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CONTROLLED CONVERGENCE THEOREMS FOR HENSTOCK-KURZWEIL-PETTIS INTEGRAL ON m-DIMENSIONAL COMPACT INTERVALS

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Abstract. In this paper we use a generalized version of absolute continuity defined by *J. Kurzweil, J. Jarník*, Equiintegrability and controlled convergence of Perron-type integrable functions, Real Anal. Exch. *17* (1992), 110–139. By applying uniformly this generalized version of absolute continuity to the primitives of the Henstock-Kurzweil-Pettis integrable functions, we obtain controlled convergence theorems for the Henstock-Kurzweil-Pettis integral. First, we present a controlled convergence theorem for Henstock-Kurzweil-Pettis integral of functions defined on m-dimensional compact intervals of $\mathbb{R}^m$ and taking values in a Banach space. Then, we extend this theorem to complete locally convex topological vector spaces.

Keywords: Henstock-Kurzweil-Pettis integral, controlled convergence theorem, complete locally convex spaces, m-dimensional compact interval

MSC 2010: 28B05, 46G10

1. Introduction

We define the Henstock-Kurzweil-Pettis integral of functions defined on a non-degenerate compact interval S of $\mathbb{R}^m$, $m \geqslant 1$ and taking values in a complete locally convex space, Definition 2.4. The Henstock-Kurzweil-Pettis integral is the generalization of the Pettis integral of a function, obtained by replacing the Lebesgue integrability of scalar functions by the Henstock-Kurzweil integrability. We refer to [10], [11] and [12] for information about Pettis integrability. For the case of Banach valued function and $m = 1$, Definition 2.4 is the same of Definition 3 in [2] or Definition 2.2 in [7].

We firstly present a controlled convergence theorem for the Henstock-Kurzweil-Pettis integral of functions defined on S and taking values in a Banach space, Theorem 3.3. Then, we extend Theorem 3.3 to complete locally convex spaces, Theorem 4.2. The convergence theorems for the Henstock-Kurzweil-Pettis integral of functions defined on *one*-dimensional compact intervals and taking values in a Banach space have been shown in [1], [2], [3], and [6]; the best of them is Theorem 5 in [2].

2. Basic definitions

Throughout this paper, X denotes a real Banach space with its norm $\|.\|$ and X^* its dual. By $B(X^*)$ the closed unit ball in X^* is denoted.

For simplicity, the letters $\mathcal{HK}$ and $\mathcal{HKP}$ stand for Henstock-Kurzweil and Henstock-Kurzweil-Pettis, respectively.

By $\mathbb{N}$ the set of all positive integers is denoted. The set of all real numbers is denoted $\mathbb{R}$, and the ambient space of this paper is $\mathbb{R}^m$, where m is a fixed positive integer. In $\mathbb{R}^m$ we use the metric induced by the maximum norm. A compact interval I in $\mathbb{R}^m$ refers to a rectangle in $\mathbb{R}^m$, that is

$$I = \{(x_1, x_2, \ldots, x_m)\colon\ a_i \leqslant x_i \leqslant b_i \text{ for } i = 1, 2, \ldots, m\}.$$

Let S be a fixed compact non-degenerate interval in $\mathbb{R}^m$. We denote by $\mathscr{S}$ the family of all closed non-degenerate subintervals of S and by $\mathcal{L}_m$ the collection of all λ_m-measurable subsets of S, where λ_m stands for the Lebesgue measure in $\mathbb{R}^m$.

A function $F\colon \mathscr{S} \to X$ is said to be an interval function. The interval function F is said to be additive if $F(I \cup J) = F(I) + F(J)$ for each nonoverlapping intervals $I, J \in \mathscr{S}$ with $I \cup J \in \mathscr{S}$. We say that intervals I and J are nonoverlapping if $\operatorname{int}(I) \cap \operatorname{int}(J) = \emptyset$, where $\operatorname{int}(I)$ denotes the interior of I.

A pair (I, s) of an interval $I \in \mathscr{S}$ and a point $s \in I$ is called tagged interval, s is the tag of I. An $\mathcal{HK}$-partition π in S is a finite collection of tagged intervals (I, s) whose corresponding intervals are non overlapping. A function $\delta\colon A \to (0, +\infty)$ is said to be a gauge on A, where A is a subset of S. We say that an $\mathcal{HK}$-partition π in S is

- ▷ an $\mathcal{HK}$-partition of S if $\bigcup\limits_{(I,s)\in\pi} I = S$,
- ▷ A-tagged if for all $(I, s) \in \pi$ we have $s \in A$,
- ▷ δ-fine, if for every tagged interval $(I, s) \in \pi$ we have $I \subset B(s, \delta(s))$, where $B(s, \delta(s))$ is the ball in $\mathbb{R}^m$ centered at s with radius $\delta(s)$.

Definition 2.1. A function $f\colon S \to X$ is called $\mathcal{HK}$-integrable on S if there exists a vector $w_f \in X$ satisfying the following property: for every $\varepsilon > 0$ there exists a gauge δ on S such that for every δ-fine $\mathcal{HK}$-partition π of S, we have

$$\Bigl\| \sum_{(I,s)\in\pi} f(s)\lambda_m(I) - w_f \Bigr\| < \varepsilon.$$

We write (HK) $\int_S f = w_f$ and call w_f $\mathcal{HK}$-integral of f over S.

The function f is said to be $\mathcal{HK}$-integrable on a subset $A \subset S$ if the function $f \cdot \chi_A \colon S \to X$ is $\mathcal{HK}$-integrable on S, where χ_A denotes the characteristic function of A. We write (HK) $\int_S f \cdot \chi_A =$ (HK) $\int_A f$ for $\mathcal{HK}$-integral of f on A.

If f is $\mathcal{HK}$-integrable on S, then by Theorem 3.3.4 in [14], the function

$$F\colon \mathscr{S} \to X, \quad F(I) = (\mathrm{HK})\int_I f, \quad I \in \mathscr{S}$$

is well defined and it is called the $\mathcal{HK}$-primitive of the function f on S; by Theorem 3.3.5 in [14], F is additive.

Definition 2.2. A family $\mathcal{M}$ of functions $f\colon S \to X$ is called $\mathcal{HK}$-*equiintegrable* if each $f \in \mathcal{M}$ is $\mathcal{HK}$-integrable and for every $\varepsilon > 0$ there exists a gauge δ on S such that for every δ-fine $\mathcal{HK}$-partition π of S, we have

$$\Bigl\| \sum_{(I,s)\in\pi} f(s)\lambda_m(I) - (\mathrm{HK})\int_S f \Bigr\| < \varepsilon,$$

for all $f \in \mathcal{M}$.

If π_1 and π_2 are two $\mathcal{HK}$-partition in S and

$$\lambda_m\Bigl(\bigcup_{(I,s)\in\pi_1} I \triangle \bigcup_{(J,t)\in\pi_2} J \Bigr) \leqslant \eta$$

(the symbol $\triangle$ denotes the symmetric difference of sets), then π_1 and π_2 are said to be η-close.

We set

$$F(\pi) = \sum_{(I,s)\in\pi} F(I)$$

for an additive interval function $F\colon \mathscr{S} \to \mathbb{R}$ and an $\mathcal{HK}$-partition π in S.

Definition 2.3. Let A be a subset of S. An additive interval function $F\colon \mathscr{S} \to \mathbb{R}$ is said to be

▷ $AC^{\triangledown}(A)$ if for every $\varepsilon > 0$ there exists a gauge δ on A and $\eta > 0$ such that for every δ-fine η-close A-tagged $\mathcal{HK}$-partitions π_1 and π_2 in S, we have

$$|F(\pi_1) - F(\pi_2)| \leqslant \varepsilon.$$

▷ $ACG^{\triangledown}(A)$ if there exists a sequence (A_i) of sets $A_i \subset S$ such that $A = \bigcup_{i=1}^{\infty} A_i$ and for every $i \in \mathbb{N}$ the function F is $AC^{\triangledown}(A_i)$

A sequence (F_n) of additive interval functions $F_n\colon \mathscr{S} \to \mathbb{R}$ is said to be

▷ $UAC^{\triangledown}(A)$ if δ and η in the definition of $AC^{\triangledown}(A)$ with F replaced by F_n are independent of n.

▷ $UACG^{\triangledown}(A)$ if there exists a sequence (A_i) of sets $A_i \subset S$ such that $A = \bigcup_{i=1}^{\infty} A_i$ and for every $i \in \mathbb{N}$ the sequence (F_n) is $UAC^{\triangledown}(A_i)$.

We denote by V a complete locally convex space with its topology τ and topological dual V'. By $\mathscr{P}$ the family of all continuous semi-norms in V is denoted. Let p be an element of $\mathscr{P}$. We denote by $\widetilde{V}_p$ the quotient vector space of the vector space V with respect to the equivalence relation $x \sim^p y \Leftrightarrow p(x-y) = 0$. The map $\varphi_p\colon V \to \widetilde{V}_p$ is the canonical quotient map. The quotient normed space $(\widetilde{V}_p, \tilde{p})$ is said to be the *normed component* of the space V with respect to p, where $\tilde{p}(\varphi_p(x)) = p(x)$, for each $x \in V$. The Banach space $(\overline{V}_p, \bar{p})$, which is the completion of the space $(\widetilde{V}_p, \tilde{p})$, is said to be the *Banach component* of the space V with respect to p. We denote by $\widetilde{V}'_p$ and $\overline{V}'_p$ the topological duals of $(\widetilde{V}_p, \tilde{p})$ and $(\overline{V}_p, \bar{p})$ respectively and by $B(\widetilde{V}'_p)$ the closed unit ball in $\widetilde{V}'_p$.

For every $p, q \in \mathscr{P}$ such that $p \leqslant q$, we denote by $\tilde{g}_{pq}$ the map defined as follows

$$\tilde{g}_{pq}\colon \widetilde{V}_q \to \widetilde{V}_p, \quad \tilde{g}_{pq}(w_q) = w_p, \quad w_q \in \widetilde{V}_q,$$

where $w_p = \varphi_p(x)$, for some vector $x \in w_q$. By $\overline{g}_{pq}$ the continuous linear extension of $\tilde{g}_{pq}$ to $\overline{V}_q$ is denoted.

Definition 2.4. Let $f\colon S \to V$ be a function such that for each $v' \in V'$ the function $v'(f)$ is $\mathcal{HK}$-integrable on S. If for each $I \in \mathscr{S}$ there exists a vector $w_I \in V$ such that for each $v' \in V'$, we have

$$v'(w_I) = (\mathrm{HK}) \int_I v'(f),$$

then f is said to be $\mathcal{HKP}$-integrable. The vector w_I is called the $\mathcal{HKP}$-integral of f over the interval I and we set $w_I = (\mathrm{HKP}) \int_I f$. The function

$$F\colon \mathscr{S} \to V, \quad F(I) = (\mathrm{HKP}) \int_I f, \quad I \in \mathscr{S}$$

is called the $\mathcal{HKP}$-primitive of the function f on S. Note that if $I, J \in \mathscr{S}$ and $\operatorname{int}(I) \cap \operatorname{int}(J) = \emptyset$, then for every $v' \in V'$, we have

$$\begin{aligned} v'\biggl((\mathrm{HKP})\int_{I\cup J} f\biggr) &= (\mathrm{HK})\int_{I\cup J} v'(f) = (\mathrm{HK})\int_I v'(f) + (\mathrm{HK})\int_J v'(f) \\ &= v'\biggl((\mathrm{HKP})\int_I f\biggr) + v'\biggl((\mathrm{HKP})\int_J f\biggr) \\ &= v'\biggl((\mathrm{HKP})\int_I f + (\mathrm{HKP})\int_J f\biggr). \end{aligned}$$

Since V is Hausdorff, the last equality yields that the $\mathcal{HKP}$-primitive F of f is additive.

3. The controlled convergence theorem for $\mathcal{HKP}$-integrable functions taking values in Banach spaces

In this section we present the controlled convergence theorem for the $\mathcal{HKP}$-integral of Banach valued functions, Theorem 3.3. Let us start with the following lemmas which make it possible to present clearly Theorem 3.3.

Lemma 3.1. *Let (f_n) be a sequence of $\mathcal{HK}$-integrable functions $f_n\colon S \to \mathbb{R}$ and let $f\colon S \to \mathbb{R}$ be a function. Assume that*

(i) *$f_n(s) \to f(s)$ a.e. in S,*

(ii) *the sequence (F_n) is $UACG^{\triangledown}(S)$, where F_n's are $\mathcal{HK}$-primitives of f_n's.*

Then the function f is $\mathcal{HK}$-integrable on S and for every $I \in \mathscr{S}$ we have

$$\lim_{n\to\infty} F_n(I) = F(I),$$

where F is the $\mathcal{HK}$-primitive of f.

Proof. By hypothesis there exists $Z \subset S$ with $\lambda_m(Z) = 0$ such that f_n converges pointwise in $E = S \setminus Z$ to f. Denote $f^{(E)} = f \cdot \chi_E$ and $f_n^{(E)} = f_n \cdot \chi_E, n \in \mathbb{N}$. Then, the sequence $(f_n^{(E)})$ converges pointwise in S to $f^{(E)}$, and by Theorem 3.3.7 in [14] we have also that each F_n is the $\mathcal{HK}$-primitive of $f_n^{(E)}$. Therefore, by Corollary 1.7 in [9], we have

$$\lim_{n\to\infty} F_n(I) = F(I),$$

for any $I \in \mathscr{S}$. □

Lemma 3.2. *Let $(f_{k,n})$ be a sequence of $\mathcal{HK}$-integrable functions $f_{k,n}\colon S \to \mathbb{R}$ and let (f_k) be a sequence of functions $f_k\colon S \to \mathbb{R}$. Assume that*

(i) *for every $k \in \mathbb{N}$, we have $f_{k,n}(s) \to f_k(s)$ a.e. in S,*

(ii) *there exists a function $f\colon S \to \mathbb{R}$ such that $f_k(s) \to f(s)$ a.e. in S,*

(iii) *the sequence $(F_{k,n})_{k,n}$ is $UACG^{\triangledown}(S)$, where $F_{k,n}$'s are $\mathcal{HK}$-primitives of $f_{k,n}$'s.*

Then f is $\mathcal{HK}$-integrable on S and for every $I \in \mathscr{S}$ we have

$$\lim_{k\to\infty}\lim_{n\to\infty} F_{k,n}(I) = F(I),$$

where F is the $\mathcal{HK}$-primitive of f.

P r o o f. Note that for every $k \in \mathbb{N}$ the sequence $(f_{k,n})_n$ satisfies the conditions of Lemma 3.1. Therefore every f_k is $\mathcal{HK}$-integrable and for every $I \in \mathscr{S}$ we have

$$\lim_{n\to\infty} F_{k,n}(I) = F_k(I), \tag{3.1}$$

where F_k's are $\mathcal{HK}$-primitives of f_k's.

We are going to show that (f_k) satisfies the conditions of Lemma 3.1. Evidently (f_k) satisfies (i). It remains to show that (F_k) is $UACG^{\triangledown}(S)$. Since $(F_{k,n})$ is $UACG^{\triangledown}(S)$, there exists a sequence (A_i) of sets $A_i \subset S$ such that $S = \bigcup\limits_{i=1}^{\infty} A_i$ and for every $i \in \mathbb{N}$ the sequence $(F_{k,n})$ is $UAC^{\triangledown}(A_i)$. Fix an arbitrary A_i. Then, for the given $\varepsilon > 0$ there exists a gauge δ on A_i and $\eta > 0$ such that for every δ-fine η-close A_i-tagged $\mathcal{HK}$-partitions π_1, π_2 in S and for all $k, n \in \mathbb{N}$, we have

$$|F_{k,n}(\pi_1) - F_{k,n}(\pi_2)| \leqslant \frac{\varepsilon}{3}. \tag{3.2}$$

Assume that an arbitrary $k \in \mathbb{N}$ and two arbitrary δ-fine η-close A_i-tagged $\mathcal{HK}$-partial-partitions π_1 and π_2 in S are given. By (3.1) there exists $n_{(k,\pi_1,\pi_2)} \in \mathbb{N}$ such that

$$|F_k(\pi_1) - F_{k,n_{(k,\pi_1,\pi_2)}}(\pi_1)| < \frac{\varepsilon}{3} \text{ and } |F_{k,n_{(k,\pi_1,\pi_2)}}(\pi_2) - F_k(\pi_2)| < \frac{\varepsilon}{3}. \tag{3.3}$$

Note that

$$\begin{aligned}|F_k(\pi_1) - F_k(\pi_2)| &\leqslant |F_k(\pi_1) - F_{k,n_{(k,\pi_1,\pi_2)}}(\pi_1)| \\ &\quad + |F_{k,n_{(k,\pi_1,\pi_2)}}(\pi_1) - F_{k,n_{(k,\pi_1,\pi_2)}}(\pi_2)| + |F_{k,n_{(k,\pi_1,\pi_2)}}(\pi_2) - F_k(\pi_2)|.\end{aligned}$$

Then, the last inequality together with (3.2) and (3.3) yields

$$|F_k(\pi_1) - F_k(\pi_2)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

By the arbitrariness of k, π_1 and π_2 we obtain that (F_k) is $UAC^{\triangledown}(A_i)$, and by the arbitrariness of A_i we infer that (F_k) is $UACG^{\triangledown}(S)$.

Consequently we obtain by Lemma 3.1 that the function f is $\mathcal{HK}$-integrable on S and for every $I \in \mathscr{S}$ we have

$$\lim_{k\to\infty} F_k(I) = F(I),$$

where F is the $\mathcal{HK}$-primitive of f. The last equality together with (3.1) yields that for every $I \in \mathscr{S}$ we have

$$\lim_{k\to\infty} \lim_{n\to\infty} F_{k,n}(I) = F(I).$$

□

The proof of Theorem 3.3 is similar in spirit to the results in [2], [10] and [11].

Theorem 3.1. *Let (f_n) be a sequence of $\mathcal{HKP}$-integrable functions $f_n\colon S \to X$ and let $f\colon S \to X$ be a function. Assume that*

(i) *for every $x^* \in X^*$, we have $x^*(f_n(s)) \to x^*(f(s))$ a.e. in S,*
(ii) *for every sequence $(x_k^*) \subset B(X^*)$, the sequence $(x_k^*(F_n))_{k,n}$ is $UACG^{\triangledown}(S)$, where F_n's are $\mathcal{HKP}$-primitives of f_n's.*

Then f is $\mathcal{HKP}$-integrable and for every $I \in \mathscr{S}$, we have

$$\lim_{n\to\infty} F_n(I) = F(I)$$

in the weak topology $\sigma(X, X^*)$, where F is the $\mathcal{HKP}$-primitive of f.

Proof. Since for every $x^* \in X^*$ the sequence $(x^*(F_n))$ is $UACG^{\triangledown}(S)$ and $x^*(F_n)$'s are $\mathcal{HK}$-primitives of $x^*(f_n)$'s, Lemma 3.1 yields that the function $x^*(f)$ is $\mathcal{HK}$-integrable on S and for every $I \in \mathscr{S}$ we have

$$\lim_{n\to\infty} x^*\Big((\mathrm{HKP})\int_I f_n\Big) = (\mathrm{HK})\int_I x^*(f). \tag{3.4}$$

Fix an arbitrary $I \in \mathscr{S}$. By C the weak closure in X of the set $\{(\mathrm{HKP})\int_I f_n\colon n \in \mathbb{N}\}$ is denoted.

Now we are going to prove that the set C is weakly compact in X. Assume, by contradiction, that the set C is not weakly compact. Since the sequence $((\mathrm{HKP})\int_I f_n)$ is weakly Cauchy in X, the set C is bounded weakly closed subset of X. Therefore, by Theorem 1, [8], there exists $\theta > 0$, a sequence $(x_n) \subset C$ and a sequence $(x_n^*) \subset B(X^*)$ such that $x_n^*(x_m) = 0$ for $n > m$, $x_n^*(x_m) > \theta$ for $m \geqslant n$. Since the set $C \setminus \{(\mathrm{HKP}) \int_I f_n \colon n \in \mathbb{N}\}$ contains at most one point, we can take subsequences $(g_n) \subset (f_n)$, $(y_n^*) \subset (x_n^*)$ with the following properties

(a) $y_n^*(G_m(I)) = 0$ for $n > m$,

(b) $y_n^*(G_m(I)) \geqslant \theta$ for $m \geqslant n$,

where G_m's are $\mathcal{HKP}$-primitives of g_m's.

Since the function f is weakly measurable and $(S, \mathcal{L}_m, \lambda_m)$ is a finite perfect measure space, we obtain by Theorem 2F in [4] that there exists a subsequence $(y_{n_k}^*) \subset (y_n^*)$ such that $(y_{n_k}^*(f))$ is almost everywhere convergent on S. According to Alaoglu Theorem there exists a weak* cluster point y_0^* of the sequence $(y_{n_k}^*)$. Hence we infer

$$y_{n_k}^*(f(s)) \to y_0^*(f(s))$$

almost everywhere in S. Thus, we have that for each $k \in \mathbb{N}$ the sequence $(y_{n_k}^*(g_m))_m$ converges pointwise almost everywhere to $y_{n_k}^*(f)$ and the sequence $(y_{n_k}^*(f))_k$ converges pointwise almost everywhere to $y_0^*(f)$. We also have that $(y_{n_k}^*(G_m))_{k,m}$ satisfies the condition (iii) of Lemma 3.2. Therefore the function $y_0^*(f)$ is $\mathcal{HK}$-integrable on S and

$$\lim_{k\to\infty} \lim_{m\to\infty} y_{n_k}^*(G_m(I)) = \int_I y_0^*(f). \tag{3.5}$$

Hence by (b), we get

$$(\mathrm{HK}) \int_I y_0^*(f) \geqslant \theta. \tag{3.6}$$

On the other hand, since y_0^* is a weak* cluster point of the sequence $(y_{n_k}^*)$, there exists a subnet (y_t^*) of $(y_{n_k}^*)$ which is weak* convergent to y_0^*. Then by (a), for each $m \in \mathbb{N}$ we obtain

$$y_0^*(G_m(I)) = \lim_t y_t^*(G_m(I)) = 0,$$

and therefore by (3.5) we get

$$\int_I y_0^*(f) = \lim_{m\to\infty} y_0^*(G_m(I)) = 0.$$

The last equality contradicts (3.6). Consequently, the set C is weakly compact in X and therefore C is weakly complete. Then, since the sequence $((\mathrm{HKP}) \int_I f_n)$ is weakly Cauchy in X, there exists $w_I \in X$ such that for every $x^* \in X^*$, we have

$$\lim_{n\to\infty} x^*\left((\mathrm{HKP}) \int_I f_n\right) = x^*(w_I).$$

The last equality together with (3.4) yields that for each $x^* \in X^*$, we have

$$x^*(w_I) = (\mathrm{HK}) \int_I x^*(f).$$

By arbitrariness of I, this means that f is $\mathcal{HKP}$-integrable and

$$\lim_{n\to\infty} F_n(I) = F(I)$$

in the weak topology $\sigma(X, X^*)$, where F is the $\mathcal{HKP}$-primitive of f. □

The following lemma allows us to ignore sets of measure zero in the notion of *$\mathcal{HK}$-equiintegrability*. The "only if part" of the lemma is proven in [5], Exercise 13.9 (there it is proven for real valued functions, but it is enough to replace the absolute value by the norm). The "if part" is straightforward.

Lemma 3.3. *Let (f_n) be a pointwise bounded sequence of functions $f_n\colon S \to X$ and let E be subset of S such that $\lambda_m(S \setminus E) = 0$. Then, the sequence (f_n) is $\mathcal{HK}$-equiintegrable if and only if the sequence $(f_n \cdot \chi_E)$ is $\mathcal{HK}$-equiintegrable.*

By Lemma 3.4 and Main Theorem 1.6 in [9], Theorem 3.3 yields the following.

Theorem 3.2. *Let (f_n) be a pointwise bounded sequence of $\mathcal{HKP}$-integrable functions $f_n\colon S \to X$ and let $f\colon S \to X$ be a function. Assume that*

(i) *for every $x^* \in X^*$, we have $x^*(f_n(s)) \to x^*(f(s))$ a.e. in S,*
(ii) *for every sequence $(x_k^*) \subset B(X^*)$, the sequence $(x_k^*(f_n))_{k,n}$ is $\mathcal{HK}$-equiintegrable.*

Then f is $\mathcal{HKP}$-integrable and for every $I \in \mathscr{S}$, we have

$$\lim_{n\to\infty} F_n(I) = F(I)$$

in the weak topology $\sigma(X, X^*)$, where F is the $\mathcal{HKP}$-primitive of f.

For the case when $m = 1$, Theorem 3.5 has been proved in a different way in [2], Theorem 5.

4. The controlled convergence theorem for $\mathcal{HKP}$-integrable functions taking values in complete locally convex spaces

The main result of the paper is Theorem 4.2 in this section. The following lemma is the key for the extension of Theorem 3.3 to Theorem 4.2.

Lemma 4.1. *A function $f\colon S \to V$ is $\mathcal{HKP}$ integrable if and only if for each $p \in \mathscr{P}$ the function $\varphi_p \circ f$ is $\mathcal{HKP}$ integrable. Moreover, the equality*

$$\varphi_p\Big((\mathrm{HKP})\int_I f\Big) = (\mathrm{HKP})\int_I \varphi_p \circ f,$$

holds for every $p \in \mathscr{P}$ and every $I \in \mathscr{S}$.

Proof. Assume that the function f is $\mathcal{HKP}$-integrable. Then, the function

$$\tilde{v}'_p \circ (\varphi_p \circ f) = (\tilde{v}'_p \circ \varphi_p) \circ f = v' \circ f,$$

is $\mathcal{HK}$-integrable, for each $\tilde{v}'_p \in \widetilde{V}'_p$ and every $p \in \mathscr{P}$.

Let p be any element of $\mathscr{P}$. For the given $I \in \mathscr{S}$, we have also

$$(\tilde{v}'_p \circ \varphi_p)\Big((\mathrm{HKP})\int_I f\Big) = (\mathrm{HK})\int_I (\tilde{v}'_p \circ \varphi_p) \circ f = (\mathrm{HK})\int_I \tilde{v}'_p \circ (\varphi_p \circ f),$$

for each $\tilde{v}'_p \in \widetilde{V}'_p$. Hence, the function $\varphi_p \circ f$ is $\mathcal{HKP}$-integrable in the Banach component $(\overline{V}_p, \bar{p})$ and $\varphi_p((\mathrm{HKP})\int_I f) = (\mathrm{HKP})\int_I \varphi_p \circ f$.

Conversely, assume that for every $p \in \mathscr{P}$ the function $\varphi_p \circ f$ is $\mathcal{HKP}$-integrable in the Banach component $(\overline{V}_p, \bar{p})$. Let I be any element of $\mathscr{S}$. We set

$$w_p^{(I)} = (\mathrm{HKP})\int_I \varphi_p \circ f, \tag{4.1}$$

for every $p \in \mathscr{P}$.

Firstly, we show that for every $v' \in V'$ the function $v' \circ f$ is $\mathcal{HK}$-integrable. Let v' be any element of V'. Since

$$V' = \{\tilde{v}'_p \circ \varphi_p \colon p \in \mathscr{P},\ \tilde{v}'_p \in \widetilde{V}'_p\}, \tag{4.2}$$

there exist $p \in \mathscr{P}$ and $\tilde{v}'_p \in \widetilde{V}'_p$ such that $v' = \tilde{v}'_p \circ \varphi_p$. Hence, the function $v' \circ f = \tilde{v}'_p \circ (\varphi_p \circ f)$ is $\mathcal{HK}$-integrable.

Secondly, we show that there exists $w_I \in V$ such that $v'(w_I) = (\mathrm{HK}) \int_I v' \circ f$, for every $v' \in V'$. Notice that for every $p \in P$ and every $\overline{v}'_p \in \overline{V}'_p$, we have

$$\overline{v}'_p(w_p^{(I)}) = (\mathrm{HK}) \int_I \overline{v}'_p \circ (\varphi_p \circ f) = (\mathrm{HK}) \int_I \tilde{v}'_p \circ (\varphi_p \circ f),$$

where $\tilde{v}'_p$ is the restriction of $\overline{v}'_p$ to $\widetilde{V}_p$.

Suppose that two arbitrary continuous semi norms p and q such that $p \leqslant q$ are given. Since $w_q^{(I)} \in \overline{V}_q$, there exists a sequence $(w_n^{(q)}) \subset \widetilde{V}^q$ such that $\lim_{n\to\infty} w_n^{(q)} = w_q^{(I)}$. Define the sequence $(w_n^{(p)})$ of vectors $w_n^{(p)} = \tilde{g}_{pq}(w_n^{(q)})$. Therefore we get

$$\lim_{n\to\infty} w_n^{(p)} = \overline{g}_{pq}(w_q^{(I)}).$$

Hence, we obtain

$$\lim_{n\to\infty} \tilde{v}'_p(w_n^p) = \overline{v}'_p(\overline{g}_{pq}(w_q(S'))), \tag{4.3}$$

for every $\tilde{v}'_p \in \widetilde{V}'_p$, where $\overline{v}'_p$ is the extension of $\tilde{v}'_p$ to $\overline{V}_p$. We have also

$$\begin{aligned}\lim_{n\to\infty} \tilde{v}'_p(w_n^p) &= \lim_{n\to\infty} (\tilde{v}'_p \circ \tilde{g}_{pq})(w_n^q) = (\mathrm{HK}) \int_I (\tilde{v}'_p \circ \tilde{g}_{pq}) \circ (\varphi_q \circ f) \\ &= (\mathrm{HK}) \int_I \tilde{v}'_p \circ (\tilde{g}_{pq} \circ \varphi_q) \circ f = (\mathrm{HK}) \int_I \tilde{v}'_p \circ (\varphi_p \circ f) = \overline{v}'_p(w_p^{(I)}),\end{aligned}$$

for every $\tilde{v}'_p \in \widetilde{V}'_p$, where $\overline{v}'_p$ is the extension of $\tilde{v}'_p$ to $\overline{V}_p$. The last equality together with (4.3) yields $\overline{v}'_p(\overline{g}_{pq}(w_q^{(I)})) = \overline{v}'_p(w_p^{(I)})$, for every $\overline{v}'_p \in \overline{V}'_p$. Therefore we obtain

$$\overline{g}_{pq}(w_q^{(I)}) = w_p^{(I)}.$$

Hence, by arbitrariness of p and q, Theorem II.5.4, [13] implies that there exists $w_I \in V$ such that for every $p \in \mathscr{P}$, we have

$$\varphi_p(w_I) = w_p^{(I)}.$$

By (4.2), for any vector $v' \in V'$ there exists $p \in \mathscr{P}$ and $\tilde{v}'_p \in \widetilde{V}'_p$ such that $v' = \tilde{v}'_p \circ \varphi_p$. Hence, we get

$$\begin{aligned}v'(w_I) = \tilde{v}'_p(\varphi_p(w_I)) = \tilde{v}'_p(w_p^{(I)}) &= (\mathrm{HK}) \int_I \tilde{v}'_p \circ (\varphi_p \circ f) \\ = (\mathrm{HK}) \int_I (\tilde{v}'_p \circ \varphi_p) \circ f &= (\mathrm{HK}) \int_I v' \circ f.\end{aligned}$$

Finally, the function f is $\mathcal{HKP}$-integrable and (HKP) $\int_I f = w_I$. The last equality together with (4.1) yields

$$\varphi_p\Big((\mathrm{HKP})\int_I f\Big) = (\mathrm{HKP})\int_I \varphi_p \circ f$$

for every $p \in \mathscr{P}$. □

Now, we are ready to prove the main theorem.

Theorem 4.1. *Let (f_n) be a sequence of $\mathcal{HKP}$-integrable functions $f_n\colon S \to V$ and let $f\colon S \to V$ be a function such that*

(i) *for each $v' \in V'$, we have $v'(f_n) \to v'(f)$ a.e. in S,*

(ii) *for every sequence $(v'_k) \subset V'$, we have $(v'_k(F_n))_{k,n}$ is $UACG^{\triangledown}(S)$, where F_n's are $\mathcal{HKP}$-primitives of f_n's.*

Then, the function f is $\mathcal{HKP}$-integrable and for every $I \in \mathscr{S}$, we have

$$\lim_{n\to\infty} F_n(I) = F(I)$$

in the weak topology $\sigma(V, V')$, where F is the $\mathcal{HKP}$-primitive of f.

Proof. Let q be an arbitrary element of $\mathscr{P}$. We have

▷ $(\varphi_q \circ f_n)$ is a sequence of $\mathcal{HKP}$-integrable functions,

▷ for each $\tilde{v}'_q \in \widetilde{V}'_q$, $\tilde{v}'_q \circ (\varphi_q \circ f_n) \to \tilde{v}'_q \circ (\varphi_q \circ f)$ a.e. in S,

▷ for each sequence $(\tilde{v}'_k) \subset B(\widetilde{V}'_q)$, $(\tilde{v}'_k \circ (\varphi_q \circ F_n))_{k,n}$ is $UACG^{\triangledown}(S)$, where $\varphi_q \circ F_n$'s are $\mathcal{HKP}$-primitives of $\varphi_q \circ f_n$'s.

Therefore, by arbitrariness of q, Theorem 3.3 implies

▷ for every $p \in \mathscr{P}$, $(\varphi_p \circ f)$ is $\mathcal{HKP}$-integrable,

▷ for each $p \in \mathscr{P}$, $I \in \mathscr{S}$ and each $\tilde{v}'_p \in \widetilde{V}'_p$

$$\lim_{n\to\infty} \tilde{v}'_p\Big((\mathrm{HKP})\int_I \varphi_p \circ f_n\Big) = \tilde{v}'_p((\mathrm{HKP})\int_I \varphi_p \circ f).$$

Hence, by Lemma 4.1, we obtain

▷ f is $\mathcal{HKP}$-integrable,

▷ for each $I \in \mathscr{S}$ and each $\tilde{v}'_p \in \widetilde{V}'_p$

$$\lim_{n\to\infty} (\tilde{v}'_p \circ \varphi_p)(F_n(I)) = (\tilde{v}'_p \circ \varphi_p)(F(I)).$$

Finally, the last equality together with (4.2) yields that for each $I \in \mathscr{S}$, we have

$$\lim_{n\to\infty} F_n(I) = F(I),$$

in the weak topology $\sigma(V, V')$. □

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