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NEW METHODS OF DETERMINING THE PARAMETERS OF THE PHOTOMETRICAL CURVE OF A COMET DUST-GAS MODEL

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НОВЫЕ МЕТОДЫ ОПРЕДЕЛЕНИЯ ПАРАМЕТРОВ ФОТОМЕТРИЧЕСКОЙ КРИВОЙ ПЫЛЕ-ГАЗОВОЙ МОДЕЛИ КОМЕТЫ

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1. INTRODUCTION

In paper [1] the method of computation was given of the basic physical parameters of a comet dust-gas model under the assumption that the total reflecting surface of the dust particles in the coma was constant. A closer analysis, however, of the statistical material [2] comprising both long- and non-periodical comets with photometrical exponents lower than 5 revealed [3] that the mentioned assumption was incorrect in the case of a great contribution of the dust coma. Under certain acceptable assumptions the photometrical exponent of the dust radiation constituent can be written in the form [3, 4]:

$$n_d(r) = 2 - \frac{r}{\nu(r) + \left(\frac{R}{\varrho}\right)^2} \cdot \frac{\mathrm{d}}{\mathrm{d}r}\,\nu(r), \tag{1}$$

where r is the heliocentric distance of the comet, $\nu(r)$ the total number of photometrically effective dust particles in the cometary atmosphere at a given distance from the Sun, R the effective radius of the comet nucleus, i. e. the radius of the monolithic nucleus, and ϱ the radius of an average meteoric particle in the coma. Since the number of particles, as a rule, increases with decreasing heliocentric distance, the exponent $n_d(r) > 2$, and is, in addition, a function of r.

Thus, the first objective of the present paper is to derive general expressions for physical parameters of the dust-gas model of a comet which would be applicable to any form of the relation $n_d = n_d(r)$.

2. LIST OF SYMBOLS USED, AND FUNDAMENTAL EQUATIONS

Let us denote by symbols without index the quantities concerning the whole coma index d will be used for the same quantities concerning the dust coma and index g for those related with the gas coma. Let us introduce the following denotations:

$I_\Delta(r)$ — the brightness of the coma in the heliocentric distance and geocentric distance $\Delta = 1$ A.U.;
$H_\Delta(r)$ — the magnitude of the comet corresponding to $I_\Delta(r)$;
$n(r)$ — the photometrical exponent defined by the well-known formula;
$\eta(r)$ — the function giving the dependence of I_Δ on the heliocentric distance (physical exponent);
I_o — the absolute brightness of the comet;
H_o — the absolute magnitude of the comet;
B — the quantity resulting from the average heat of evaporation of gases L, the gas constant R_o, and the absolute temperature of the nucleus surface in $r = 1$ A.U., T_0;
k — the ratio of the absolute brightness of the dust- and gas coma;
ψ — the ratio of the brightness of the dust- and gas coma in a given heliocentrical distance.

Irrespective of the phase-angle the following relations are applicable:

$$I_\Delta = I_{\Delta d} + I_{\Delta g}, \tag{2}$$

$$I_{\Delta d} = I_{od} . r^{-\eta_d}, \tag{3}$$

$$I_{\Delta g} = I_{og} . r^{-\eta_g}, \tag{4}$$

$$I_\Delta = I_o . r^{-\eta}, \tag{5}$$

so that

$$I_\Delta = I_o \cdot \frac{kr^{-\eta_d} + r^{-\eta_g}}{1 + k}, \tag{6}$$

$$n(r) = \frac{n_d kr^{-\eta_d} + n_g r^{-\eta_g}}{kr^{-\eta_d} + r^{-\eta_g}}. \tag{7}$$

These relations apply to arbitrary forms of the functions $\eta_d = \eta_d(r)$ and $\eta_g = \eta_g(r)$ which are related to the corresponding photometrical exponents by differential equations of the form ($i = d, g$)

$$\ln r \cdot \frac{\mathrm{d}\eta_i(r)}{\mathrm{d} \ln r} + \eta_i(r) = n_i(r). \tag{8}$$

If we insert for n_g and η_g expressions from generalized [6] Levin's formula [5]

$$\left.\begin{aligned} n_g &= \frac{\alpha}{2} + \alpha B r^\alpha, \\ \eta_g &= \frac{\alpha}{2} + B\frac{r^\alpha - 1}{\ln r}, \end{aligned}\right\} \tag{9}$$

where

$$\alpha = -\frac{\ln \dfrac{T}{T_o}}{\ln r}, \tag{10}$$

we obtain the following expression for the magnitude of the comet in the heliocentric distance r:

$$H_\Delta(r) = H_o + 2.5 \log \frac{1 + k}{kr^{-n_d} + r^{-\frac{\alpha}{2}} \exp\left[B(1 - r^\alpha)\right]} . \tag{11}$$

Hence, the photometrical curve of the comet is characterized by three parameters, H_0, B, k, called the physical parameters.

3. DERIVATION OF THE GENERAL FORM OF THE EXPRESSIONS FOR THE PHYSICAL PARAMETERS

Expanding (11) into a series of the form:

$$H_\Delta(r) = \sum_{i=0}^{p} a_i \left(\log \frac{r}{r_o}\right)^i \tag{12}$$

(r_o is the geometrical mean of the heliocentric distances for which measurements of the comet brightness were carried out) and neglecting the terms with $p > 2$ we obtain, with respect to

$$\frac{\mathrm{d}}{\mathrm{d} \log r}\left(r^{-n_d}\right) = - r^{-n_d} \cdot \frac{n_d(r)}{\mathrm{mod}} ,$$

the following expressions for the coefficients a_i:

$$\left.\begin{aligned} a_o &= H_\Delta(r_o), \\ a_1 &= \frac{5}{4(1+\psi)}\left[\alpha + n_d\psi + 2\alpha B r_o^\alpha\right], \\ a_2 &= \frac{5}{16\,\mathrm{mod.}(1+\psi)^2}\left[-\frac{\alpha^2}{4}\psi + \psi n_d(\alpha - n_d) + \psi\,\mathrm{mod}\,(1+\psi)\frac{\mathrm{d}\,n_d}{\mathrm{d}\log r} + \right. \\ &\quad \left. + (\alpha + 2\psi n_d)\,\alpha B r_o^\alpha - \psi\alpha^2 B^2 r_o^{2\alpha}\right]. \end{aligned}\right\} \tag{13}$$

Quantities ψ, n_d and $\frac{\mathrm{d}\,n_d}{\mathrm{d}\log r}$ must be taken in r_o. Eliminating $\psi(r_o)$ from the second and third equations of (13) we obtain the quadratic equation for B:

$$\alpha^2 B^2 r_o^{2\alpha}\,\Delta\nu + \alpha B r_o^\alpha \Delta\mu + \mu_d\nu - \mu\nu_d + \frac{1}{4}\alpha^2\Delta\nu = 0, \tag{14}$$

where

$$\left.\begin{aligned} \mu &= \frac{4}{25}a_1^2 - \frac{4}{5}\,\mathrm{mod}\,a_2, \\ \mu_d &= n_d^2 - \mathrm{mod}\frac{\mathrm{d}n_d}{\mathrm{d}\log r}, \\ \nu &= \frac{\alpha}{2} - \frac{2}{5}a_1, \\ \nu_d &= \frac{\alpha}{2} - n_d, \\ \Delta\mu &= \mu_d - \mu, \\ \Delta\nu &= \nu_d - \nu. \end{aligned}\right\} \tag{15}$$

Then the sought for root of (14) is equal to:

$$B = \frac{1}{2\alpha r_o^\alpha . \Delta\nu} \left[\Delta\mu + \left\{ (\Delta\mu)^2 - 4\Delta\nu \left(\nu\mu_d - \nu_d\mu + \frac{\alpha^2}{4}\Delta\nu\right) \right\}^{1/2} \right]. \quad (16)$$

For the ratio ψ we obtain the expression

$$\psi = \frac{\frac{1}{2}\alpha + \alpha B r_o^\alpha - \frac{2}{5}a_1}{\frac{2}{5}a_1 - n_d}, \quad (17)$$

and

$$k = \psi . r_o^{\eta_d - \frac{\alpha}{2}} . \exp\left[B\left(1 - r_o^\alpha\right)\right]. \quad (18)$$

The fundamental equation (11) together with (13) and the other equations gives the expression for the absolute brightness H_o.

Thus, equations (16), (18) and (11) together with the other equations make it possible to determine the physical parameters of the comet designed according to the dust-gas model for an arbitrary form of the dependence of the photometrical exponent of the dust coma on the heliocentric distance.

In the special case, when

$$n(r_o) = n_d(r_o)$$

equation (16) becomes irrelevant. The heat of evaporation is now given by the requirement of a finite solution of equation (17), so that

$$B = \frac{1}{r^\alpha}\left(\frac{2a_1}{5\alpha} - \frac{1}{2}\right), \quad (20)$$

and by inserting (19) and (20) into the last equation of (13) we determine:

$$\psi = \frac{\frac{\alpha}{2\,\mathrm{mod}}\left(a_1 - \frac{5}{4}\alpha\right) - a_2}{a_2 - \frac{5}{4}\frac{\mathrm{d}\,n_d}{\mathrm{d}\log r}}. \quad (21)$$

The parameters k and H_o will be determined in the same way as before.

4. FORM OF THE DEPENDENCE OF THE PHOTOMETRICAL EXPONENT OF THE DUST COMA ON THE HELIOCENTRIC DISTANCE

The form of the dependence of the total photometrical exponent on the heliocentric distance derived from an abundant material in paper [3] and presented in Fig. 1 of the present paper, may be used for the determination of the photometrical and physical exponent of the dust part of the cometary atmosphere, provided that we know the probable values of the average parameters B and k, applicable to the set of investigated comets. After modification of equations (5), (6) and (7) we obtain for n_d and η_d the following expressions:

$$n_d = n + \frac{n - n_g}{(1 + k)r^{\eta_g - \eta} - 1}, \quad (22)$$

$$\eta_d = -\frac{\log\left[(1+k)r^{-\eta} - r^{-\eta_0}\right] - \log k}{\log r}. \quad (23)$$

It was in this way that the measurements $n = n(r)$, summarily given in [3], were treated for $\alpha = 0.5$ [7, 8] and for two different combinations of the parameters B and k:

I) $\overline{B} = 5.5 \qquad \overline{k} = 1.75$

II) $\overline{B} = 7.0 \qquad \overline{k} = 1.00$

The obtained exponents n_d are plotted against the heliocentric distance in Fig. 1, where full circles indicate the Case I, and open circles the Case II. The curves in Fig. 1 represent the dependences given by the expressions:

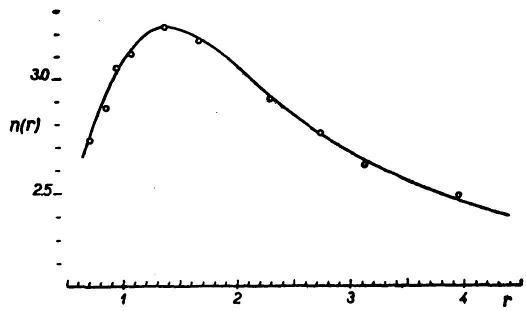

Fig. 1. Statistical dependence of the total photometrical exponent of long-periodical and non-periodical comets on the heliocentric distance.

Case I

$n_d = 3.16 - (r - 1.16)^2 \qquad$ for $r \leq 1.68$ A.U.

$n_d = 2 + 2.5r^{-2} \qquad$ for $r \geq 1.68$ A.U.

Case II

$n_d = 2.57 - 0.84\,(r - 1.45)^2 \qquad$ for $r \leq 1.56$ A.U.

$n_d = 2 + 0.70r^{-1/2} \qquad$ for $r \geq 1.56$ A.U.

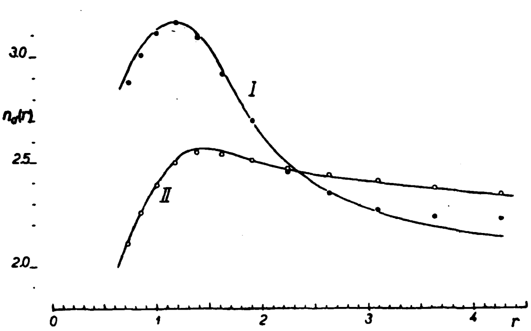

Fig. 2. Dependence of the photometrical exponent of the dust coma on the heliocentric distance for two combinations of parameters B and k.

The corresponding physical exponents are as follows:

Case I

$$\eta_d = 1.81 + 0.79\,\frac{r-1}{\log r}\,(1 - 0.275r) \qquad \text{for } r \leq 1.68 \text{ A.U.}$$

$$\eta_d = 2 + \frac{0.44}{\log r}\left(1 - \frac{1.234}{r^2}\right) \qquad \text{for } r \geq 1.68 \text{ A.U.}$$

Case II

$$\eta_d = 0.78 + 0.88 \frac{r - 1}{\log r}(1 - 0.208r) \qquad \text{for } r \leq 1.56 \text{ A.U.}$$

$$\eta_d = 2 + \frac{0.58}{\log r}\left(1 - \frac{1.044}{r^{1/2}}\right) \qquad \text{for } r \geq 1.56 \text{ A.U.}$$

We may assume that both curves in Fig. 2 are certain limits of the most probable course of the dependence $n_d = n_d(r)$.

5. CHANGES IN THE RATIO OF INTENSITY OF BOTH COMA CONSTITUENTS WITH VARYING HELIOCENTRIC DISTANCE

By investigating the course of the ratio of intensity of both coma parts which, according to (18), is given by the expression (for $\alpha = 0.5$):

$$\psi(r) = k . r^{1/4-\eta_d} . \exp\left[B\left(r^{1/2} - 1\right)\right], \tag{24}$$

we find that this function reaches the extreme value in the heliocentric distance r_1; then there applies the relation

$$n_d(r_1) = n_g(r_1) \tag{25}$$

which is independent of the basic parameters k and H_0. Hence it is possible to determine the relation between the heat of evaporation of molecules and the photometrical exponent of the dust coma in a given heliocentric distance, as

$$B = \left[2n_d(r_1) - \frac{1}{2}\right] \cdot r_1^{-1/2}. \tag{26}$$

The problem of determining r_1 will be discussed in the following paragraph.

Let us continue by determining the type of the extreme at the heliocentric distance r_1. The second derivative of (24) regarding r in connection with the expressions for $n_d = n_d(r)$ gives, after elimination of B by inserting from (26), the following inequalities in the Case I:

$$r_1^2 - 0.77r_1 + 0.52 \gtrless 0 \qquad \text{for } r_1 \leq 1.68 \text{ A.U.}$$

$$1.75 + \frac{12.5}{r_1^2} \gtrless 0 \qquad \text{for } r_1 \geq 1.68 \text{ A.U.}$$

and in the Case II:

$$r_1^2 - 0.97r_1 + 1.52 \gtrless 0 \qquad \text{for } r_1 \leq 1.56 \text{ A.U.}$$

$$1.75 + \frac{1.4}{r_1^{1/2}} \gtrless 0 \qquad \text{for } r_1 \geq 1.56 \text{ A.U.}$$

For an arbitrary real and positive r_1 there applies the uppper sign, so that the sought for extreme $\psi(r)$ is always the minimum.

The relation between parameter B and the heliocentric distance r_1 for the

Cases I, II of the dependence $n_d = n_d(r)$ as well as for $n_d \equiv 2$ is presented in Fig. 3. In general, the distance of the minimum ratio ψ decreases with increased heat of evaporation. Interesting is the form of the curve in the Case II: the whole wide interval of the heliocentric distances r_1 from 0.1 A.U. to 1.3 A.U. corresponds to the narrow range of the evaporation heat from 2800 cal/ /mol to 3500 cal/mol only.

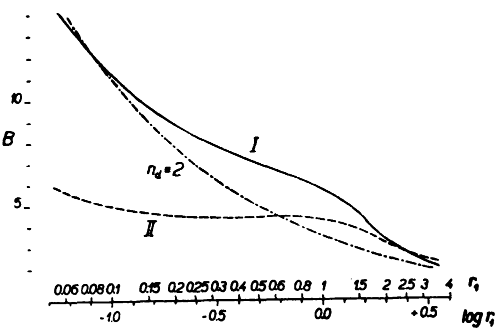

Fig. 3. Correlation of the heat of evaporation of molecules with the position of the minimum on curve $\psi = \psi(r)$ for various forms of function $n_d = n_d(r)$.

For an α different from 0.5 the described method cannot be applied, becausethe most probable values of $\overline{B}$ and $\overline{k}$ are not known, not even approximately. Nevertheless, we can expect that the maximum photometrical exponent of the dust coma will be somewhat higher for $\alpha < 0.5$.

6. THE HELIOCENTRIC DISTANCE r_1, AND DETERMINATION OF THE PHYSICAL PARAMETERS

The determination of the heliocentric distance r_1 corresponding to the minimum value of ratio ψ is a problem of application of suitable methods which demonstrate the contribution of both constituents to the total radiation of a comet. These methods are as follows:

a) spectral investigation of comets;

b) polarization measurements of the cometary light;

c) measurements of the monochromatic brightness of comets and determination of the photoelectrical indices.

Since all these measurements are rather difficult to obtain, they can be applied only to the brightest comets. If we succeed in determining the heliocentric distance r_1 and ratio ψ of the intensities of both constituents in r_1, the further computation of the physical parameters will be simple.

From condition (25) applying to the moment of minimum ratio ψ there follows immediately — e. g. from equation (7) — that either partial exponent may be replaced in the corresponding heliocentric distance by the total photometrical exponent which can be derived from the photometrical curve of the integral radiation of a comet. This case has been discussed in fine of paragraph 3, and therefore equation (26) can be substituted by equation (20); the latter equation and the known photometrical exponent directly give the heat of evaporation.

Then, in addition to a_1, from the measurements of the integral comet brightness also the coefficient a_2 is derived in series (10) expanded in r_1. For the distance r_1 in close vicinity of $r = 1$ A.U.

$$\eta(r_1) = n(r_1) - 0.4a_2(r_1) \log r_1, \tag{27}$$

as follows from equation (8) regarding (12). Moreover we obtain from equations (5) and (6) with respect to the expression (18):

$$\eta_d(r_1) = \frac{\log\,[(1 + \psi) r_1^{\eta} - r_1^{\eta_g}] - \log \psi}{\log r_1}. \qquad (28)$$

Inserting $\eta(r_1)$ from (27), $\psi(r_1)$ from the spectral, polarization or monochromatic measurements and B from (20) we derive $\eta_d(r_1)$ and there is nothing that would prevent us from determining parameter k from (18). The derivation of the absolute magnitude H_0 is, then, quite simple. The expression (21) may serve for the determination of the derivative $\frac{\mathrm{d}n_d}{\mathrm{d}\log r}$, so that the described method gives the photometrical exponents as well as their derivatives for both constituents of the coma in the heliocentric distance r_1.

7. CONCLUSIONS

The present paper is concerned with the method of computing the basic physical paramaters of the photometrical curve of a comet designed according to the dust-gas model; it is assumed that the gas coma complies with Levin's generalized formula, while the dust-coma intensity changes with heliocentric distance according to an arbitrary known relation. Next a brief description is given of the derivation of the most probable form of this relation from the statistical material on the total photometrical exponents; for this purpose the most probable values of the evaporation heat and of the ratio of intensities of both parts of the coma must be adopted.

The final part of this paper is devoted to the method of determining the physical parameters of both constituents of the cometary atmosphere. This method is based on the determination of the heliocentric distance of a comet, at which the contribution of the dust part to the total radiation of the coma is minimum. However, it is applicable only to comets with a sufficiently small perihelion distance and in which the heat of evaporation of molecules does not exceed 4500—5000 cal/mol. The considerable requirements expected to be met by the observational methods must be considered a disadvantage: if this physical method is to be used, some data at least must be available on the monochromatic intensities, the spectrum or the polarization of the light of the comet. The degree of completeness of these data is then reflected in the accuracy and reliability of the results.

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Souhrn

Práce podává metodu odvození fyzikálních parametrů vystupujících ve výrazu pro závislost jasnosti komety na heliocentrické distanci v případě, že plynná komponenta záření se řídí zobecněnou formulí Levinovou, a pro libovolný ale známý tvar změn jasu komponenty prachové. Tyto změny, vyjádřené fotometrickým exponentem n_d, se dají statisticky stanovit alespoň v jistém přiblížení.

Zajímavá metoda, kterou však lze aplikovat jen na komety s menší hodnotou výpar-

ného tepla molekul, se zakládá na stanovení heliocentrické distance, v níž je podíl prachové komponenty na celkovém záření komety minimální. Tato metoda ovšem předpokládá alespoň částečné údaje o monochromatických intensitách, o spektru a polarizaci světla komety.

Резюме

В работе приведен метод выведения физических параметров выступающих в выражении для зависимости блеска комет от гелиоцентрического расстояния в случае что газовая составляющая дается обобщенной формулой Левина и для любой известной формы изменений блеска пылевой составляющей. Эти изменения, выраженные фотометрическим показателем степени n_d, могут быть определены хотя бы в некотором приближении.

Интересный метод, который однако может быть применен лишь для комет с меньшим значением теплоты десорбции молекул основан на опредении гелиоцентрического расстояния соответствующего минимальному участию пылевой составляющей в общем блеске кометы. Этот метод однако требует по крайней мере частичные данные о монохроматических интенсивностях, о спектре и о поляризации света кометы.