \in \mathbb{N}$ and a point $x_i \in C$ for each $i \leqslant n$ such that $C \subset \bigcup\{V_{x_i}\colon i \leqslant n\}$. If $Y = \bigcup\{V_{x_i}\colon i \leqslant n\}$, then the subspace $Y$ has a point-countable base $\mathscr{B}$. Thus $C$ is metrizable. The subspace $C$ is separable, since $C$ is compact and metrizable. Let $D$ be a countable dense subset of $C$. Thus $\mathscr{B}' = \{B\colon B \in \mathscr{B}$ and $B \cap C \neq \emptyset\}$ is countable. So $C = \bigcap\{\bigcup \mathscr{F}\colon \mathscr{F} \subset \mathscr{B}',\ C \subset \bigcup \mathscr{F}$, and $|\mathscr{F}| < \omega\}$, and hence $C$ is a $G_\delta$-set of $X$. □

By Proposition 3.17 and Theorem 3.7, we have:

**Theorem 3.18.** *If $G$ is a non-locally compact topological group and has a compactification $bG$ such that every point of the remainder $bG \setminus G$ has a neighborhood in $bG \setminus G$, which has a point-countable base, then $G$ and $bG \setminus G$ are separable and metrizable.*

**Corollary 3.19** ([3, Theorem 10]). *If $G$ is a non-locally compact topological group and has a compactification $bG$ such that the remainder $bG \setminus G$ has a point-countable base, then $G$ and $bG \setminus G$ are separable and metrizable.*

## 4. Results on some remainders of topological groups with point-countable weak bases

In this part, we will mainly discuss the properties of a non-locally compact topological group $G$ which has a compactification $bG$ such that the remainder $bG \setminus G$ has a point-countable weak base and has a dense subset $D$ such that every point of the set $D$ has countable pseudo-character in the remainder $bG \setminus G$ (or the subspace $D$ has countable $\pi$-character).

Let us recall the definition of a weak base of a space $X$. A collection $\mathscr{B} = \bigcup\{\mathscr{B}_x \colon x \in X\}$ is called a *weak base* [25] of $X$, if for any $x \in X$ the following conditions hold:

(1) for each $x \in X$, $\mathscr{B}_x$ is closed under finite intersections and $x \in \bigcap \mathscr{B}_x$;
(2) a subset $U$ of $X$ is open if and only if for any $x \in U$ there is $B \in \mathscr{B}_x$ such that $x \in B \subset U$.

Recall that a space $X$ is *Fréchet* if for any point $x$ is the closure of a subset $A$ of $X$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of $A$ which converges to the point $x$. A space $X$ is *sequential* if a subset $A$ of $X$ is closed if and only if the set $A$ contains all the limit points of the convergent sequences of $A$. Let $X$ be a topological space, for a subset $A \subset X$, denote $[A]_\omega = \bigcup\{\overline{C} \colon C \subset A \text{ and } |C| \leqslant \omega\}$. Recall that a space $X$ has *countable tightness* if for any point $x$ in the closure of a subset $A$ of $X$, there is a countable subset $C \subset A$ such that $x \in \overline{C}$. We denote this by $t(X) \leqslant \omega$. It is well known that a Fréchet space is sequential and a sequential space has countable tightness.

**Lemma 4.1** ([25, Theorem 1.10]). *If $\mathscr{B} = \bigcup\{\mathscr{B}_x \colon x \in X\}$ is a weak base of a Hausdorff Fréchet space $X$, then $\mathscr{B}^* = \{B^o \colon B \in \mathscr{B}\}$ is a base of $X$.*

**Lemma 4.2** ([21, Corollary 8]). *If $X$ is a countably compact Hausdorff space with a point-countable weak base, then $X$ is a compact metrizable space.*

**Lemma 4.3** ([17, Lemma 2.1]). *If $\mathscr{P} = \bigcup\{\mathscr{P}_x \colon x \in X\}$ is a weak base of a space $X$ and $F$ is a closed subset of $X$, then $\mathscr{P}' = \bigcup\{\mathscr{P}'_x \colon x \in F\}$ is a weak base of the subspace $F$, where $\mathscr{P}'_x = \{F \cap P \colon P \in \mathscr{P}_x\}$ for each $x \in F$.*

**Proposition 4.4.** *Let $X$ be a $T_1$-space and let $A$ be a subset of $X$. If $t(X) \leqslant \omega$, then the set $[A]_\omega$ is a closed subset of $X$; if $X$ is countably compact, then the set $[A]_\omega$ is countably compact.*

Proof. Suppose $t(X) \leqslant \omega$. If $x \in \overline{[A]_\omega}$, then there is a countable subset $B \subset [A]_\omega$ such that $x \in \overline{B}$. For each $b \in B$ there is a countable set $C_b \subset A$ such that $b \in \overline{C_b}$. So $x \in \overline{\bigcup\{C_b\colon\ b \in B\}} \subset [A]_\omega$. Thus $[A]_\omega$ is a closed subset of $X$ if $t(X) \leqslant \omega$.

Suppose $X$ is countably compact. For any infinite countable subset $B$ of $[A]_\omega$, there is a countable subset $C \subset A$ such that $B \subset \overline{C} \subset [A]_\omega$. Thus the set $B$ has an accumulation point in $\overline{C}$, hence $[A]_\omega$ is countably compact if $X$ is countably compact. □

**Lemma 4.5.** *Let $G$ be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of $G$ in a compactification $bG$ of $G$ such that $Y = bG \setminus G$ has countable tightness. If there is an open subset $U$ of $Y$ such that every closed countably compact subset which is contained in $U$ is compact and there is a subspace $M \subset Y$ such that $U \subset \overline{M}^Y$ and $M$ has a dense subspace $D$ which has countable $\pi$-character, then $G$ is metrizable.*

Proof. Let $U_0$ be an open subset of $bG$ such that $U_0 \cap Y = U$ and let $U_1$ be an open subset of $bG$ such that $\overline{U_1} \subset U_0$, hence $\overline{U_1} \cap Y \subset U$. If $D_1 = U_1 \cap D$, then the set $D_1$ is dense in the subspace $U_1$. Denote $[D_1]_\omega = \bigcup\{\overline{C}\colon\ C \subset D_1$ and $|C| \leqslant \omega\}$. By Proposition 4.4 the set $[D_1]_\omega$ is a countably compact subspace of $bG$. Suppose $[D_1]_\omega \cap G = \emptyset$, then $[D_1]_\omega \subset \overline{U_1} \cap Y \subset U$. By Proposition 4.4 the set $[D_1]_\omega$ is closed in the subspace $Y$. Since $[D_1]_\omega \subset U$, the set $[D_1]_\omega$ is compact. Thus $\overline{U_1} = \overline{D_1} \subset [D_1]_\omega \subset Y$. This contradicts $U_1 \cap G \neq \emptyset$, so $[D_1]_\omega \cap G \neq \emptyset$. If $x \in [D_1]_\omega \cap G$, then there is a countable subset $C \subset D_1$ such that $x \in \overline{C}$.

The set $U_0 \cap D$ is dense in $U_0$, since $U \subset \overline{M}^Y$ and $D$ is a dense subset of $M$. The subspace $U_0 \cap D$ is an open subspace of $D$, the subspace $U_0 \cap D$ has countable $\pi$-character. Thus every point of $U_0 \cap D$ has a countable $\pi$-base in $U_0$ by Lemma 3.3. The point $x \in \overline{C} \subset \overline{U_1} \subset U_0$. For each $z \in C$ let $\mathscr{V}_z$ be a countable $\pi$-base of the point $z$ in $U_0$. If $\mathscr{B} = \bigcup\{\mathscr{V}_z\colon\ z \in C\}$, then $\mathscr{B}$ is a countable family of open subsets of $U_0$. Thus $\{B \cap G\colon\ B \in \mathscr{B}\}$ is a countable $\pi$-base of the point $x$ in $G$. Thus $G$ is metrizable by Lemma 3.5. □

**Corollary 4.6.** *Let $G$ be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of $G$ in a compactification $bG$ of $G$ such that $Y = bG \setminus G$ has countable tightness. If there is a point $y \in Y$ and an open neighborhood $U(y)$ of $y$ in $Y$ such that every closed countably compact subset which is contained in $U(y)$ is compact and $U(y)$ has a dense subspace $D$ which has countable $\pi$-character, then $G$ is metrizable.*

By the proof of Theorem 5.1 in [4], we have:

**Lemma 4.7.** *If a non-locally compact topological group $G$ has a compatification $bG$ such that the remainder $Y = bG \setminus G$ has a point $y$ which has countable pseudo-character in $Y$, then $G$ is a paracompact $p$-space or the point $y$ has a countable open neighborhood base in $bG$.*

**Theorem 4.8.** *Let $G$ be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of $G$ in a compactification $bG$ of $G$ such that every compact subset of $Y$ is metrizable and $Y$ has countable tightness. If there is a point $y \in Y$ and an open neighborhood $U(y)$ of $y$ in $Y$ such that every closed countably compact subset which is contained in $U(y)$ is compact and there is a dense subspace $D$ of $U(y)$ such that every point of $D$ has countable pseudo-character in $Y$ (or the subspace $D$ has countable $\pi$-character), then $G$ is locally separable and metrizable.*

P r o o f. If there is a dense subspace $D$ of $U(y)$ such that every point $d$ of $D$ has countable pseudo-character in $Y$, then $G$ is a paracompact $p$-space or every point $d$ of $D$ has a countable open neighborhood base in $bG$ by Lemma 4.7. If $G$ is a paracompact $p$-space, then $G$ is a locally separable and metrizable space by Lemma 3.2. If every point $d$ of $D$ has a countable open neighborhood base in $bG$, then the subspace $D$ has countable $\pi$-character. If the subspace $D$ has countable $\pi$-character, then $G$ is metrizable by Corollary 4.6, hence $G$ is a locally separable and metrizable space by Lemma 3.2. □

Recall that a *neighborhood assignment* for a space $X$ is a function $\varphi$ from $X$ to the topology of the space $X$ such that $x \in \varphi(x)$ for any $x \in X$. A space $X$ is called a *$D$-space* if for any neighborhood assignment $\varphi$ for $X$ there exists a closed discrete subset $D$ of $X$ such that $X = \bigcup\{\varphi(d)\colon d \in D\}$ [10]. Every metrizable space is a $D$-space.

**Lemma 4.9** ([12], [22]). *If $X$ is a countably compact space that is the union of a countable family of $D$-spaces, then $X$ is compact.*

**Theorem 4.10.** *Let $G$ be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of $G$ in a compactification $bG$ of $G$ such that $Y = bG \setminus G$ has countable tightness. If the remainder $Y$ is the union of a countable family $\{X_i\colon i \in \mathbb{N}\}$ of $D$-spaces such that for each $i \in \mathbb{N}$ there is a dense subspace $D_i$ of $X_i$ such that the subspace $D_i$ has countable $\pi$-character (or every point of $D_i$ has countable pseudo-character in $Y$), then $G$ is a paracompact $p$-space.*

P r o o f. If there is a dense subspace $D_i$ of $X_i$ such that every point of $D_i$ has countable pseudo-character in $Y$ for each $i \in \mathbb{N}$, then $G$ is a paracompact $p$-space or every point $y$ of $D_i$ has a countable open neighborhood base in $bG$ by Lemma 4.7.

If every point $y$ of $D_i$ has a countable open neighborhood base in $bG$ for each $i \in \mathbb{N}$, then the subspace $D_i$ has countable $\pi$-character. In what follows, we show that $G$ is a paracompact $p$-space if there is a dense subspace $D_i$ in $X_i$ such that the subspace $D_i$ has countable $\pi$-character for each $i \in \mathbb{N}$.

Since every closed subspace of a $D$-space is a $D$-space, every closed countably compact subspace of $Y$ is compact by Lemma 4.9.

If there is some $i \in \mathbb{N}$ and an open subset $U$ of $Y$ such that $U \subset \overline{X_i}^Y$, then $G$ is metrizable by Lemma 4.5, otherwise, $X_i$ is a nowhere dense subset of $Y$ for each $i \in \mathbb{N}$. For $i \in \mathbb{N}$, assuming that there is an open subset $U_j$ of $bG$ for each $j \leqslant i$ such that $\overline{U_j} \subset U_{j-1} (U_0 = bG)$, $U_j \subset bG \setminus \bigcup\{\overline{X_m} \colon m \leqslant j\}$, and $U_j \cap Y \neq \emptyset$.

The set $(U_i \setminus \overline{X_{i+1}}) \cap Y \neq \emptyset$, there is an open subset $U_{i+1}$ of $bG$ such that $\overline{U_{i+1}} \subset U_i$ and $U_{i+1} \cap Y \neq \emptyset$. Thus $U_{i+1} \subset bG \setminus \bigcup\{\overline{X_m} \colon m \leqslant i+1\}$. So we have a sequence $\{U_i \colon i \in \mathbb{N}\}$ of open subsets of $bG$ such that $\overline{U_{i+1}} \subset U_i$ and $U_i \subset bG \setminus \bigcup\{\overline{X_m} \colon m \leqslant i\}$. Thus $E = \bigcap\{\overline{U_i} \colon i \in \mathbb{N}\} = \bigcap\{U_i \colon i \in \mathbb{N}\} \neq \emptyset$, and $E \subset G$. Thus the family $\{U_i \cap G \colon i \in \mathbb{N}\}$ is a countable base of open neighborhoods of the set $E$ in $G$. Every topological group that contains a non-empty compact subset with a countable base of open neighborhoods is a paracompact $p$-space [24]. Thus $G$ is a paracompact $p$-space. □

In [21] Peng proved that every space with a point-countable weak base is a $D$-space.

**Corollary 4.11.** *Let $G$ be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of $G$ in a compactification $bG$ of $G$ such that $Y = bG \setminus G$ has countable tightness. If the remainder $Y$ is the union of a countable family $\{X_i \colon i \in \mathbb{N}\}$ of spaces such that for each $i \in \mathbb{N}$ the space $X_i$ has a point-countable weak base and there is a dense subspace $D_i$ of $X_i$ such that the subspace $D_i$ has countable $\pi$-character (or every point of $D_i$ has countable pseudo-character in $Y$), then $G$ is a paracompact $p$-space.*

**Theorem 4.12.** *Let $G$ be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of $G$ in a compactification $bG$ of $G$ such that $Y = bG \setminus G$ has countable tightness. If the remainder $Y$ is the union of a finite family $\{X_i \colon i \leqslant n\}$ of $D$-spaces such that for each $i \leqslant n$ there is a dense subspace $D_i$ of $X_i$ such that the subspace $D_i$ has countable $\pi$-character, then $G$ is metrizable.*

P r o o f. Since every closed subspace of a $D$-space is a $D$-space, every closed countably compact subspace of $Y$ is compact by Lemma 4.9. Since the family $\{X_i \colon$

$i \leqslant n\}$ is finite and is a cover of $bG \setminus G$, there are an open subset $U \subset Y$ and some $i \leqslant n$ such that $U \subset \overline{X_i}$, hence $G$ is metrizable by Lemma 4.5. □

**Corollary 4.13.** *Let $G$ be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of $G$ in a compactification $bG$ of $G$ such that $Y = bG \setminus G$ has countable tightness. If the remainder $Y$ is the union of a finite family $\{X_i \colon i \leqslant n\}$ of spaces such that for each $i \leqslant n$ the space $X_i$ has a point-countable weak base and there is a dense subspace $D_i$ of $X_i$ such that the subspace $D_i$ has countable $\pi$-character, then $G$ is metrizable.*

**Lemma 4.14.** *Let $G$ be a non-locally compact topological group, and $bG$ be a compactification of $G$ such that the remainder $bG \setminus G$ has a point-countable weak base and has a dense subset $D$ such that every point of the set $D$ has countable pseudo-character in the remainder $bG \setminus G$ (or the subspace $D$ has countable $\pi$-character), then $G$ is locally separable and metrizable.*

P r o o f. A space with a point-countable weak base is sequential, hence it has countable tightness. Thus the remainder $bG \setminus G$ has countable tightness. If every point of $D$ has countable pseudo-character in $Y$, then $G$ is a paracompact $p$-space or every point $y$ of $D$ has a countable open neighborhood base in $bG$ by Lemma 4.7.

By Lemma 4.2 and Lemma 4.3 every compact subset of $bG \setminus G$ is metrizable. Thus $G$ is locally separable and metrizable if $G$ is a paracompact $p$-space by Lemma 3.2. If every point $y$ of $D$ has a countable open neighborhood base in $bG$, then the subspace $D$ has countable $\pi$-character. If the subspace $D$ has countable $\pi$-character in $Y$, then $G$ is metrizable by Corollary 4.13. Since $G$ is metrizable and every compact subset of $bG \setminus G$ is metrizable, $G$ is locally separable and metrizable by Lemma 3.2. □

We recall that a space is a *$M$-space* if and only if it is the inverse image of a metric space by a quasi-perfect map.

**Lemma 4.15** ([16, Corollary 13]). *Let $f \colon X \to Y$ be a closed map such that $X$ has a point-countable weak base. If $Y$ is a $M$-space, then $Y$ is metrizable.*

By Lemma 4.15, we have:

**Corollary 4.16.** *If $X$ is a Lindelöf $p$-space with a point-countable weak base, then $X$ is metrizable.*

**Theorem 4.17.** *Let $G$ be a non-locally compact topological group, and $bG$ be a compactification of $G$ such that the remainder $Y = bG \setminus G$ has a point-countable weak base and has a dense subset $D$ such that every point of the set $D$ has countable pseudo-character in the remainder $bG \setminus G$ (or the subspace $D$ has countable $\pi$-character), then $G$ and $bG \setminus G$ are separable and metrizable.*

P r o o f. $G$ is locally separable and metrizable by Lemma 4.14.

If $Y$ is a Fréchet space, then $Y$ has a point-countable base by Lemma 4.1. Thus $G$ and $bG \setminus G$ are separable and metrizable by Corollary 3.19.

Suppose $Y$ is not a Fréchet space, there exists a subset $A$ of $Y$ such that the set $B = \bigcup\{C \cup \{x_C\}\colon C$ is a convergence sequence of $A$ which converges to the point $x_C\}$ is not a closed subset of $Y$. Since $Y$ has a point-countable weak base, the space $Y$ is a sequential space. Since the set $B$ is not a closed subset of $Y$, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ of $B$ such that the sequence $\{y_n\}_{n\in\mathbb{N}}$ converges to a point $y \notin B$. For each $n \in \mathbb{N}$ the point $y_n \in B$, so there exists a sequence $\{y_{nk}\}_{k\in\mathbb{N}}$ of $A$ such that $\{y_{nk}\}_{k\in\mathbb{N}}$ converges to the point $y_n$. The point $y \notin B$, then there is no subsequence of $\{y_{nk}\colon n, k \in \mathbb{N}\}$ converging to the point $y$, otherwise $y \in B$.

$G$ is locally separable and metrizable, hence $G = \bigoplus\limits_{\alpha\in\Lambda} G_\alpha$ by Proposition 2.3, where $\{G_\alpha\colon \alpha \in \Lambda\}$ is a discrete family of separable and metrizable subspaces of $G$.

Denote by $F$ the set of all accumulation points for $\{G_\alpha\colon \alpha \in \Lambda\}$ in $bG$. Thus $F \subset Y$ and $F$ is a compact subset of $Y$. Since $Y$ has a point-countable weak base, the subspace $F$ has a point-countable weak base by Lemma 4.3 and $F$ is metrizable by Lemma 4.2.

If $\{y_{n_p}\}_{p\in\mathbb{N}}$ is a subsequence of the sequence $\{y_n\}_{n\in\mathbb{N}}$, then $\{y_{n_p}\}_{p\in\mathbb{N}}$ converges to $y$. Thus the point $y$ is in the closure of $\{y_{n_pk}\colon p \in \mathbb{N},\ k \in \mathbb{N}\}$. So the point $y$ is in the closure of $\{y_{mk}\colon m \in N_1,\ k \in \mathbb{N}\}$ if the subset $N_1$ of $\mathbb{N}$ is infinite.

Denote $L = \{m\colon m \in \mathbb{N}$ and $|\{k\colon y_{mk} \in F, k \in \mathbb{N}\}| = \omega\}$. Suppose $|L| = \omega$, then the point $y$ is in the closure of the set $\{y_{mk}\colon m \in L$ and $y_{mk} \in F\}$. Thus $y \in F$. The set $F$ is metrizable, so there is a sequence of the set $\{y_{mk}\colon m \in L$ and $y_{mk} \in F\}$ converging to the point $y$. A contradiction. Thus $|L| < \omega$.

Without loss of generality, we assume $\{y_{nk}\colon k \in \mathbb{N},\ n \in \mathbb{N}\} \subset Y \setminus F$. Then there exists an open subset $U_{nk}$ of $bG$ such that $y_{nk} \in U_{nk}$ and $\overline{U_{nk}} \cap F = \emptyset$ for each $k \in \mathbb{N}$ and for each $n \in \mathbb{N}$. Thus $|\{\alpha\colon U_{nk} \cap G_\alpha \neq \emptyset, \alpha \in \Lambda\}| < \omega$. If $U = \bigcup\{U_{nk}\colon n, k \in \mathbb{N}\}$, then $U$ is an open subset of $bG$. The set $U$ intersects with at most countably many $G_\alpha$. We denote by $U \cap G = \bigcup\{U \cap G_{\alpha_i}\colon i \in \mathbb{N}\}$. If we let $M = \bigcup\{G_{\alpha_i}\colon i \in \mathbb{N}\}$, then $M$ is separable and $U \cap G \subset M$. Since $\overline{G} = bG$, $\overline{U \cap G} = \overline{U}$. Thus $\overline{U} \subset \overline{M}$. The set $M$ is a closed subset of $G$, so $\overline{M} \setminus M \subset bG \setminus G$, hence $\overline{M} \setminus M = \overline{M} \cap (bG \setminus G)$. The set $\overline{M} \setminus M$ has a point-countable weak base by Lemma 4.3. Since $M$ is separable and metrizable, $\overline{M} \setminus M$ is a Lindelöf $p$-space. $\overline{M} \setminus M$ has a point-countable weak base, thus it is metrizable by Corollary 4.16. Since $y \in \overline{M} \setminus M$, there exists a subsequence of $\{y_{nk}\colon\ n,\ k \in \mathbb{N}\}$ which converges to $y$. Thus $y \in B$. This contradicts $y \notin B$.

Thus $Y$ is a Fréchet space, hence $G$ and $bG \setminus G$ are separable and metrizable. □

By the proof of Theroem 4.17, we have:

**Theorem 4.18.** *Let $X$ be a locally separable and metrizable space. If $bX$ is a compactification of $X$ such that every Lindelöf $p$-subspace of the remainder $bX \setminus X$ is metrizable, then the remainder $bX \setminus X$ is a Fréchet space.*

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