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On fixed sets under continuous transformations


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ON FIXED SETS UNDER CONTINUOUS TRANSFORMATIONS

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1. PRELIMINARIES

All spaces under consideration are supposed to be $T_1$-spaces, i.e. finite sets are closed. $\overline{M}$, or merely $\overline{M}$, will be used to denote the closure of a subset $M$ of a space $P$. If $\mathcal{M}$ is a family of subsets of a space $P$, then $\overline{\mathcal{M}}$, or merely $\mathcal{M}$, will be used to denote the family of all $\overline{M}$, $M \in \mathcal{M}$. A centred family of sets is a family $\mathcal{M}$ of sets with the finite intersection property, i.e., the intersection of every finite subfamily of $\mathcal{M}$ is non-void. $\beta (P)$ will be used to denote the Čech-Stone compactification of a completely regular space $P$.

A transformation of a space $P$ is a continuous mapping from $P$ to $P$. If $f$ is a transformation of $P$, then $f^n$ denotes the identity transformation of $P$, i.e. $f^n(x) = x$ for each $x$ in $P$, and by induction

$$f^{n+1} = f \circ f^n \quad (n = 0, 1, 2, \ldots)$$

Let $f$ be a transformation of a space $P$. A subset $M$ of $P$ is fixed under $f$, or merely $f$-fixed, if $f[M] \subseteq M$, or equivalently, if the restriction $f/M$ of $f$ to $M$ is a transformation of $M$.

Definition 1. Let $f$ be a transformation of a space $P$. A minimal $f$-fixed set is a non-void closed $f$-fixed subset $F$ of $P$ which contains no non-void closed $f$-fixed proper subset, i.e., if $\emptyset \neq F_1 \subseteq F$ and $F_1$ is a closed $f$-fixed set, then $F_1 = F$.

In the present note we shall investigate the existence and properties of minimal $f$-fixed sets. In the sequel, the following trivial assertions (a) and (b) will be used without references:

(a) Let $f$ be a transformation of a space $P$. The intersection of $f$-fixed sets is a $f$-fixed set. The closure of an $f$-fixed set is an $f$-fixed set.

Indeed, if $f[M] \subseteq M$ for each $M$ in $\mathcal{M}$, then

$$f \left( \bigcap \mathcal{M} \right) \subseteq \bigcap \{ f[M]; M \in \mathcal{M} \} \subseteq \bigcap \mathcal{M}$$

and if $f[M] \subseteq M$, then by continuity of $f$ we have

$$f[M] \subseteq f[M]$$

and consequently, $f[M] \subseteq M$. 

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Let \( f \) be a transformation of \( P \) and let \( x \) be a point of \( P \). The sets
\[
M(f, x) = \{ f^n(x); n = 0, 1, 2, \ldots \}
\]
and
\[
F(f, x) = M(f, x)
\]
are \( f \)-fixed. Moreover, if \( M \) is \( f \)-fixed and \( x \in M \), then \( M(f, x) \subset M \). If \( M \) is closed and \( f \)-fixed and if \( x \in M \), then \( F(f, x) \subset M \).

Thus we have proved.

**Theorem 1.** Let \( f \) be a transformation of a space \( P \). A non-void subset \( F \) of \( P \) is a minimal \( f \)-fixed set if, and only if,
\[
x \in F \Rightarrow F(f, x) = F
\]
In consequence, if \( F \) is a minimal \( f \)-fixed set, then \( f[F] = F \).

2. Existence of Minimal Fixed Sets

**Theorem 2.** If \( f \) is a transformation of a compact space \( P \), then every non-void closed \( f \)-fixed subset of \( P \) contains a minimal \( f \)-fixed set. In consequence, if \( P \neq \emptyset \), then there exists at least one minimal \( f \)-fixed set.

**Proof.** Let \( \mathcal{M} \) be the family of all non-void closed \( f \)-fixed subsets of \( P \). The family \( \mathcal{M} \) is ordered by inclusion. Clearly, minimal elements of \( \mathcal{M} \) are precisely minimal \( f \)-fixed sets. Thus we have to prove that for every \( F \) in \( \mathcal{M} \) there exists a minimal \( F_0 \in \mathcal{M} \) with \( F_0 \in \mathcal{M} \). It is sufficient to show that every linearly ordered subfamily \( \mathcal{R} \) of \( \mathcal{M} \) has a lower bound. But it is obvious. Indeed, \( N_0 = \cap \mathcal{R} \) is \( f \)-fixed as an intersection of \( f \)-fixed sets and \( N_0 \) is non-void, since \( P \) is a compact space. Clearly \( N_0 \subset N \) for each \( N \) in \( \mathcal{R} \).

Let us recall that a Lipschitz transformation of a metric space \((P, \varrho)\) is a transformation \( f \) of \( P \) such that
\[
x, y \in P \Rightarrow \varrho(f(x), f(y)) \leq \alpha \varrho(x, y).
\]
The following result is well-known.

**Theorem 3.** If \( f \) is a Lipschitz transformation of a complete metric space \((P, \varrho)\) with constant \( \alpha < 1 \), then there exists a point \( x \) in \( P \) such that \( f(x) = x \) and for every \( y \) in \( P \)
\[
\lim_{n \to \infty} \varrho(f^n(y), x) = 0,
\]
that is,
\[
y \in P \Rightarrow x \in F(f, y)
\]

We shall prove a generalization of the preceding theorem. Clearly, a transformation \( f \) of a metric space \((P, \varrho)\) is a Lipschitz transformation with constant \( \alpha \) if and only if
\[
M \subset P \Rightarrow d(f[M]) \leq \alpha d(M)
\]
where
\[
d(M) = \sup \{ \varrho(x, y); x \in M, y \in M \},
\]
\[
d(\emptyset) = 0.
\]
Definition 2. A diameter on a space $P$ is a non-negative function $d$ (values of $d$ are non-negative real numbers and $\infty$) defined for all $M \subseteq P$ and satisfying the following axioms:

- $(d_1)$ $M \subseteq N \subseteq P \Rightarrow d(M) \leq d(N)$
- $(d_2)$ $x \in P \Rightarrow d((x)) = 0$
- $(d_3)$ $d(M) = \inf \{d(U); U \text{ open}, U \supseteq M\}$
- $(d_4)$ $d(M) = d(M)$

A $d$-Cauchy family is a centred family $\mathcal{M}$ of subsets of $P$ such that

$$\inf \{d(M); M \in \mathcal{M}\} = 0.$$ 

A diameter $d$ on $P$ will be called complete, if $\bigcap M \neq \varnothing$ for every $d$-Cauchy family $\mathcal{M}$.

Definition 3. Let $d$ be a diameter on a space $P$. A transformation $f$ of $P$ will be called a Lipschitz transformation of $P$ with constant $\alpha$, if

$$M \subseteq P \Rightarrow d(f(M)) \leq \alpha d(M).$$

Theorem 4. Let $d$ be a complete diameter on a space $P$. If $f$ is a Lipschitz transformation of $P$ with constant $\alpha$, then every non-void closed $f$-fixed set of finite diameter contains a minimal $f$-fixed set.

Proof. Let $F$ be a non-void closed $f$-fixed set and let $d(F) < \infty$. Let us define by induction $F_0 = F$ and

$$F_{n+1} = f[F_n] \quad (n = 0, 1, 2, \ldots)$$

Then $F_n \supseteq F_{n+1} \neq \varnothing$ for all $n$ and

$$\lim_{n \to \infty} d(F_n) = 0$$

It is easy to see that

$$K = \bigcap \{F_n; n = 1, 2, \ldots\}$$

is a compact non-void $f$-fixed set. $K$ is $f$-fixed, since every $F_n$ is $f$-fixed. The diameter $d$ being complete, from (4) it follows at once that $K$ is non-void. To prove compactness of $K$, let $\mathcal{M}$ be a centred family of subsets of $P$. Clearly

$$M \in \mathcal{M} \Rightarrow d(M) \leq d(F_n) = d(F_n) \leq \alpha d(F_n)$$

and consequently,

$$M \in \mathcal{M} \Rightarrow d(M) = 0$$

The diameter $d$ being complete, we have

$$\varnothing \neq \bigcap \mathcal{M}^P = \bigcap \mathcal{M}^K.$$ $

Thus $K$ is a compact $f$-fixed subspace of $P$. By Theorem 2 there exists a minimal $(f/K)$-fixed subset $K_0$ of $K$. Clearly $K_0$ is a minimal $f$-fixed set and $K_0 \subseteq K \subseteq F$. The proof is complete.

It is easy to see that a metric space $(P, \rho)$ is complete if and only if the diameter $d$ defined by (3) is complete in the sense of Definition (3). A transformation $f$ of $(P, \rho)$ is Lipschitz with constant $\alpha$ if and only if $f$ is a Lipschitz transformation of $(P, d)$ with constant $\alpha$. 

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Now the Theorem 3 follows easily from Theorem 4. Indeed, it is easy to see that
\[ d(F(f, y)) = d(M(f, y)) < \infty \]
Thus every \( F(f, y) \) contains a minimal \( f \)-fixed set.
If \( d(F) < \infty \) and \( f[F] = F \), then \( d(F) = 0 \) because \( d(F) \leq ad(F) \). But
\( d(M) = 0 \) if and only if \( M \) is at most one-point. Thus minimal \( f \)-fixed sets are
one-point and there exists only one minimal set. The fact
\[ \lim_{n \to \infty} d(F(f, f^n(y))) = 0 \]
follows from the fact that
\[ \lim_{n \to \infty} d(F(f, f^n(y))) = 0. \]

Let us recall that a completely regular space \( P \) is said to be topologically
complete in the sense of E. Čech, or merely complete, if \( P \) is a \( G_\delta \) in the Čech-
Stone compactification \( \beta(P) \) of \( P \). If \( P \) is complete and \( P \subset R, P = R \), then \( P \) is a \( G_\delta \) in \( R \).

**THEOREM. 5.** A completely regular space \( P \) is complete if and only if there
exists a complete diameter on \( P \).

Proof. In [2] it is proved that if there exists on \( P \) a complete diameter
satisfying only (d1), (d2) and (d3), then \( P \) is a complete space. Conversely,
let \( P \) be a \( G_\delta \) in a compactification \( K \) of \( P \). Let us choose open sets \( U_n \) in \( K \)
such that \( U_n \supset U_{n+1} \) and
\[ P = \bigcap \{ U_n; n = 1, 2, \ldots \}. \]
Let \( M \subset P \). If \( M \) and \( K - U_n \) are completely separated for no \( n \), then we
put \( d(M) = 1 \). (Of course, subsets \( N_1, N_2 \) of \( K \) are completely separated, if
there exists a continuous real-valued function \( f \) on \( K \) with \( f[N_1] = (0) \) and
\( f[N_2] = (1) \).) In the other case, let
\( d(M) = \inf \{ 1/n; M \) and \( K - U_n \) are completely separated \}. Clearly the con­
ditions (d1)—(d4) are fulfilled. It is sufficient to prove that \( d \) is complete.
Let \( \mathcal{R} \) be a \( d \)-Cauchy family. Let us choose \( M_n \in \mathcal{R} \) with \( d(M_n) < 1/n \).
Clearly we have \( \overline{M_n} \subset U_n \). It follows
\[ \emptyset \neq \bigcap \mathcal{R} \subset \bigcap \{ \overline{M_n}; n = 1, 2, \ldots \} \subset P. \]
But for every \( M \subset P \) we have
\[ \overline{M} \cap P = \overline{M} \]
and consequently,
\[ \bigcap \mathcal{R} = \bigcap \overline{M} \]
The proof is complete.

Note. To prove Theorem 4, it is sufficient to assume that \( d \) satisfies only
the conditions (d1) and (d4).

3. A QUOTIENT SPACE ASSOCIATED WITH A GIVEN TRANSFORMATION

Let \( f \) be transformation of a space \( Q \). Let \( \mathcal{D} \) be the family of subsets of \( Q \)
consisting from all minimal \( f \)-fixed sets and all one-point sets \( (x) \), where \( x \)
belongs to no minimal \( f \)-fixed set. Clearly, \( \mathcal{D} \) is a decomposition of \( Q \). Let us
consider the quotient space $K = Q/\mathcal{D}$. Let $\pi$ be the projection of $Q$ onto $K$. Clearly
\[ \pi(x) = \pi(y) = \pi(f(x)) = \pi(f(y)). \]

Let us define a transformation $\Phi$ of $K$ as follows
\[ \Phi(\pi(x)) = \pi(f(x)), \]
that is,
\[ \Phi \circ \pi = \pi \circ f. \]

Evidently, $\Phi$ is a mapping from $K$ to $K$. Both mappings $f$ and $\pi$ are continuous, and consequently, $\pi \circ f$ and hence $\Phi \circ \pi$ is continuous. $\pi$ being the quotient-mapping and $\Phi \circ \pi$ being continuous, the mapping $\Phi$ is continuous. Thus $\Phi$ is a transformation of $K$.

Proposition 1. If $M$ is a $f$-fixed subset of $Q$, then $\pi[M]$ is a $\Phi$-fixed set. Indeed, clearly
\[ \Phi[\pi[M]] = \pi[f[M]] \subset \pi[M]. \]

Proposition 2. If $N$ is a $\Phi$-fixed subset of $K$, then $\pi^{-1}[N]$ is a $f$-fixed subset of $Q$.
Proof. Clearly
\[ \pi[f(\pi^{-1}[N])] = \Phi[\pi[\pi^{-1}[N]]] = \Phi[N] \subset N. \]

It follows
\[ \pi^{-1}[\pi[F]] = F, \]
because
\[ \pi^{-1}[\pi[F]] = F. \]

The proof is complete.

If $F$ is a minimal $f$-fixed set, then $F \in \mathcal{D}$ and $f[F] = F$, and consequently
\[ \Phi(F) = F. \]

Thus minimal $f$-fixed sets are fixed points of $\Phi$. Conversely, fixed points of $\Phi$ are minimal $f$-fixed sets.

Thus we have proved the following assertion.

**THEOREM 6.** Let $f$ be a transformation of a space $Q$. There exists a quotient mapping $\pi$ of $Q$ onto a space $K$ and a transformation $\Phi$ of $K$ such that

1. If $F \subset Q$ is a minimal $f$-fixed set, then $\pi[F]$ is a fixed point of $\Phi$.
2. If $F \subset Q$ is $f$-fixed, then $\pi[F]$ is $\Phi$-fixed.
3. If $F \subset K$ is $\Phi$-fixed, then $\pi^{-1}[F]$ is $f$-fixed.
4. $\Phi \circ \pi = \pi \circ f$.

4. **ORRICULAR SPACES**

In this section we shall investigate spaces which are minimal $f$-fixed sets under a transformation of a space.

Let $f$ be a transformation of a space $P$ and let $F$ be a minimal $f$-fixed set. If $x \in P$, then $F(f, x) = F$ (See 2).

Definition. An orbicular transformation of a space $P$ is a transformation $f$ of $P$ such that
\[ x \in P \Rightarrow F(f, x) = P. \]
A space $P$ will be called orbicular, if there exists an orbicular transformation of $P$ and $P$ has at least two points.

Examples. If a space $P$ has the fixed-point property, that is, if every continuous transformation $f$ has a fixed point, then $P$ is not an orbicular space. For example, a closed interval of real-numbers (i.e., simple arc) and Euclidean cubes are not. On the other hand a simple closed curve (i.e., any space homeomorphical with the circle) is an orbicular space. It is easy to construct an orbicular transformation of the circle $K = \{ z; |z| = 1 \}$ of the complex plane. Let

$$f_\eta(z) = e^{i(\pi + \eta)}.$$ 

If $\eta = \alpha \pi, \alpha$ irrational, then $f_\eta$ is an orbicular mapping of $K$.

Proposition 3. Let $f$ be an orbicular transformation of a countably compact space $P$. Let $\{y_n\}$ be a sequence of points in $P$ such that

$$f(y_{n+1}) = y_n \ (n = 1, 2, \ldots)$$

Then the set $Y$ of all $y_n, n = 1, 2, \ldots$ is dense in $P$.

Proof. For every $k = 1, 2, \ldots$, let $Y_k$ be the closure of the set $\{y_n; n \geq k\}$. Put

$$F = \bigcap_{k=1}^{\infty} Y_k.$$ 

The space $P$ being countably compact, the set $F$ is non-void. By our assumption

$$f[Y_{k+1}] \subset Y_k$$ 

and hence

$$f[F] \subset \bigcap_{k=1}^{\infty} f[Y_{k+1}] \subset \bigcap_{k=1}^{\infty} Y_k = F.$$ 

Thus $F$ is a non-void closed $f$-fixed subset of $P$. $f$ being an orbicular mapping, $F = P$. In consequence, $Y$ is dense in $P$ because $\overline{Y} \supseteq Y_k \supseteq F$.

Proposition 4. Let $f$ be an orbicular transformation of an infinite space $P$: Let $x_0 \in P, X_0 = (x_0)$ and

$$X_{n+1} = f^{-1}[X_n] \ (n = 0, 1, 2, \ldots)$$

Then the sequence $\{X_n\}$ is disjoint.

Proof. Let us suppose that the proposition is not true. Let $n$ be the least integer such that $X_n \cap X_m \neq \emptyset$ for some $m > n$. Let $p$ be the least positive integer, with

$$X_n \cap X_{n+p} \neq \emptyset.$$ 

If $n > 0$, then

$$X_{n-1} \cap X_{n+p-1} \neq \emptyset$$

since

$$f[X_n] = X_{n-1}, f[X_{n+p}] = X_{n+p-1}.$$ 

But it is impossible. Thus $n = 0$. Since $X_0 = (x_0)$, we have $x_0 \in X_p$, and by definition of $X_k$ we have $f^p(x_0) = x_0$. In consequence

$$F(f, x_0) = \{ x_0, f(x_0), \ldots, f^{p-1}(x_0) \}.$$ 

The transformation $f$ being orbicular,

$$F(f, x_0) = P.$$
but it is impossible, since $P$ is an infinite set and $F(f, x_0)$ is a finite set. This contradiction completes the proof.

**Theorem 7.** Let $f$ be an orbicular transformation of a space $P$. Then either $f^2$ is an orbicular transformation of $P$ or there exist two disjoint subsets $F_1$ and $F_2$ of $P$ such that $F_1 \cup F_2 = P$, $f[F_1] = F_1$, $f[F_2] = F_2$, and $f^2[F_i]$ is an orbicular transformation of $F_i$ ($i = 1, 2$). In particular, if $P$ is connected, then $f^2$ is an orbicular transformation of $P$.

**Proof.** Let us suppose that $f^2$ is not an orbicular transformation of $P$. Thus there exists a point $x$ in $P$ such that the set

$$M = M(f^2, x) = \{f^{2n}(x); n = 0, 1, 2, \ldots\}$$

is not dense in $P$. Consider also the set

$$N = \{f^{2n+1}(x); n = 0, 1, 2, \ldots\}.$$ 

Since $f$ is an orbicular mapping, $M(f, x)$ is dense in $P$ and hence $(M(f, x) = M \cup N)$

$$M \cup N = P.$$ 

Clearly

$$f[M] \subset N, \quad f[N] \subset M,$$

and hence

$$f[M \cap N] \subset M \cap N.$$ 

Thus the set $M \cap N$ is $f$-fixed. The transformation $f$ being orbicular, we have either $M \cap N = \emptyset$ or $M \cap N = P$. By our assumption $M$ is not dense in $P$, and consequently, $M \cap N = \emptyset$. Moreover, from (3) it follows that $f[M] \subset M$ and $f[N] \subset N$. It remains to prove, that the restrictions $f^2/M$ and $f^2/N$ are orbicular. Let $y$ be an element of $M$. The set $M(f, y)$ is dense in $P$ and $M$ is open in $P$. Thus

$$M \cap M(f, y)$$

is a dense subset of $M$. But

$$M \cap M(f, y) = M(f^2, y).$$

Thus the restriction $f^2/M$ of $f$ to $M$ is an orbicular transformation of $M$. Analogously we can prove that $f^2/N$ is orbicular.

The preceding theorem has the following generalization.

**Theorem 8.** Let $f$ be an orbicular transformation of $P$ and let $p$ be a positive integer. Then either $f^p$ is an orbicular transformation of $P$ or there exist disjoint closed sets $F_0, \ldots, F_{p-1}$ such that

(a) $F_0 \cup \ldots \cup F_{p-1} = P$ 

(b) $f[F_0] \subset F_1, \ldots, f[F_{p-1}] \subset F_{p-1}, f[F_{p-1}] \subset F_0$.

(c) The restriction $f^p/F_i$ of $f^p$ to $F_i$ ($i = 0, 1, \ldots, p-1$) is an orbicular transformation of $F_i$.

In consequence, if $P$ has at most $p-1$ different components, then $f^p$ is an orbicular transformation of $P$.

As an immediate consequence of Theorem 8 we have the following

**Theorem 9.** If $f$ is an orbicular transformation of a connected space $P$, then $f^p$ is an orbicular transformation of $P$ for every positive integer $p$.

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Proof of Theorem 8. Let us suppose that the theorem is not true. Hence, there exists an orbicular transformation \( f \) of a space \( P \) and a positive integer \( p \) such that \( f^p \) is not an orbicular transformation of \( P \) and for no disjoint closed sets \( F_0, \ldots, F_{p-1} \) the conditions (a), (b) and (c) are fulfilled. Since \( f^p \) is not orbicular, there exists a point \( x \) in \( P \) such that the set

\[
M_0 = \{f^m(x); n = 0, 1, 2, \ldots\}
\]

is not dense in \( P \). Put

\[
M_1 = f[M_0] = \{f^{m+1}(x); n = 0, 1, 2, \ldots\}
\]

\[
M_{p-1} = f[M_{p-2}] = \{f^{m(p+1)-1}(x); n = 0, 1, 2, \ldots\}
\]

Since

\[
M_0 \cup \ldots \cup M_{p-1} = M(f, x)
\]

we have

\[
\overline{M}_0 \cup \ldots \cup \overline{M}_{p-1} = P.
\]

By continuity of \( f \) we have

\[
f[\overline{M}_0] \subset \overline{M}_1, \ldots, f[\overline{M}_{p-1}] \subset \overline{M}_{p-1}
\]

Clearly

\[
f[M_{p-1}] \subset M_0
\]

and hence also

\[
f[\overline{M}_{p-1}] \subset \overline{M}_0.
\]

The set

\[
F = \overline{M}_0 \cap \ldots \cap \overline{M}_{p-1}
\]

is \( f \)-invariant, since

\[
f[F] \subset \overline{M}_1 \cap \ldots \cap \overline{M}_{p-1} \cap \overline{M}_0 \subset F.
\]

Since \( f \) is an orbicular transformation of \( P \), we have either \( F = \emptyset \) or \( F = P \). If \( F = P \), then \( \overline{M}_0 = F = P \), which is impossible because \( M_0 \) is not dense in \( P \) (by our assumption). Hence \( F = \emptyset \). Let \( i + 1 \) be the least integer such that

\[
\overline{M}_0 \cap \ldots \cap \overline{M}_{i+1} = \emptyset.
\]

Put

\[
F_0 = \overline{M}_0 \cap \ldots \cap \overline{M}_i \neq \emptyset
\]

\[
F_1 = f[F_0], \ldots, F_{p-1} = f[F_{p-2}].
\]

It is easy to see that the sets \( F_0, \ldots, F_{p-1} \) are disjoint and \( f[F_{p-1}] \subset \overline{F}_0 \).

It follows that the set

\[
F = F_0 \cup \ldots \cup F_{p-1}
\]

is \( f \)-fixed (and of course closed) and hence \( F = P \). Clearly \( F_i \) satisfy the conditions (a) and (b). To prove (c) it is sufficient to notice that \( F_i \) are also open in \( P \). The proof is complete.

**Theorem 10.** Let \( f \) be an orbicular transformation of an infinite Hausdorff compact space \( K \). Let us choose points \( x_n \in K, n = 0, \pm 1, \pm 2, \ldots \), so that

\[
f(x_n) = x_{n+1} (n = 0, \pm 1, \pm 2, \ldots)
\]
Let $X$ be the set of all $x$ and let $\beta(X)$ be the Čech-Stone compactification of $X$. Let $g$ be the Čech-Stone continuous mapping of $\beta(X)$ onto $K$ (that is, the restriction of $g$ to $X$ is the identity mapping). Then there exists an orbicular transformation $\Phi$ of $\beta(X)$ such that

$$f \circ g = g \circ \Phi$$

The preceding equality uniquely defines continuous $\Phi$, and $\Phi$ is the unique continuous extension of $f/X$ to $\beta(X)$.

Proof. It is easy to see that $f[X] = X$ and hence $f/X$ is a continuous mapping from $X$ onto $X$. By Čech-Stone theorem, there exists a continuous mapping $\Phi$ from $\beta(X)$ onto $\beta(f[X]) = \beta(X)$. Thus $\Phi$ is a transformation of $\beta(X)$. Now we shall prove (4), i.e. the commutativity of the following diagram:

$$
\begin{array}{ccc}
K & \stackrel{f}{\longrightarrow} & K \\
\uparrow & & \uparrow \\
\beta(X) & \longrightarrow & \beta(X)
\end{array}
$$

Since $g/X$ is the identity mapping, we have

$$f \circ g/K = f/X.$$ 

Since $\Phi/X = f/X$, we have also

$$(g \circ \Phi)/X = g \circ (\Phi/X) = g \circ (f/X) = f/X$$

The last equality follows from the fact that $f[X] \subset X$. Since $f \circ g$ and $g \circ \Phi$ are continuous, $X$ is dense in $\beta(X)$ and the restrictions to $X$ are identical, we have $f \circ g = g \circ \Phi$.

It remains to prove that $\Phi$ is an orbicular transformation. Let $F$ be a minimal $\Phi$-fixed set in $\beta(X)$. Then

$$f[g[F]] = g[\Phi[F]] = g[F].$$

Thus $g[F]$ is a non-void closed $f$-fixed subset of $P$, and consequently, $g[F] = K$. In particular, $X \subset g[F]$, and hence, $X \subset F$. $X$ being dense in $\beta(X)$, $F = \beta(X)$.

Corollary. An orbicular space may fail to be a homogeneous space (in any usual sense). Indeed, if $K$ is a circle, then $K$ is a metrizable space, and in particular, every point of $K$ is of a countable character. Let $X$ be the set from theorem 10. Clearly every point of $X$ has a countable character in $\beta(X)$. On the other hand, no point of $\beta(X) - X$ has a countable pseudocharacter in $\beta(X)$. It can be noticed that in this case the space $\beta(X)$ is totally disconnected.

Problem. I do not know whether there exists a connected (locally connected) orbicular compact space which is not homogeneous.

Note. Let us recall that the character of a point $x$ of a space $P$ is the least cardinal of a local base at $x$ in $P$. The pseudocharacter is the least cardinal $m$ such that there exists a family $\mathfrak{M}$ of open subsets of $P$ such that the cardinality of $\mathfrak{M}$ is $m$ and the intersection of $\mathfrak{M}$ is the one-point set $\{x\}$. Let us recall, if $P$ is a compact space then characters and pseudocharacters of points coincide. For further informations see [3].

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O SAMODRUŽNÝCH MNOŽINÁCH PŘI SPOJITÝCH TRANSFORMACÍCH

Souhrn

Transformaci prostoru \( P \) rozumíme spojité zobrazení prostoru \( P \) do sebe. V článku se studují minimální uzavřené invariantní množiny při dané transformaci a dále prostory, které mohou být minimální uzavřenou samodružnou množinou při nějaké transformaci nějakého kompaktního prostoru. Přitom minimální uzavřenou samodružnou množinou transformace \( f \) prostoru \( P \) se rozumí neprázdná uzavřená množina \( F \subseteq P \), která je samodružná, to zn. \( f[F] = F \) a která neobsahuje žádnou neprázdnou uzavřenou samodružnou množinu \( F_1 \neq F \). Zřejmě každý pevný bod je minimální uzavřenou samodružnou množinou.

Snaží se seznámí se nahlédnout, že každá transformace kompaktního prostoru má minimální uzavřené samodružné množiny. Dokazuje se, že některé transformace topologicky úplného úplně regulárního prostoru mají minimální uzavřené samodružné množiny. Tato věta obsahuje jako speciální případ známou větu o existenci pevného bodu Lipschitzovských transformací úplného metrického prostoru.

Transformaci \( f \) prostoru \( P \) nazýváme cirkulární, jestliže pro každou postupnou interaci \( f(x), f^2(x) = f(f(x)), \ldots \) bodu \( x \) tvoří hustou množinu v \( P \). Prostor nazýváme cirkulárním, jestliže příslušející cirkulární transformaci. Zřejmě právě cirkulární prostory mohou být minimální uzavřenou samodružnou množinou při transformacích prostorů. Například kružnice je cirkulárním prostorem. Dále zřejmě všechny konečné prostory jsou cirkulární. Dá se dokázat, že Cantorovo discontuum je cirkulárním prostorem.

V článku se dokazuje m. j. několik výsledků o mocninách cirkulárních zobrazení.

**REFERENCES**