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Variational formulations I: Statics of mechanical systems.

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Abstract. Two improvements of variational formulations of mechanics are proposed. The first consists in a modification of the definition of equilibrium. The second consists in adding elements of control by external devices. In the present note the proposed improvements are applied to variational principles of statics. Numerous examples are given.

Introduction

The fundamental concept in variational formulations of physical theories is that of equilibrium. In the current literature on mechanics an equilibrium configuration is a configuration at which a function such as internal energy or action assumes a local minimum. This definition is too narrow. It excludes the treatment of dissipative systems. A definition of equilibrium based on the response to virtual displacements is proposed. This proposal does net affect the treatment of potential unconstrained systems. It allows the treatment of dissipative systems. Applying constraints to virtual displacements and not to configurations is a natural consequence of this proposal. A different interpretation of non holonomic constraints is obtained as one of the results. This modified version of non holonomic constraints applies to statics as well as dynamics.

The study of motions of an isolated object in a configuration space is the subject of geometric formulations of mechanics. Let Q be an affine configuration space modelled on a vector space V. For a potential unconstrained system a motion

$$\mathfrak{q}\colon \mathbb{R} \to Q$$

is required to satisfy the Hamilton principle

$$\delta \int_{-\infty}^{\infty} L \circ (\mathfrak{q}, \dot{\mathfrak{q}}) = 0 \,.$$

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Here

$$L\colon Q\times V\to\mathbb{R}$$

is the Lagrangian and

$$\dot{\mathfrak{q}}\colon \mathbb{R}\to V$$

is the velocity. The Hamilton principle must be satisfied for all variations

$$\delta\mathfrak{q}\colon \mathbb{R}\to V$$

with compact support. Variations with compact support are used in order to make the integration meaningful. The Euler-Lagrange equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}-\frac{\partial L}{\partial q}\right)\circ(\mathfrak{q},\dot{\mathfrak{q}},\ddot{\mathfrak{q}})=0$$

follow from the variational principle.

The formulation of mechanics based on the Hamilton principle is suitable for studying motions of isolated systems such as planets. Modern formulations of mechanics should treat boundary value problems and should include elements of control theory. A motion is typically observed in a precise time interval $[t_0,t_1]$. The observed object is not created at the initial moment t_0 and does not disappear at the terminal moment t_1. The past motion of the object interacts with the motion in the time interval $[t_0,t_1]$ by supplying the initial momentum p_0 and the terminal momentum p_1 is passed onto the future motion. This type of interaction is well described by the variational principle

$$\delta\int_{t_0}^{t_1} L\circ(\mathfrak{q},\dot{\mathfrak{q}})=\langle p_1,\delta\mathfrak{q}(t_1)\rangle-\langle p_0,\delta\mathfrak{q}(t_0)\rangle \tag{1}$$

with free variations of the boundary configurations. This principle leads to the equations

$$p_0=\frac{\partial L}{\partial\dot{q}}(\mathfrak{q}(t_0),\dot{\mathfrak{q}}(t_0))$$

and

$$p_1=\frac{\partial L}{\partial\dot{q}}(\mathfrak{q}(t_1),\dot{\mathfrak{q}}(t_1))$$

in addition to the Euler-Lagrange equations satisfied inside the interval $[t_0,t_1]$.

The variational principle (1) provides a theoretical background for ballistics. It is not general enough for treatig guided missiles and not even cars or planes. External forcess applied to the object during the interval $[t_0,t_1]$ must be included. An external force represented by

$$\mathfrak{f}\colon\mathbb{R}\to V^*.$$

appears in the variational principle

$$\delta\int_{t_0}^{t_1} L\circ(\mathfrak{q},\dot{\mathfrak{q}})=-\int_{t_0}^{t_1}\langle\mathfrak{f},\delta\mathfrak{q}\rangle+\langle p_1,\delta\mathfrak{q}(t_1)\rangle-\langle p_0,\delta\mathfrak{q}(t_0)\rangle.$$

Equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right) \circ (\mathfrak{q}, \dot{\mathfrak{q}}, \ddot{\mathfrak{q}}) = \mathfrak{f}, \tag{2}$$

$$p_0 = \frac{\partial L}{\partial \dot{q}}(\mathfrak{q}(t_0), \dot{\mathfrak{q}}(t_0)),$$

and

$$p_1 = \frac{\partial L}{\partial \dot{q}}(\mathfrak{q}(t_1), \dot{\mathfrak{q}}(t_1))$$

follow from the principle. The equation (2) is to be satisfied in the interval $[t_0, t_1]$.

Control by external forces and boundary momenta is not the only form of control. We suggest that at least this form of control be explicitly included in modern formulations of mechanics.

This note is a part of a series of notes on variational formulations of physical theories. Static mechanical systems are considered. Formulations of dynamics of mechanical systems and field theories will follow.

Statics of mechanical systems is hardly present in modern literature. Static systems appeared in catastrophe theory. Equilibrium configurations of isolated systems defined as minima of internal energy functions were studied. Some elements of control were present. All proposed improvements are fully implemented in the present note.

1 Equilibria

1.1 Two simple examples

Example 1. Let Q be an affine space modelled on a vector space V with a Euclidean metric $g\colon V \to V^*$. A material point with configuration $q \in Q$ is connected with a spring of spring constant k to a fixed point $q_0 \in Q$. The configuration $q = q_0$ is the only stable configuration of the material point.

Example 2. The material point with configuration $q \in Q$ in Example 1 is subject to friction. The friction is measured by the coefficient ρ. The set

$$\{q \in Q; \|q - q_0\| \leqslant \rho/k\}$$

is the set of equilibrium configurations.

Definitions of equilibrium:

A) A stable equilibrium configuration is a configuration at which the internal energy of the system assumes its minimum value.

B) A configuration q is a stable equilibrium configuration if the work of each process starting at q and not ending at q is positive.

Definition A) applies to the first example. The internal energy is the function

$$U\colon Q \to \mathbb{R}\colon q \mapsto \frac{k}{2}\|q - q_0\|^2.$$

It assumes its minimum value at the configuration $q = q_0$. Definition A) does not apply to the second example.

Definition B) applies to both examples. In the first example the work of a process starting at q_1 and ending at q_2 equals $U(q_2) - U(q_1)$. This work is always positive unless $q_1 = q_0$. In the second example the work of a process from q_1 to q_2 equals

$$U(q_2) - U(q_1) + \rho \times [\text{length of process}]. \tag{3}$$

If $q_1 = q$, $q_2 = q + \Delta q \neq q$, then

$$\begin{aligned} &U(q_2) - U(q_1) + \rho \times [\text{length of process}] \\ &= \frac{k}{2}\|q - q_0 + \Delta q\|^2 - \frac{k}{2}\|q - q_0\|^2 + \rho\|\Delta q\| \\ &= k\langle g(q - q_0), \Delta q\rangle + \frac{k}{2}\|\Delta q\|^2 + \rho\|\Delta q\|\,. \end{aligned}$$

Let

$$\|q - q_0\| > \rho/k.$$

Choose Δq in the direction opposite to $(q - q_0)$ and assume that the process is the straight segment from q to $q + \Delta q$. We have

$$\begin{aligned} &U(q_2) - U(q_1) + \rho \times [\text{length of process}] \\ &= -k\|q - q_0\|\|\Delta q\| + \rho\|\Delta q\| + \frac{k}{2}\|\Delta q\|^2. \end{aligned}$$

This quantity is negative if $\|\Delta q\|$ is small enough since

$$-k\|q - q_0\|\|\Delta q\| + \rho\|\Delta q\| < 0.$$

It follows that q is not a configuration of equilibrium.

Let

$$\|q - q_0\| \leqslant \rho/k.$$

The quantity

$$\rho \times [\text{length of process}]$$

is always positive. It assumes its lowest value

$$\rho \times [\text{length of process}] = \rho\|\Delta$$

for given q and Δq if the process is a segment of a straight line. The lowest value of the term

$$U(q_2) - U(q_1) = U(q + \Delta q) - U(q)$$

with a given $\|\Delta q\|$ is obtained when Δq points in the direction opposite to $(q - q_0)$. In this case

$$k\langle g(q - q_0), \Delta q\rangle = -k\|q - q_0\|\|\Delta q\| \geqslant 0.$$

If the process is the segment of a straight line from q to $q + \Delta q$ and the vector Δq points in the direction of $-(q - q_0)$, then

$$\begin{aligned} &U(q_2) - U(q_1) + \rho \times [\text{length of process}] \\ &= -k\|q - q_0\|\|\Delta q\| + \frac{k}{2}\|\Delta q\|^2 + \rho\|\Delta q\| > 0 \end{aligned}$$

In all other cases the value of the expression (3) is higher. It follows that q is a configuration of equilibrium.

The two examples were designed to show that variational formulations have a wider area of applicability if based on Definition B). This definition appears in the Levi-Civita formulations of mechanics. It is not present in modern geometric formulations.

1.2 Precise definitions of local equilibria

Let Q be the *configuration space* of a system. A *virtual displacement trajectory* (a *trajectory* for short) is a submanifold $\mathfrak{c} \subset Q$ homeomorphic to the interval $\mathbb{R}_+ = [0, \infty) \subset \mathbb{R}$. The submanifold $\mathfrak{c}$ the image of an embedding

$$\mathfrak{q} \colon \mathbb{R}_+ \to Q.$$

The point $q = \mathfrak{q}(0)$ is the *initial point* of the trajectory and the trajectory will be denoted by $(q, \mathfrak{c})$.

The embedding $\mathfrak{q}$ is called a *parameterization* of $(q, \mathfrak{c})$. The set of virtual displacement trajectories will be denoted by $\mathcal{P}(Q)$.

There is a *work function*

$$W_{(q,\mathfrak{c})} \colon \mathfrak{c} \to \mathbb{R}$$

defined on each trajectory $(q, \mathfrak{c})$. We introduce the mapping

$$W \colon \mathcal{P}(Q) \to \bigcup_{(q,\mathfrak{c}) \in \mathcal{P}(Q)} C^\infty(\mathbb{R}|\mathfrak{c}) \colon (q, \mathfrak{c}) \mapsto W_{(q,\mathfrak{c})}.$$

This mapping characterizes the system.

A configuration $q \in Q$ is a *local stable equilibrium configuration* if for each displacement trajectory $(q, \mathfrak{c})$ the work function $W_{(q,\mathfrak{c})}$ has a local minimum at q.

Let $\mathfrak{c}$ be parameterized by an embedding

$$\mathfrak{q} \colon \mathbb{R}_+ \to Q.$$

The work function can be converted to a function

$$\tilde{W}_{\mathfrak{q}} \colon \mathbb{R}_+ \to \mathbb{R} \colon s \mapsto W_{(q,\mathfrak{c})}(\mathfrak{q}(s))$$

of the parameter. The first order necessary condition of equilibrium for a configuration q states that for each trajectory $(q, \mathfrak{c})$ the derivative of the work function $\tilde{W}_{\mathfrak{q}}$ satisfies

$$\mathrm{D}\tilde{W}_{\mathfrak{q}}(0) \geqslant 0. \tag{4}$$

This condition is parameterization independent.

Only the first order differential conditions are used in variational formulations of physical theories. For the purpose of studying the first order differential criteria virtual displacemements are well represented by vectors $\delta q \in \mathsf{T}Q$ tangent to trajectories and the set of work functions is represented by a *work form*

$$\sigma\colon \mathsf{T}Q \to \mathbb{R}$$

derived from the differentials of work functions. The work form is positive homogeneous in the sense that

$$\sigma(k\delta q) = k\sigma(\delta q)$$

if $k \geqslant 0$. The condition (4) assumes the form

$$\sigma(\delta q) \geqslant 0$$

for each vector δq at q.

1.3 Constraints

An unconstrained system is characterized by a work function

$$W_{(q,\mathfrak{c})}\colon \mathfrak{c} \to \mathbb{R} \tag{5}$$

defined on each trajectory $(q,\mathfrak{c})$. The mapping

$$W\colon \mathcal{P}(Q) \to \bigcup_{(q,\mathfrak{c})\in\mathcal{P}(Q)} C^\infty(\mathbb{R}|\mathfrak{c})\colon (q,\mathfrak{c}) \mapsto W_{(q,\mathfrak{c})}.$$

is also used. *Constraints* are conditions imposed on trajectories by specifying a subset $\mathcal{C}$ of the set of all displacement trajectories. Trajectories in $\mathcal{C}$ are said to be *admissible*. A work function (5) is assigned to admissible trajectories. Let C^0 be the set of initial configurations of all admissible displacement trajectories. Constraints are said to be *holonomic* if $\mathcal{C}$ is the set of all displacement trajectories included in C^0. In other cases constraints are said to be *non holonomic*.

A system is characterized by the pair $(\mathcal{C}, W)$, with

$$W\colon \mathcal{C} \to \bigcup_{(q,\mathfrak{c})\in\mathcal{C}} C^\infty(\mathbb{R}|\mathfrak{c})\colon (q,\mathfrak{c}) \mapsto W_{(q,\mathfrak{c})}.$$

A configuration $q \in C^0$ is a *local stable equilibrium configuration* of a constrained system if for each displacement trajectory $(q,\mathfrak{c}) \in \mathcal{C}$ the work function $W_{(q,\mathfrak{c})}$ has a local minimum at q.

For the purpose of formulating the first differential order necessary condition of local equilibrium the system is characterized by a *virtual work function*

$$\sigma\colon C^1 \to \mathbb{R}$$

defined on a *constraint set* $C^1 \subset \mathsf{T}Q$. For each

$$q \in C^0 = \tau_Q(C^1)$$

the set

$$C_q^1 = C^1 \cap \mathsf{T}_q Q$$

is a cone in the sense that if

$$\delta q \in C_q^1,$$

then

$$\lambda \delta q \in C_q^1$$

for each

$$\lambda \geqslant 0.$$

A vector δq is said to be *tangent* to a set $C^0 \in Q$ if there is a curve

$$\gamma\colon \mathbb{R} \to Q$$

such that $\gamma([0,\infty)) \subset C^0$ and $\delta q = \mathsf{t}\gamma(0)$. The set of vectors tangent to C^0 is the *tangent set* of C^0 denoted by $\mathsf{T}C^0$. Constraints are said to be *holonomic* if $C^1 = \mathsf{T}C^0$. Otherwise constraints are said to be *non holonomic*. The inclusion

$$C^1 \subset \mathsf{T}C^0$$

is usually verified.

The virtual work function is a homogeneous form in the sense that

$$\sigma(\lambda\delta q) = \lambda\sigma(\delta q)$$

if

$$\lambda \geqslant 0.$$

The necessary condition of local equilibrium states that a configuration $q \in C^0$ is an *equilibrium configuration* of the static system

$$(C^1, \sigma)$$

if the inequality

$$\sigma(\delta q) \geqslant 0$$

is satisfied for each virtual displacement

$$\delta q \in C_q^1.$$

2 Control of mechanical system by external forces

2.1 Composed systems

Let two static systems with the same configuration space Q be characterized by

$$({C^1}_1, \sigma_1)$$

and

$$({C^1}_2, \sigma_2)$$

respectively. Then the system constructed by coupling the two systems is characterized by

$$(C^1, \sigma)$$

with

$$C^1 = {C^1}_1 \cap {C^1}_2$$

and

$$\sigma = \sigma_1 | C^1 + \sigma_2 | C^1.$$

Certain regularity is assumed in this construction of the coupled system. Some possible irregularities will be discussed separately. The construction of the coupled system is certainly valid when one of the systems is unconstrained.

2.2 Control

Equilibrium configurations of an isolated system are not of much interest. A static system is usually subjected to *control* by being coupled to an external system. The work function σ together with the constraint set C^1 provides complete information on the response of a static system to control. Equilibrium configurations $q \in C^0 \cap F^0$ of a static system characterized by (C^1, σ) coupled to an external system represented by (F^1, φ) are determined by the *virtual work principle*

$$\sigma(\delta q) + \varphi(\delta q) \geqslant 0 \quad \text{for each virtual displacement } \delta q \in C^1_q \cap F^1_q.$$

2.3 The Legendre-Fenchel transformation, the constitutive set

A static system is said to be *regular* if $C^1 = \mathsf{T}Q$, there is a function

$$U \colon Q \to \mathbb{R},$$

and the virtual work form is derived from the potential U according to

$$\sigma \colon \mathsf{T}Q \to \mathbb{R} \colon \delta q \mapsto \langle \mathrm{d}U, \delta q \rangle.$$

Control by regular external systems is of special interest. Equilibrium configurations $q \in C^0$ of a static system (C^1, σ) controlled by a regular system represented by $(\mathsf{T}Q, \mathrm{d}U)$ are determined by

$$\sigma(\delta q) + \langle \mathrm{d}U, \delta q \rangle \geqslant 0 \quad \text{for each virtual displacement } \delta q \in C^1_q. \tag{6}$$

Note that only the differential $\mathrm{d}U(q)$ of the potential U appears in the virtual work principle (6). Two controlling regular systems $(\mathsf{T}Q, \mathrm{d}U_1)$ and $(\mathsf{T}Q, \mathrm{d}U_2)$ will have the same effect at q if

$$\mathrm{d}U_2(q) = \mathrm{d}U_1(q).$$

This equality establishes an equivalence relation of controlling regular systems at q. A suitable representant of the equivalence class of a system $(\mathsf{T}Q, \mathrm{d}U)$ at q is the covector

$$f = -\mathrm{d}U(q) \in \mathsf{T}^*_q Q. \tag{7}$$

Due to the presence of constraints two different covectors f_1 and f_2 in $\mathsf{T}^*_q Q$ will still have the the same effect if

$$\langle f_2, \delta q\rangle = \langle f_1, \delta q\rangle \quad \text{for each virtual displacement } \delta q \in C^1_q.$$

This could lead to a further classification of controlling devices different for different controlled systems. The covector (7) is a completely universal characteristic of a regular controlling system $(\mathsf{T}Q, \mathrm{d}U)$ at q. An *external force* will be the term used for this covector.

An alternative representation of a static system (C^1, σ) is provided by the *constitutive set*

$$S = \{f \in \mathsf{T}^*Q; q = \pi_Q(f) \in C^0, \forall_{\delta q \in C^1_q} \sigma(\delta q) - \langle f, \delta q\rangle \geqslant 0\} \tag{8}$$

The passage from the objects (C^1, σ) characterizing a system to the constitutive set S is the *Legendre-Fenchel transformation* known in convex analysis. The constitutive set provides a complete characterization of a *convex* system. For a convex system the objects C^1 and σ can be reconstructed from the constitutive set.

3 Examples of static systems

The geometric structure used in formulations of statics with external forces is the diagram

$$\begin{array}{c} (\mathsf{T}^*Q, \langle\,,\rangle) \\ {\scriptstyle \pi_Q}\big\downarrow \\ Q \end{array} \tag{9}$$

It is the cotangent fibration of the configuration space Q with the canonical pairing

$$\langle\,,\rangle \colon \mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q \to \mathbb{R}.$$

If Q is an affine space modelled on a vector space V, then the cotangent bundle is identified with $Q \times V^*$ and the mapping π_Q is the canonical projection

$$\pi_Q \colon Q \times V^* \to Q \colon (q, f) \mapsto q.$$

The component f of an element (q, f) of the phase space T^*Q is the external force applied to the material point at configuratiom q. The tangent bundle $\mathsf{T}Q$ is identified with the product $Q \times V$ and the tangent projection is represented by the canonical projection

$$\tau_Q \colon Q \times V \to Q \colon (q, \delta q) \mapsto q.$$

The fibre product of the cotangent bundle wth the tangent bundle is the space of elements $(q, f), (q, \delta q)$ in $(Q \times V^*) \times (Q \times V)$. The pairing $\langle\,,\rangle$ is defined by

$$\langle (q, f), (q, \delta q)\rangle = \langle f, \delta q\rangle.$$

The diagram (9) takes the form

$$\begin{array}{c}(Q \times V^*, \langle, \rangle) \\ \pi_Q \Big\downarrow \\ Q\end{array}$$

The response of a static system to control by esternal forces is described by the constitutive set (8).

Example 3. A material point with configuration q in an affine space Q is tied to a fixed point $q_0 \in Q$ with a spring of spring constant k. The model space is a Euclidean vector space V with a metric tensor

$$g \colon V \to V^*.$$

The system is regular. The internal energy of the system is the function

$$U \colon Q \to \mathbb{R} \colon q \mapsto \frac{k}{2}\|q - q_0\|^2.$$

This function generates the constitutive set

$$S = \{(q, f) \in Q \times V^*; f = kg(q - q_0)\}.$$

Example 4. A material point with configuration q in a Euclidean affine space Q is tied to a fixed point with configuration q_0 with a rigid rod of length a. The configuration q is constrained to the sphere

$$C^0 = \{q \in Q; \|q - q_0\| = a\}.$$

This is a system with a holonomic bilateral constraint. The set

$$C^1 = \{(q, \delta q) \in Q \times V; \|q - q_0\| = a, \langle g(q - q_0), \delta q\rangle = 0\}$$

of admissible virtual displacements is the tangent set $\mathsf{T}C^0$ of the holonomic constraint C^0. With the virtual work form $\sigma = 0$ the constitutive set is the set

$$S = \{(q, f) \in Q \times V^*; \|q - q_0\| = a, f = a^{-2}\langle f, q - q_0\rangle g(q - q_0)\}.$$

Example 5. The rigid rod of the Example 4 is replaced by a flexible string of length a. The configuration q is constrained to the closed ball

$$C^0 = \{q \in Q; \|q - q_0\| \leqslant a\}.$$

This is a system with a holonomic unilateral constraint. The set

$$C^1 = \{(q, \delta q) \in Q \times V; \|q - q_0\| \leqslant a, \langle g(q - q_0), \delta q\rangle \leqslant 0 \text{ if } \|q - q_0\| = a\}$$

of admissible virtual displacements is the tangent set $\mathsf{T}C^0$ of the configuration constraint C^0. With the virtual work form $\sigma = 0$ the constitutive set is the set

$$\begin{aligned} S = \{(q, f) \in Q \times V^*; \|q - q_0\| \leqslant a, f = 0 \text{ if } \|q - q_0\| \leq a, \\ f = \|f\| a^{-1} g(q - q_0) \text{ if } \|q - q_0\| = a\}.\end{aligned}$$

Example 6. Let Q be a Riemannian manifold with a metric tensor

$$g\colon \mathsf{T}Q \to \mathsf{T}^*Q.$$

A material point with configuration $q \in Q$ is subject to homogeneous, isotropic friction. The virtual work form is the mapping

$$\sigma\colon \mathsf{T}Q \to \mathbb{R}\colon \delta q \mapsto \rho\sqrt{\langle g(\delta q), \delta q\rangle}$$

with $\rho \geqslant 0$. The principle of virtual work is the inequality

$$\rho\sqrt{\langle g(\delta q), \delta q\rangle} \geqslant 0$$

satisfied for each virtual displacement $\delta q \in \mathsf{T}Q$. This inequality is obviously satisfied at each $q \in Q$ for each virtual displacement $\delta q \in \mathsf{T}_qQ$. Hence each configuration is an equilibrium configuration of the system. A covector $f \in \mathsf{T}^*Q$ is in the constitutive set if the inequality

$$\rho\|\delta q\| - \langle f, \delta q\rangle \geqslant 0$$

is satisfied for each virtual displacement δq such that $\tau_Q(\delta q) = \pi_Q(f)$. Let f be in the constitutive set. By using $\delta q = g^{-1}(f)$ in the preceding inequality we arrive at

$$\|f\|^2 \leqslant \rho\|f\|.$$

Hence,

$$\|f\| \leqslant \rho.$$

The inequality

$$\langle f, \delta q\rangle \leqslant \|f\|\|\delta q\|$$

is the result of the Schwarz inequality applied to the pair of vectors $g^{-1}(f)$ and δq such that $\tau_Q(\delta q) = \pi_Q(f)$. If $\|f\| \leqslant \rho$, then

$$\langle f, \delta q\rangle \leqslant \rho\|\delta q\|.$$

Hence, f is in the constitutive set. We conclude that the constitutive set of the system is the set

$$S = \{f \in \mathsf{T}^*Q; \|f\| \leqslant \rho\}.$$

Example 7. This is the affine version of Example 6. Let the configuration space Q be an affine space modelled on a Euclidean vector space V. The material point is not constrained and is subject to isotropic static friction. The virtual work is the function

$$\sigma\colon Q \times V \to \mathbb{R}\colon (q, \delta q) \mapsto \rho(q)\|\delta q\| = \rho(q)\sqrt{\langle g(\delta q), \delta q\rangle}.$$

The set

$$S = \{(q, f) \in Q \times V^*; \forall_{\delta q \in V}\, \rho(q)\|\delta q\| \geqslant \langle f, \delta q\rangle\} \tag{10}$$

is the constitutive set. Let $(q, f) \in S$. By setting $\delta q = g^{-1}(f)$ in the inequality

$$\rho(q)\|\delta q\| \geqslant \langle f, \delta q\rangle$$

we obtain the inequality

$$\rho(q)\|f\| \geqslant \|f\|^2.$$

Hence,

$$S \subset \{(q,f) \in Q \times F; \|f\| \leqslant \rho(q)\}.$$

Let (q,f) satisfy the inequality

$$\|f\| \leqslant \rho(q).$$

The relation

$$\langle f, \delta q\rangle \leqslant |\langle f, \delta q\rangle| \leqslant \|f\|\|\delta q\| \leqslant \rho(q)\|\delta q\|$$

is derived from the Schwarz inequality

$$|\langle f, \delta q\rangle| \leqslant \|f\|\|\delta q\|.$$

We have shown that

$$S = \{(q,f) \in Q \times F; \|f\| \leqslant \rho(q)\}.$$

Example 8. The material point with configuration $q \in Q$ in Example 3 is subject to friction. The virtual work form is the mapping

$$\sigma \colon Q \times V \to \mathbb{R} \colon (q, \delta q) \mapsto k\langle g(q - q_0), \delta q\rangle + \rho(q)\|\delta q\|.$$

The constitutive set is the set

$$S = \{(q,f) \in Q \times V^*; \forall_{\delta q \in V} k\langle g(q-q_0), \delta q\rangle + \rho(q)\|\delta q\| \geqslant \langle f, \delta q\rangle\}.$$

This set is the set constitutive set (10) of Example 7 with f replaced by

$$f - kg(q - q_0).$$

The expression

$$S = \{(q,f) \in Q \times V^*; \|f - kg(q - q_0)\| \leqslant \rho(q)\}$$

for the constitutive set is the result.

Example 9. Let M be an affine plane modelled on a Euclidean vector space V. The configuration space of a skate is the set $Q = M \times D$, where D is the projective space of directions in the affine space M. We use the Euclidean metric in M to identify the space D with the unit circle

$$D = \{\vartheta \in V; \langle g(\vartheta), \vartheta\rangle = 1\}.$$

Virtual displacements are elements of the space $M \times V \times \mathsf{T}D$, where

$$\mathsf{T}D = \{(\vartheta, \delta\vartheta) \in D \times V; \langle g(\vartheta), \delta\vartheta\rangle = 0\}.$$

The skate is a system with non holonomic constraints. The set C^0 is the entire space Q. The constraint consists in restricting virtual displacements in M to those parallel to the direction specified by an element of D. Thus

$$C^1 = \{(x, \delta x, \vartheta, \delta\vartheta) \in M \times V \times \mathsf{T}D; \exists_{\lambda\in\mathbb{R}} \delta x = \lambda\vartheta\}.$$

The constitutive set is a subset of the space $Q \times V^* \times \mathsf{T}^*D$. The space T^*D is specified as the set of pairs (ϑ, τ), where ϑ is in D and τ is in the quotient space $V^*/\mathsf{T}^\circ_\vartheta D$, where the space $\mathsf{T}^\circ_\vartheta D$ is the polar of the space $\mathsf{T}_\vartheta D \subset V$. The quotient space $V^*/\mathsf{T}^\circ_\vartheta D$ is dual to $\mathsf{T}_\vartheta D$. The set

$$\begin{aligned} S &= \{(x, f, \vartheta, \tau) \in Q \times V^* \times \mathsf{T}^*D; \langle f, \delta x\rangle + \langle \tau, \delta\vartheta\rangle = 0 \\ &\quad \text{for each } (x, \delta x, \vartheta, \delta\vartheta) \in C^1\} \\ &= \{(x, f, \vartheta, \tau) \in \mathsf{T}^*Q; \langle f, \vartheta\rangle = 0, \tau = 0\} \end{aligned}$$

is the constitutive set of the system with the virtual work form $\sigma = 0$. Let the skate be subject to friction represented by a non negative function $\rho\colon Q \to \mathbb{R}$. The virtual work is the function

$$\sigma\colon C^1 \to \mathbb{R}\colon (x, \delta x, \vartheta, \delta\vartheta) \mapsto \rho(x, \vartheta)\|\delta x\| = \rho(x, \vartheta)\sqrt{\langle g(\delta x), \delta x\rangle}.$$

The set

$$S = \{(x, f, \vartheta, \tau) \in \mathsf{T}^*Q; \forall_{(x,\delta x,\vartheta,\delta\vartheta)\in C^1} \rho(x, \vartheta)\|\delta x\| \geqslant \langle f, \delta x\rangle + \langle \tau, \delta\vartheta\rangle\}$$

is the constitutive set. The equality $\tau = 0$ is obtained by setting $\delta x = 0$ in the inequality

$$\rho(x, \vartheta)\|\delta x\| \geqslant \langle f, \delta x\rangle + \langle \tau, \delta\vartheta\rangle$$

with arbitrary $\delta\vartheta$. By setting $\delta x = \lambda\vartheta$ we arrive at the inequality

$$\rho(x, \vartheta)|\lambda| \geqslant \lambda\langle f, \vartheta\rangle$$

for each $\lambda \in \mathbb{R}$. The inequality must be satisfied for $\lambda = \langle f, \vartheta\rangle$. Hence

$$\rho(x, \vartheta)|\langle f, \vartheta\rangle| \geqslant \langle f, \vartheta\rangle^2$$

and $|\langle f, \vartheta\rangle| \leqslant \rho(x, \vartheta)$. If $|\langle f, \vartheta\rangle| \leqslant \rho(x, \vartheta)$, then

$$\rho(x, \vartheta)|\lambda| \geqslant |\lambda||\langle f, \vartheta\rangle| \geqslant \langle f, \lambda\vartheta\rangle$$

for each $\lambda \in \mathbb{R}$. It follows that the virtual work principle is satisfied. In conclusion we obtain the expression

$$S = \{(x, f, \vartheta, \tau) \in \mathsf{T}^*Q; |\langle f, \vartheta\rangle| \leqslant \rho(x, \vartheta), \tau = 0\}$$

for the constitutive set of the system.

Example 10. Let Q be the affine physical space. The example gives a formal description of experiments performed by Coulomb in his study of static friction. Let a material point be constrained to the set

$$C^0 = \{q \in Q; \langle g(k), q - q_0\rangle \geqslant 0\},$$

where q_0 is a point in Q and $k \in V$ is a unit vector. The boundary

$$\partial C^0 = \{q \in Q; \langle g(k), q - q_0\rangle = 0\}$$

is a plane passing through q_0 and orthogonal to k. In its displacements along the boundary the point encounters friction proportional to the component of the external force pressing the point against the boundary. The system is characterized by the virtual work function $\sigma = 0$ defined on the non holonomic constraint

$$\begin{aligned} C^1 = \{(q, \delta q) \in Q \times V; \langle g(k), q - q_0\rangle \geqslant 0, \\ \langle g(k), \delta q\rangle \geqslant \nu\sqrt{\|\delta q\|^2 - \langle g(k), \delta q)\rangle^2} \text{ if } \langle g(k), q - q_0\rangle = 0\}, \end{aligned}$$

where $\nu > 0$ is the coefficient of friction. The inequality

$$\langle g(k), \delta q\rangle \geqslant \nu\sqrt{\|\delta q\|^2 - \langle g(k), \delta q)\rangle^2}$$

defines a cone in the tangent space $\mathsf{T}_q Q$. The axis of the cone is the vector k and the angle 2ϑ such that $\nu = \cot\vartheta$ is the aperture. The principle of virtual work states that (q, f) is in the constitutive set S if and only if the inequality

$$\langle f, \delta q\rangle \leqslant 0$$

is satisfied for each $(q, \delta q) \in C^1$. If the material point is not on the boundary, then $\langle g(k), q - q_0\rangle > 0$. The virtual displacements are not constrained and a pair $(q, f) \in Q \times V^*$ is in the constitutive set S if and only if $f = 0$. If the material point is on the boundary, then $\langle g(k), q - q_0\rangle = 0$. We show that in this case a pair (q, f) is in the constitutive set if and only if the inequality

$$\sqrt{\|f\|^2 - \langle f, k\rangle^2\rangle} + \nu\langle f, k\rangle \leqslant 0$$

is satisfied. If $f = -\|f\|g(k)$, then (q, f) is in the constitutive set and $\|f\|^2 - \langle f, k\rangle^2 = 0$. Let (q, f) be in the constitutive set and let $\|f\|^2 - \langle f, k\rangle^2 \neq 0$. The virtual displacement $(q, \delta q)$ with

$$\delta q = g^{-1}(f) - \langle f, k\rangle k + \nu\sqrt{\|f\|^2 - \langle f, k\rangle^2}k$$

is in C^1 since

$$\langle g(k), \delta q\rangle = \nu\sqrt{\|f\|^2 - \langle f, k\rangle^2}.$$

From the principle of virtual work and

$$\langle f, \delta q\rangle = \|f\|^2 - \langle f, k\rangle^2 + \nu\sqrt{\|f\|^2 - \langle f, k\rangle^2}\langle f, k\rangle$$

it follows that

$$\|f\|^2 - \langle f, k\rangle^2 + \nu\sqrt{\|f\|^2 t - \langle f, k\rangle^2}\langle f, k\rangle \leqslant 0$$

and

$$\sqrt{\|f\|^2 - \langle f, k\rangle^2)} + \nu\langle f, k\rangle \leqslant 0$$

since $\|f\|^2 - \langle f, k\rangle^2 > 0$.

The Schwarz inequality

$$|\langle g(u), v\rangle - \langle g(k), u\rangle\langle g(k), v\rangle| \leqslant \sqrt{\|u\|^2 - \langle g(k), u\rangle^2}\sqrt{\|v\|^2 - \langle g(k), v\rangle^2}$$

for the bilinear symmetric form

$$(u, v) \mapsto (u - \langle g(k), u\rangle k | v - \langle g(k), v\rangle k) = \langle g(u), v\rangle - \langle g(k), u\rangle\langle g(k), v\rangle$$

applied to the pair $(g^{-1}(f), \delta q)$ leads to the inequality

$$\langle f, \delta q\rangle - \langle f, k\rangle\langle g(k), \delta q\rangle \leqslant \sqrt{\|f\|^2 - \langle f, k\rangle^2}\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2}.$$

If

$$\sqrt{\|f\|^2 - \langle f, k\rangle^2)} + \nu\langle f, k\rangle \leqslant 0$$

and

$$\langle g(k), \delta q\rangle \geqslant \nu\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2},$$

then

$$\sqrt{\|f\|^2 - \langle f, k\rangle^2}\sqrt{\|\delta q\|^2 - \langle g(k), \delta q\rangle^2} \leqslant -\langle f, k\rangle\langle g(k), \delta q\rangle.$$

It follows that $\langle f, \delta q\rangle \leqslant 0$. Hence, (q, f) is in the constitutive set S. We have shown that the set

$$S = \{(q, f) \in Q \times V^*; \langle g(k), q - q_0\rangle \geqslant 0, f = 0 \text{ if } \langle g(k), q - q_0\rangle > 0$$
$$\text{and } \sqrt{\|f\|^2 - \langle f, k\rangle^2} + \nu\langle f, k\rangle \leqslant 0 \text{ if } \langle g(k), q - q_0\rangle = 0\}$$

is the constitutive set of the system. The inequality

$$\sqrt{\|f\|^2 - \langle f, k\rangle^2} + \nu\langle f, k\rangle \leqslant 0$$

means that the vector $g^{-1}(f)$ is inside a cone in the tangent space $\mathsf{T}_q Q$. The vector $-k$ is the axis of the cone and the angle 2ϑ such that $\nu = \cot\vartheta$ is the aperture.

4 Partial control of static systems

We have considered control of static systems through interaction with systems with the same configuration space. This is not always the case. One can in general associate three distinct configuration spaces with a static system: the *internal configuration space* $\bar{Q}$, the *control configuration space* Q, and the *observed configuration space* $\tilde{Q}$. There are differential relations connecting the three spaces.

We will consider the cases when a static system with a configuration space $\bar{Q}$ is controlled by external devices in a configuration space Q and the relation between the two spaces is a differential fibration $\eta\colon \bar{Q} \to Q$. The configuration space $\bar{Q}$ of the controlled system is the internal configuration space and the configuration space Q of the controlling devices is the control configuration space. We will refer to such situations as cases of *partial control*. The observed configuration space $\tilde{Q}$ will coincide either with Q or with $\bar{Q}$.

4.1 Families of functions

An *internal energy function*

$$\bar{U} \colon \bar{Q} \to \mathbb{R}$$

is interpreted as a *family* of functions defined on fibres of the fibration η. The symbol $(\bar{U}, \eta)$ is used to denote this family.

A generating family $(\bar{U}, \eta)$ generates the constitutive set

$$S = \{f \in \mathsf{T}^*Q; \exists_{\bar{q} \in \bar{Q}} \eta(\bar{q}) = \pi_Q(f) \forall_{\delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q}} \langle \mathrm{d}\bar{U}, \delta\bar{q}\rangle = \langle f, \mathsf{T}\eta(\delta\bar{q})\rangle\} \tag{11}$$

of a partially controlled system.

We denote by $\mathsf{V}\bar{Q}$ the subbundle

$$\{\delta\bar{q} \in \mathsf{T}\bar{Q}; \mathsf{T}\eta(\delta\bar{q}) = 0\}$$

of vertical vectors. The set

$$Cr(\bar{U}, \eta) = \{\bar{q} \in \bar{Q}; \langle \mathrm{d}\bar{U}, \delta\bar{q}\rangle = 0 \text{ for each } \delta\bar{q} \in \mathsf{V}_{\bar{q}}\bar{Q}\}$$

is called the *critical set* of the family. If $\bar{q}$ satisfies the conditions stated in the definition of S, then the equality $\langle \mathrm{d}\bar{U}(\bar{q}), \delta\bar{q}\rangle = 0$ is obtained with $\delta q = 0$ and any vertical vector $\delta\bar{q} \in \bar{Q}_{\bar{q}}$. It follows that $\bar{q} \in Cr(\bar{U}, \eta)$.

There is a mapping

$$\kappa(\bar{U}, \eta) \colon Cr(\bar{U}, \eta) \to \mathsf{T}^*Q$$

characterized by

$$\langle \kappa(\bar{U}, \eta)(\bar{q}), \delta q\rangle = \langle \mathrm{d}\bar{U}, \delta\bar{q}\rangle$$

for each $\delta q \in \mathsf{T}_{\eta(\bar{q})}Q$ and each $\delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q}$ such that $\mathsf{T}\eta(\delta\bar{q}) = \delta q$. The constitutive set is the image of $\kappa(\bar{U}, \eta)$. Note that if

$$\kappa(\bar{U}, \eta)(\bar{q}) = f,$$

then

$$\pi_Q(f) = \eta(\bar{q}).$$

The constitutive set (11) describes the relation between the controlling force and the controlled configuration. It is used when the controlled configuration is the observed configuration. If the internal configuration is observed, then the constitutive set

$$\tilde{S} = \{(\bar{q}, f) \in \bar{Q} \times \mathsf{T}^*Q; \bar{q} \in Cr(\bar{U}, \eta), f = \kappa(\bar{U}, \eta)(\bar{q})\}$$

should be used.

4.2 Reduction of generating families

Let $(\bar{U}, \eta)$ be a family generating the set (11). We have the following obvious proposition.

Proposition 1. *Let $\bar{q} \in Cr(\bar{U}, \eta)$. The single point set*

$$S_{\bar{q}} = \{f \in \mathsf{T}^*Q; \pi_Q(f) = \eta(\bar{q}) \forall_{\delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q}} \mathrm{d}\bar{U}(\delta\bar{q}) = \langle f, \mathsf{T}\eta(\delta\bar{q})\rangle\}.$$

is represented in the form

$$S_{\bar{q}} = \{f \in \mathsf{T}^*Q; \pi_Q(f) = \eta(\bar{q}) \forall_{\delta q \in \mathsf{T}_{\eta(\bar{q})}Q} \sigma_{\bar{q}}(\delta q) = \langle f, \delta q\rangle\},$$

where

$$\sigma_{\bar{q}} \colon \mathsf{T}_{\eta(\bar{q})}Q \to \mathbb{R} \colon \delta q \mapsto \mathrm{d}\bar{U}(\delta\bar{q}), \delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q}, \mathsf{T}\eta(\delta\bar{q}) = \delta q. \tag{12}$$

It follows from the above proposition that if $Cr(\bar{U}, \eta)$ is the image of a section $\zeta \colon Q \to \bar{Q}$ of the fibration η then the family $(\bar{U}, \eta)$ generating the set S in (11) can be replaced by the function

$$\sigma \colon \mathsf{T}Q \to \mathbb{R} \colon (\delta q) \mapsto \sigma_{\zeta(\tau_Q(\delta q))}(\delta q),$$

where $\sigma_{\zeta(\tau_Q(\delta q))}$ is the function $\sigma_{\bar{q}}$ defined in the the formula (12) with $\bar{q} = \zeta(\tau_Q(\delta q))$. It is obvious that $\sigma = \mathrm{d}(\bar{U} \circ \zeta)$. Thus the set S is generated by the function $U = \bar{U} \circ \zeta$.

4.3 Examples

Example 11. Three material points with configurations q_0, q, and q' in the affine space Q are interconnected with springs with spring constants k_1, k_2, and k_3. The point q_0 is fixed and not controlled. The two points q and q' are not constrained. The configuration q' is not controlled. The internal configuration space is the affine space $\bar{Q} = Q \times Q$ of internal configurations $\bar{q} = (q, q')$ modelled on $V \times V$. The control configuration space is the space Q of controlled configurations q and V is the model space. The canonical projection

$$\eta \colon \bar{Q} \to Q \colon \bar{q} = (q, q') \mapsto q$$

is the relation between the two spaces. The internal energy is the function

$$\bar{U} \colon \bar{Q} \to \mathbb{R} \colon \bar{q} = (q, q') \mapsto \frac{k_1}{2}\|q - q_0\|^2 + \frac{k_2}{2}\|q' - q_0\|^2 + \frac{k_3}{2}\|q' - q\|^2.$$

The internal energy defines a family $(\bar{U}, \eta)$ of functions on fibres of the projection η. The critical set

$$Cr(\bar{U}, \eta) = \{\bar{q} = (q, q') \in \bar{Q}; (k_2 + k_3)(q' - q_0) - k_3(q - q_0) = 0\}$$

of the family is the image of the section

$$\zeta \colon Q \to \bar{Q} \colon q = q \mapsto (q, q_0 + k_3(k_2 + k_3)^{-1}(q - q_0))$$

of the projection η. The constitutive set is the set

$$S = \left\{(q, f) \in Q \times V^*; f = \frac{k_1k_2 + k_1k_3 + k_2k_3}{k_2 + k_3} g(q - q_0)\right\}.$$

Note that the presence of the material point with configuration q' can be ignored. This is due to the fact that the critical set is the image of a section of the projection η. The constitutive set is generated by the reduced internal energy function

$$U = \bar{U} \circ \zeta \colon Q \to \mathbb{R} \colon q \mapsto \frac{1}{2} \frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{k_2 + k_3} \|q - q_0\|^2.$$

This is the internal energy function

$$U \colon Q \to \mathbb{R} \colon q \mapsto \frac{k}{2} \|q - q_0\|^2.$$

of Example 3 with

$$k = \frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{k_2 + k_3}.$$

Example 12. The present example gives a simplified discrete model of the buckling of a rod. One end of the rod is a point in an affine space Q with configuration q constrained to the half-line

$$L = \{q \in Q; q - q_0 = \langle g(u), q - q_0 \rangle u, \langle g(u), q - q_0 \rangle > 0\}$$

starting at a point q_0 in the direction of a unit vector u. The other end is a point with configuration q' constrained to the plane

$$P = \{q' \in Q; \langle g(u), q' - q_0 \rangle = 0\}$$

through q_0 perpendicular to u. The rod can be compressed or extended in length but not bent. Its relaxed length is a and the elastic constant is k. The buckling of the rod is simulated by displacements of its end point in the plane P tied elastically to the point q_0 with a spring of spring constant k'. The configuration space $\bar{Q}$ is the product $Q \times Q$ with holonomic constraints represented by

$$C^0 = \{(q, q') \in \bar{Q}; q \in L, q' \in P\}.$$

The set

$$\begin{aligned} C^1 = \{&(q, q', \delta q, \delta q') \in \mathsf{T}\bar{Q}; q \in L, q' \in P, \\ &\delta q = \langle g(u), \delta q \rangle u, \langle g(u), \delta q' \rangle = 0\} \end{aligned}$$

of admissible virtual displacements is the tangent set of C^0. The internal energy of the system is the function

$$\bar{U} \colon C^0 \to \mathbb{R} \colon (q, q') \mapsto \frac{k}{2} (\|q - q'\| - a)^2 + \frac{k'}{2} \|q' - q_0\|^2.$$

The configuration q' is not controlled. The internal energy defines a family $(\bar{U}, \eta)$ of functions on fibres of

$$\eta \colon C^0 \to L \colon (q, q') \mapsto q.$$

The critical set is the union of sets

$$Cr_1(\bar{U},\eta) = \{(q,q') \in \bar{Q}; q \in L, q' = q_0\}$$

and

$$Cr_2(\bar{U},\eta) = \{(q,q') \in \bar{Q}; q \in L, q' \in P, (k+k')\|q'-q\| = ka\}.$$

The critical set $Cr_1(\bar{U},\eta)$ is the image of the section

$$\zeta_1 \colon L \to C^0 \colon q \mapsto (q,q_0).$$

The reduced internal energy

$$U_1 = \bar{U} \circ \zeta_1 \colon L \to \mathbb{R} \colon q \mapsto \frac{k}{2}(\|q-q_0\|-a)^2$$

generates the constitutive set

$$S_1 = \{(q,f) \in Q \times V^*; q \in L, \langle f,u\rangle = k(\|q-q_0\|-a)\}$$

The critical set $Cr_2(\bar{U},\eta)$ is not the image of a section of η. A reducion of the internal energy is still possible since the internal energy written in the form

$$\bar{U} \colon C^0 \to \mathbb{R} \colon (q,q') \mapsto \frac{k}{2}(\|q-q'\|-a)^2 + \frac{k'}{2}(\|q-q'\|^2 - \|q-q_0\|^2)$$

is a function only of the distance $\|q-q'\|$, and on the critical set $Cr_2(\bar{U},\eta)$ this distance is determined by

$$\|q-q'\| = \frac{ka}{k+k'}.$$

The result of the reduction is the function

$$U_2 \colon L \to \mathbb{R} \colon q \mapsto -k'\|q-q_0\|^2 + \text{Constant}.$$

It generates the constitutive set

$$S_2 = \{(q,f) \in Q \times V^*; q \in L, (k+k')\|q-q_0\| < ka, \\ \langle f,u\rangle = -k'\|q-q_0\|\}.$$

The constitutive set $S = S_1 \cup S_2$ is not a submanifold of $Q \times V^*$.

Example 13. A material point with configuration q' in the affine space Q is connected to a fixed point q_0 with a rigid rod of length a. A second material point with configuration q is tied elastically to q' with a spring of spring constant k. The configuration q' is not controlled. The internal configuration space $\bar{Q}$ is the product $Q \times Q$ with holonomic constraints represented by

$$C^0 = \{(q,q') \in \bar{Q}; \|q'-q_0\| = a\}.$$

The set

$$C^1 = \{(q,q',\delta q,\delta q') \in Q \times Q \times V \times V; \\ \|q'-q_0\| = a, \langle g(q'-q_0), \delta q'\rangle = 0\}$$

is the tangent set of C^0. The control configuration space is the space Q and the canonical projection

$$\eta\colon \bar{Q} \to Q\colon (q, q') \mapsto q$$

is the relation between the two spaces. The internal energy is the function

$$\bar{U}\colon C^0 \to \mathbb{R}\colon (q, q') \mapsto \frac{k}{2}\|q - q'\|^2$$

and

$$\begin{aligned} Cr(\bar{U}, \eta) = \{&(q, q') \in \bar{Q}; \|q' - q_0\| = a, \\ &q' - q = \langle g(q' - q_0), q' - q\rangle a^{-2}(q' - q_0)\}. \\ = \{&(q, q') \in \bar{Q}; \|q' - q_0\| = a, \\ &q' - q_0 = \pm a(q - q_0)\|q - q_0\|^{-1} \text{ if} q \neq q_0\}. \end{aligned}$$

is the critical set. The set

$$\begin{aligned} S = \{&(q, f) \in Q \times V^*; \|f\| = ka \text{ if } q = q_0, \\ &f = k(1 \pm a\|q - q_0\|^{-1})g(q - q_0) \text{ if } q \neq q_0\} \end{aligned}$$

is the constitutive set of the family $(\bar{U}, \eta)$. Note that the critical set is not the image of a section of η. For each control configuration q we have two different internal equilibrium configurations (q, q') if $q \neq q_0$ and an infinity of internal equilibrium configurations if $q = q_0$. The external force necessary to maintain the control configuration q depends on the internal configuration. Thus even if the internal configuration is not directly observed its presence can not be ignored. The constitutive set is the image of the injective mapping

$$\kappa_{(\bar{U},\eta)}\colon Cr(\bar{U}, \eta) \to Q \times V^*\colon (q, q') \mapsto (q, kg(q - q')).$$

If the internal configuration is observed, then the set

$$\tilde{S} = \{(q, q', f) \in Q \times Q \times V^*; (q, q') \in Cr(\bar{U}, \eta), f = kg(q - q')\}$$

can be used to describe the relation between the controlling force and the observed internal cofiguration.

4.4 Families of forms

A *generating family* of forms consists of a differential fibration

$$\eta\colon \bar{Q} \to Q$$

and a form

$$\bar{\sigma}\colon \mathsf{T}\bar{Q} \to \mathbb{R}.$$

The form $\bar{v}\sigma$ defines a family $(\bar{v}\sigma, \eta)$ of forms $\bar{v}\sigma_q$ on fibres of the fibration η. Each form $\bar{v}\sigma_q$ is the restriction of the form $\bar{v}\sigma$ to the set

$$\{\delta\bar{q} \in \mathsf{T}\bar{Q}; \eta(\tau_{\bar{Q}}(\delta\bar{q})) = q\}.$$

We denote by $\mathsf{V}\bar{Q}$ the subbundle

$$\{\delta\bar{q} \in \mathsf{T}\bar{Q}; \mathsf{T}\eta(\delta\bar{q}) = 0\}$$

of vertical vectors. The set

$$Cr(\bar{\sigma}, \eta) = \{\bar{q} \in \bar{Q}; \bar{\sigma}(\delta\bar{q}) \geqslant 0 \text{ for each } \delta\bar{q} \in \mathsf{V}_{\bar{q}}\bar{Q}\}$$

is called the *critical set* of the family.

A generating family $(\bar{\sigma}, \eta)$ generates the set

$$S = \{f \in \mathsf{T}^*Q; q = \pi_Q(f) \in Q, \exists_{\bar{q}\in\bar{Q}_q} \text{ if } \delta q \in \mathsf{T}_qQ, \\ \delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q}, \text{ and } \mathsf{T}\eta(\delta\bar{q}) = \delta q, \text{ then } \bar{\sigma}(\delta\bar{q}) \geqslant \langle f, \delta q\rangle\}.$$

If $\bar{q}$ satisfies the conditions stated in the definition of S, then the inequality $\bar{\sigma}(\delta\bar{q}) \geqslant 0$ is obtained with $\delta q = 0$ and any vertical vector $\delta\bar{q} \in \mathsf{V}_{\bar{q}}\bar{Q}$. It follows that $\bar{q} \in Cr(\bar{\sigma}, \eta)$. Consequently,

$$S = \bigcup_{\bar{q}\in Cr(\bar{\sigma},\eta)} S_{\bar{q}} \ ,$$

where

$$S_{\bar{q}} = \{f \in \mathsf{T}^*Q; q = \pi_Q(f) = \eta(\bar{q}), \text{ if } \delta q \in \mathsf{T}_qQ, \delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q} \\ \text{and } \mathsf{T}\eta(\delta\bar{q}) = \delta q, \text{ then } \bar{\sigma}(\delta\bar{q}) \geqslant \langle f, \delta q\rangle\}.$$

It can be shown that if $\bar{q} \in Cr(\bar{\sigma}, \eta)$, then the set $S_{\bar{q}}$ is not empty. The relation

$$\kappa(\bar{\sigma}, \eta)\colon Cr(\bar{\sigma}, \eta) \to \mathsf{T}^*Q$$

defined by

$$\operatorname{graph} \kappa(\bar{\sigma}, \eta) = \{(\bar{q}, f) \in Cr(\bar{\sigma}, \eta) \times \mathsf{T}^*Q; f \in S_{\bar{q}}\}$$

generalizes the mapping $\kappa(\bar{U}, \eta)$ introduced in Section 4.1. The constitutive set is the image of the relation. We refer to the set $S_{\bar{q}}$ as the *contribution* to the constitutive set S from the critical point $\bar{q}$.

4.5 Examples

Example 14. Let the point with configuration q' of Example 11 be subject to friction. The virtual work form is the family $(\bar{s}, \eta)$ with

$$\bar{\sigma}\colon Q \times Q \times V \times V \to \mathbb{R}\colon (q, q', \delta q, \delta q') \to k_1\langle g(q - q_0), \delta q\rangle \\ + k_2\langle g(q' - q_0), \delta q'\rangle + k_3\langle g(q' - q), \delta q' - \delta q\rangle + \rho\|\delta q'\|$$

and

$$\eta\colon \bar{Q} \to Q\colon (q, q') \mapsto q.$$

With a suitable choice of $\delta q'$ the expression

$$\langle -(k_2 + k_3)g(q' - q_0) + k_3 g(q - q_0), \delta q'\rangle$$

in the definition

$$\begin{aligned} Cr(\bar{\sigma},\eta) &= \{(q,q') \in \bar{Q}; \forall_{\delta q' \in V} \langle k_2 g(q'-q_0) + k_3 g(q'-q), \delta q' \rangle + \rho \|\delta q'\| \geqslant 0\} \\ &= \{(q,q') \in \bar{Q}; \forall_{\delta q' \in V} \langle -(k_2+k_3) g(q'-q_0) + k_3 g(q-q_0), \delta q' \rangle \leqslant \rho \|\delta q'\|\} \end{aligned}$$

of the critical set can reach its maximum

$$\|(k_2+k_3)(q'-q_0) - k_3(q-q_0)\| \|\delta q'\|.$$

Hence,

$$Cr(\bar{\sigma},\eta) = \{(q,q') \in \bar{Q}; \|(k_2+k_3)(q'-q_0) - k_3(q-q_0)\| \leqslant \rho\}$$

The critical set is not the image of a section of η. The pair $(q,f) \in Q \times V^*$ is in the constitutive set if the inequality

$$\begin{aligned} k_1 \langle g(q-q_0), \delta q\rangle &+ k_2 \langle g(q'-q_0), \delta q'\rangle \\ &+ k_3 \langle g(q'-q), \delta q' - \delta q\rangle + \rho\|\delta q'\| - \langle f, \delta q\rangle \geqslant 0 \end{aligned}$$

is satisfied for some $q' \in Q$ and all $(\delta q, \delta q') \in V \times V$. If the inequality is satisfied, then (q,q') is in the critical set and $\delta q'$ can be set to 0. The resulting inequality

$$(k_1+k_3)\langle g(q-q_0), \delta q\rangle - k_3 \langle g(q'-q_0), \delta q\rangle - \langle f, \delta q\rangle \geqslant 0$$

has the solution

$$f = (k_1+k_3) g(q-q_0) - k_3 g(q'-q_0).$$

Combining this result with the definition of the critical set we obtain the final expression

$$S = \left\{(q,f) \in Q \times V^*; \left\| f - \frac{k_1k_2 + k_1k_3 + k_2k_3}{k_2+k_3} g(q-q_0) \right\| \leqslant \frac{k_3}{k_2+k_3}\rho \right\}$$

for the constitutive set. The presence of the internal configuration q' can not be ignored. If it is known, then the force f is obtained from (14). The internal configuration q' can be observed. The set

$$\begin{aligned} \tilde{S} = \{(q,q',f) \in \bar{Q} \times V; &\|(k_2+k_3)(q'-q_0) - k_3(q-q_0)\| \leqslant \rho, \\ &f = (k_1+k_3) g(q-q_0) - k_3 g(q'-q_0)\} \end{aligned}$$

includes the information about the internal configuration.

Example 15. The material point with configuration q' in Example 13 is subject to friction. The family $(\bar{U},\eta)$ of functions is replaced by a family of forms $(\bar{\sigma},\eta)$ with

$$\bar{\sigma}\colon C^1 \to \mathbb{R}\colon (q,q',\delta q,\delta q') \mapsto k\langle g(q-q'), \delta q - \delta q'\rangle + \rho\|\delta q'\|.$$

The set

$$\begin{aligned} Cr(\bar{\sigma},\eta) = \{(q,q') \in \bar{Q}; &\|q'-q_0\| = a, \forall_{\delta q' \in V} \\ &\text{if } \langle g(q'-q_0), \delta q'\rangle = 0, \text{ then } k\langle g(q-q'), \delta q'\rangle \leqslant \rho\|\delta q'\|\} \end{aligned}$$

is the critical set of the family. The maximum value of the expression

$$k\langle g(q-q'),\delta q'\rangle$$

is

$$k\|(q-q')-a^{-2}\langle g(q-q'),q'-q_0\rangle(q'-q_0)\|\|\delta q'\|.$$

Hence,

$$Cr(\bar{\sigma},\eta)=\{(q,q')\in\bar{Q};k\|(q-q')-a^{-2}\langle g(q-q'),q'-q_0\rangle(q'-q_0)\|\leqslant\rho\}. \quad (13)$$

The critical set is not a section of the projection η. The expression

$$(q-q')-a^{-2}\langle g(q-q'),q'-q_0\rangle(q'-q_0)$$

is the component of $q-q'$ orthogonal to $q'-q_0$. If $q\neq q_0$, then $q'\in C^0$ must be such that the length this of component does not exceed ρ/k. If $q=q_0$, then all configurations $q'\in C^0$ are in the critical set. The pair $(q,f)\in Q\times V^*$ is in the constitutive set if the inequality

$$k\langle g(q-q'),\delta q-\delta q'\rangle+\rho\|\delta q'\|-\langle f,\delta q\rangle\geqslant 0$$

is satisfied for some $q'\in C^0$ and all $(\delta q,\delta q')\in V\times V$ such that $\langle g(q'-q_0),\delta q'\rangle=0$. If the inequality is satisfied, then (q,q') is in the critical set and terms with $\delta q'$ can be discarded. The resulting inequality

$$k\langle g(q-q'),\delta q\rangle-\langle f,\delta q\rangle\geqslant 0$$

leads to

$$f=kg(q-q'). \quad (14)$$

The set

$$\tilde{S}=\{(q,q',f)\in\bar{Q}\times V^*;k\|(q-q')-a^{-2}\langle g(q-q'),q'-q_0\rangle(q'-q_0)\|\leqslant\rho \\ f=kg(q-q')\}$$

contains the information about the force in terms of the internal configuration q'. The description of the constitutive set obtained from (13) and (14) is too complicated to be useful.

5 Clean composition

Let C_1 and C_2 be subsets of Q. If the intersection $C_1\cap C_2$ is not empty, we say that it is *clean* if

$$\mathsf{T}C_1\cap\mathsf{T}C_2=\mathsf{T}(C_1\cap C_2).$$

Example 16. We consider the composition of two holonomic systems. The constraints and the constitutive set for the first system are represented by the sets

$$C^0{}_1=\{q\in Q;\|q-q_1\|=a\},$$

$$C^1{}_1 = \{(q, \delta q) \in Q \times V; \|q - q_1\| = a, \langle g(q - q_1), \delta q\rangle = 0\},$$

and

$$S_1 = \{(q, f) \in Q \times V^*; \|q - q_1\| = a, f = a^{-2}\langle f, q - q_1\rangle g(q - q_1)\}.$$

For the second system we have

$$C^0{}_2 = \{q \in Q; \|q - q_2\| = a\},$$

$$C^1{}_2 = \{(q, \delta q) \in Q \times V; \|q - q_2\| = a, \langle g(q - q_2), \delta q\rangle = 0\},$$

and

$$S_2 = \{(q, f) \in Q \times V^*; \|q - q_2\| = a, f = a^{-2}\langle f, q - q_2\rangle g(q - q_2)\}.$$

If the distance $\|q_2 - q_1\|$ between the centres of the spheres $C^0{}_1$ and $C^0{}_2$ is less than $2a$, then the composed system is a system with holonomic constraints. The intersection of the constraints is clean since

$$\begin{aligned} C^1 = C^1{}_1 \cap C^1{}_2 = \{(q, \delta q) \in Q \times V; \|q - q_1\| = a, \|q - q_2\| = a, \\ \langle g(q - q_1), \delta q\rangle = 0, \langle g(q - q_2), \delta q\rangle = 0\} \end{aligned}$$

is the tangent set $\mathsf{T}C^0$ of the intersection

$$C^0 = C^0{}_1 \cap C^0{}_2 = \{q \in Q; \|q - q_1\| = a, \|q - q_2\| = a\}.$$

The constitutive set

$$\begin{aligned} S = \{(q, f) \in Q \times V^*; \|q - q_1\| = a, \|q - q_2\| = a, \langle f, \delta q\rangle = 0 \text{ for each} \\ \delta q \in V \text{ such that } \langle g(q - q_1), \delta q\rangle = 0 \text{ and } \langle g(q - q_2), \delta q\rangle = 0\}. \end{aligned}$$

is obtained from the principle of virtual work. At each $q \in C^0$ the set

$$S_q = \{f \in V^*; (q, f) \in S\}$$

is the sum

$$\{f \in V^*; (q, f) \in S_1\} + \{f \in V^*; (q, f) \in S_2\}.$$

If $\|q_2 - q_1\| = 2a$, then the set

$$C^0 = C^0{}_1 \cap C^0{}_2 = \{q \in Q; \|q - q_1\| = a, \|q - q_2\| = a\}$$

has only one element $q = q_1 + \frac{1}{2}(q_2 - q_1)$. The intersection $C^1{}_1 \cap C^1{}_2$ is the set

$$\left\{(q, \delta q) \in Q \times V; q = q_1 + \frac{1}{2}(q_2 - q_1), \langle g(q_2 - q_1), \delta q\rangle = 0\right\}.$$

The intersection of constraints is not clean since this intersection is not the tangent set of C^0. With

$$C^1 = \mathsf{T}C^0 = \left\{(q, f) \in Q \times V^*; q = q_1 + \frac{1}{2}(q_2 - q_1), \delta q = 0\right\}$$

the principle of virtual work produces the constitutive set

$$S = \left\{ (q, f) \in Q \times V^*; q = q_1 + \frac{1}{2}(q_2 - q_1) \right\}.$$

This is not the correct constitutive set for the composed system. The reason of this failure is that the approximative assumption of perfect rigidity of the separate constraints is no longer realistic in the case of a composition with

$$\mathsf{T}{C^0}_1 \cap \mathsf{T}{C^0}_2 \neq \mathsf{T}({C^0}_1 \cap {C^0}_2).$$

To obtain a complete description of the composed system the precise elastic properties of the constraints must be known. A partial characterization of the system is provided by the constitutive set

$$S_1 \cap S_2 = \left\{ (q, f) \in Q \times V^*; q = q_1 + \frac{1}{2}(q_2 - q_1), f = a^{-2}\langle f, q - q_1 \rangle g(q - q_1) \right\}$$

generated by the non holonomic constraint $C^1 = {C^1}_1 \cap {C^1}_2$. Note that this constraint is not integrable since the inclusion $C^1 \subset \mathsf{T}C^0$ does not hold.

6 A geometric setting for catastrophe theory

6.1 The framework

The traditonal approach to statics consists in studying equilibrium configurations of isolated systems. Catastrophe theory introduces elements of control to this approach. Families of isolated static systems are considered instead of separate single systems. Variations of equilibria within the family are studied. Applicability of this theory is somewhat limited since only unconstrained potential systems are considered.

We adapt the framework established in Section 4.1. to the catastrophe theory point of view. The base Q of the differential fibration

$$\eta \colon \bar{Q} \to Q$$

is the *control space*. The control configurations are not controlled by external forces. They are directly set by an external control mechanism. Fibres of the fibration are *behaviour spaces*. An internal energy function

$$\bar{U} \colon \bar{Q} \to \mathbb{R}$$

is interpreted as a family $Cr(\bar{U}, \eta)$ of potentials on the behaviour spaces parameterized by control configurations. The potential

$$U_q \colon \bar{Q}_q \to \mathbb{R}$$

corresponding to a control configuration $q \in Q$ is the restriction of $\bar{U}$ to the fibre $\bar{Q}_q = \eta^{-1}(q)$. The critical set

$$Cr(\bar{U}, \eta) = \{\bar{q} \in \bar{Q}; \langle \mathrm{d}\bar{U}, \delta\bar{q} \rangle = 0 \text{ for each } \delta\bar{q} \in \mathsf{V}_{\bar{q}}\bar{Q}\}$$

with

$$\mathsf{V}\bar{Q} = \{\delta\bar{q} \in \mathsf{T}\bar{Q}; \mathsf{T}\eta(\delta\bar{q}) = 0\}$$

is the *catastrophe manifold*. Each element $\bar{q}$ of the catastrophe manifold is an equilibrium configuration for the potential $U_{\eta(\bar{q})}$. A *catastrophe* is a singularity of the *catastrophe map*

$$\chi\colon Cr(\bar{U}, \eta) \to Q$$

obtained as the restriction of the projection η to $Cr(\bar{U}, \eta)$. A singularity occurs at a point $\bar{q} \in Cr(\bar{U}, \eta)$ at which the rank of the tangent mapping

$$\mathsf{T}\chi\colon \mathsf{T}Cr(\bar{U}, \eta) \to \mathsf{T}Q$$

changes. The change of multiplicity of critical points projecting onto the same configuration q is also an indication of a singularity.

The framework requires a obvious extension to families of holonomically constrained potentials in order to accomodate examples we want to present.

6.2 Examples

Example 17. In Example 13 we used the internal energy

$$\bar{U}\colon C^0 \to \mathbb{R}\colon (q, q') \mapsto \frac{k}{2}\|q - q'\|^2$$

defined on the holonomic constraint

$$C^0 = \{(q, q') \in \bar{Q}; \|q' - q_0\| = a\}.$$

The critical set

$$\begin{aligned} Cr(\bar{U}, \eta) = \{(q, q') \in \bar{Q}; \|q' - q_0\| = a, \\ q' - q = \langle g(q' - q_0), q' - q\rangle a^{-2}(q' - q_0)\}. \end{aligned}$$

was obtained. This set is now interpreted as the catastrophe manifold. Let D be the unit sphere

$$\{\vartheta \in V; \langle g(\vartheta), \vartheta\rangle = 1\}.$$

The critical set is the image of the injective mapping

$$\gamma\colon \mathbb{R} \times D \to \bar{Q}\colon (r, \vartheta) \mapsto (q_0 + (a + r)\vartheta, q_0 + a\vartheta).$$

The set

$$R \times \mathbb{R} \times \{(\vartheta, \delta\vartheta) \in V \times V; \vartheta \in D, \langle g(\vartheta), \delta\vartheta\rangle = 0\}$$

is the tangent set $\mathsf{T}(\mathbb{R} \times D)$. The tangent mapping

$$\begin{aligned} \mathsf{T}\gamma\colon & \mathsf{T}(\mathbb{R} \times D) \to Q \times Q \times V \times V \\ & \colon (r, \delta r, \vartheta, \delta\vartheta) \mapsto (q_0 + (a + r)\vartheta, q_0 + a\vartheta, \delta r\vartheta + (a + r)\delta\vartheta, a\delta\vartheta) \end{aligned}$$

is injective. It follows that γ is an injective immersion. The mapping

$$\chi\colon \mathbb{R}\times D \to Q\colon (r,\vartheta)\mapsto q_0+(a+r)\vartheta$$

represents the catastrophe map. It is obtained as the composition $\eta\circ\gamma$. The rank of the tangent mapping

$$\mathsf{T}\chi\colon \mathsf{T}(\mathbb{R}\times D)\to Q\times V\colon (r,\delta r,\vartheta,\delta\vartheta)\mapsto (q_0+(a+r)\vartheta, \delta r\vartheta+(a+r)\delta\vartheta)$$

is 3 if $a+r\neq 0$ and 1 if $a+r=0$. This indicates a singularity at $q=q_0$. Specialists will refuse to recognize this singularity as a catastrophe since, as we will see in the next example, it is not stable.

Example 18. We consider a modified version of Example 13. Let

$$k\colon V\to V$$

be a linear mapping positve and symmetric in the sense that

$$\langle g(k(\delta q_1)),\delta q_2\rangle = \langle g(k(\delta q_2)),\delta q_1\rangle$$

for each pair of vectors δq_1 and δq_2 and

$$\langle g(k(\delta q)),\delta q\rangle > 0$$

unless $\delta q=0$. We use the internal energy

$$\bar{U}\colon C^0\to\mathbb{R}\colon (q,q')\mapsto \frac{1}{2}\langle g(k(q-q')),q-q'\rangle$$

defined on the holonomic constraint

$$C^0=\{(q,q')\in\bar{Q};\|q'-q_0\|=a\}.$$

The critical set

$$\begin{aligned} Cr(\bar{U},\eta)=\{(q,q')\in\bar{Q};&\|q'-q_0\|=a,\\ &q'-q=a^{-2}\langle g(k(q'-q)),q'-q_0\rangle k^{-1}(q'-q_0)\}.\end{aligned}$$

is obtained. If $(q,q')\in Cr(\bar{U},\eta)$ and $q=q_0$, then

$$\|q'-q_0\|=a \tag{15}$$

and

$$q'-q_0=a^{-2}\langle g(k(q'-q_0)),q'-q_0\rangle k^{-1}(q'-q_0). \tag{16}$$

A configuration q' in the set

$$\{q'\in Q;\|q'-q_0\|=a\}$$

satisfies the equality (16) if $q'-q_0$ is an eigenvector of k. The number of such eigenvectors depends on the number of eigenvalues of k. If k has three distinct eigenvalues, then the number is 6. For q sufficiently far from q_0 there are two configurations

$$q' = q_0 \pm a\|k(q - q_0)\|^{-1}k(q - q_0)$$

satisfying (15) and the equation

$$k(q - q') = a^{-2}\langle g(k(q - q')), q' - q_0\rangle(q' - q_0)$$

approximated by

$$k(q - q_0) = a^{-2}\langle g(k(q - q_0)), q' - q_0\rangle(q' - q_0).$$

It is clear that the system described in Example 13 and Example 17 is not topologically stable.

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