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ON A PROBLEM CONNECTED WITH THE TRANSPORTATION PROBLEM

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In this paper we discuss some questions arising in the transportation problem when solution has to be bounded by given constants.

As in [1] we use great Latin letters to denote real matrices of type (m, n). If $A = (a_{ij})$, $B = (b_{ij})$ and if $a_{ij} \leq b_{ij}$ for every $i = 1, 2, \ldots m$ and for every $j = 1, 2, \ldots, n$, we write $A \leq B$. If the sum of all elements in every line (row or column) of A equals the sum of all elements in the corresponding line of B, we write $A \sim B$. The null-matrix will be denoted by O.

PROBLEM. *Given $A \geq O$ and $B \geq O$ we have to decide whether there exists a matrix X with $O \leq X \leq B$ and $X \sim A$.*

We shall use the following notions: Given $U = (u_{ij})$ and $V = (v_{ij})$ we write $U \prec V$ if and only if

$$v_{ij} \geq 0 \quad \Rightarrow \quad v_{ij} \geq u_{ij} \geq 0 \qquad \text{and}$$

$$v_{ij} \leq 0 \quad \Rightarrow \quad v_{ij} \leq u_{ij} \leq 0$$

is true for every $i = 1, 2, \ldots, m$ and for every $j = 1, 2, \ldots, n$. If $U \prec V$ then $V - U \prec V$.

If for a given $V \sim O$ we have

$$V = U_1 + U_2 + \ldots + U_r \tag{1}$$

and $O \neq U_k \sim O$, $U_k \prec V$ for every $k = 1, 2, \ldots, r$ then we call (1) a *standard decomposition* of V. If $O \neq V \sim O$ is an *integer* matrix (with integer elements only) and if no standard decomposition (1) of V with *integer* matrices U_k is possible except the trivial one with $r = 1$ and $U_1 = V$, then we call V a *basic matrix*. In [1] all basic matrices are found ([1], Corolary 2, 1, page 192): *$V = (v_{ij})$ is basic if and only if there are two sequences of indices $i_1, i_2, \ldots, i_s$ and $j_1, j_2, \ldots, j_s$, each of them containing distinct numbers only, such that*

$$v_{i_1j_1} = -v_{i_1j_2} = v_{i_2j_2} = -v_{i_2j_3} = \ldots = -v_{i_{s-1}j_s} = v_{i_sj_s} = -v_{i_sj_1} = -1$$

and $v_{ij} = 0$ in all other cases.

This may be proved in connection with the following important lemma ([1], Theorem 2, 1, page 191):

LEMMA: *For every F such that $O \neq F \sim O$ it is always possible to find a standard decomposition*

(2) $$F = \varrho_1 U_1 + \varrho_2 U_2 + \ldots + \varrho_r U_r$$

with $\varrho_k > 0$ and with basic matrices U_k for every $k = 1, 2, \ldots, r$.

Using this lemma we may prove

THEOREM *1: Suppose $A \geq O$, $B \geq O$. Let $A \nleq B$, say $a_{i_1 j_1} > b_{i_1 j_1}$ for some fixed i_1 and j_1. Then if there is a matrix X such that $X \sim A$ and $O \leq X \leq B$, then there exists a basic matrix $U = (u_{ij})$ such that*

1) $$u_{ij} > 0 \;\Rightarrow\; a_{ij} < b_{ij}$$
$$u_{ij} < 0 \;\Rightarrow\; a_{ij} > 0$$

for every $i = 1, 2, \ldots, m$ and for every $j = 1, 2, \ldots, n$;

2) $u_{i_1 j_1} < 0$

Proof: Let $F = X - A$ so that $X = A + F$ and $O \neq F \sim O$. Using our lemma we find some standard decomposition (2) of F. We have $f_{i_1 j_1} < 0$ and consequently taking $U = U_k$ for suitable k we have $u_{i_1 j_1} < 0$. Now if $u_{ij} > 0$ then $f_{ij} > 0$ and $a_{ij} < x_{ij} \leq b_{ij}$. If $u_{ij} < 0$ then $f_{ij} < 0$ and $0 \leq x_{ij} < a_{ij}$.

REMARK 1: *If a basic matrix $U = (u_{ij})$ satisfies the conditions of theorem 1 then it is always possible to find a number $\varrho > 0$ such that*

$$u_{ij} > 0 \;\Rightarrow\; a_{ij} + \varrho u_{ij} \leq b_{ij}, \quad u_{ij} < 0 \;\Rightarrow\; a_{ij} + \varrho u_{ij} \geq 0$$

for every $i = 1, 2, \ldots, m$ and for every $j = 1, 2, \ldots, n$.

For any two matrices $U = (u_{ij})$, $V = (v_{ij})$ let $p_1(U, V)$ denote the set of all pairs (i, j) such that $u_{ij} > v_{ij}$ and $p_2(U, V)$ the set of all pairs (i, j) such that $u_{ij} \leq v_{ij}$. By $s(U, V)$ we denote the sum of all $u_{ij} - v_{ij}$ where $(i, j) \in p_1(U, V)$.

Thus the conditions 1) and 2) of theorem 1 may be written as $p_1(U, O) \subset p_1(B, A)$ and $(i_1, j_1) \in p_1(O, U) \subset p_1(A, O)$.

The conditions of remark 1. have the form $p_1(U, O) \subset p_2(A + \varrho U, B)$ and $p_1(O, U) \subset p_2(O, A + \varrho U)$. Notice that the following is true: $p_1(A + \varrho U, B) \subset p_1(A, B)$, $s(A + \varrho U, B) < s(A, B)$.

SOLUTION OF THE PROBLEM*): Our solution of the problem (for formulation see above) is based on a certain construction of a sequence $A = A_0, A_1, \ldots, A_k, \ldots$ such that $O \leq A_k \sim A$. $(k = 0, 1, \ldots)$. This sequence is constructed term by term and will stop in the following two cases:

1) We come to a matrix A_k such that $A_k \leq B$. Then our problem is solved by $X = A_k$.

2) We come to a matrix $A_k \nleq B$ and choosing some fixed pair $(i_1, j_1) \in p_1(A_k, B)$ we prove that there is no basic matrix U such that

(3) $$p_1(U, O) \subset p_1(B, A_k) \quad \text{and} \quad (i_1, j_1) \in p_1(O, U) \subset p_1(A_k, O).$$

Then from theorem 1 we conclude that our problem has no solution.

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*) for rational matrices A and B.

If for some A_k neither 1) nor 2) is satified then we construct A_{k+1} in the following way: We have already chosen some $(i_1, j_1) \in p_1(A_k, B)$ and we have found a basic matrix U such that (3). Using remark 1 we find greatest $\varrho > 0$ such that $p_1(U, O) \subset p_2(A_k + \varrho U, B)$ and $p_1(O, U) \subset p_2(O, A_k + \varrho U)$. Then putting $A_{k+1} = A_k + \varrho U$ we have $O \leq A_{k+1} \sim A_k \sim A$, $p_1(A_{k+1}, B) \subset \subset p_1(A_k, B)$ and $s(A_{k+1}, B) < s(A_k, B)$.

From that it follows that if A and B have *rational* elements then our sequence must be finite so that after a finite number of steps we come to the case 1) or 2).

Let us now discuss the case 2) in greater detail. Let $A \not\leq B$ and $(i_1, j_1) \in \in p_1(A, B)$. We have to decide whether there exists a basic matrix U such that

$$p_1(U, O) \subset p_1(B, A) \tag{4}$$

and

$$(i_1, j_1) \in p_1(O, U) \subset p_1(A, O)$$

On the set $\mathfrak{J} = \{1, 2, \ldots, n\}$ we define a binary relation α: for $j, j' \in \mathfrak{J}$ we write $j\alpha j'$ if and only if there exists an index $i = 1, 2, \ldots, m$ such that $(i, j) \in p_1(A, O)$ and $(i, j') \in p_1(B, A)$. Let $\mathfrak{J}_1$ be the set of all $j \in \mathfrak{J}$ such that $(i_1, j) \in p_1(B, A)$. Let $\overline{\mathfrak{J}_1}$ be the least subset of $\mathfrak{J}$ containing $\mathfrak{J}_1$ such that if $j \in \overline{\mathfrak{J}_1}$ and $j\alpha j'$ then $j' \in \overline{\mathfrak{J}_1}$. Now we can prove

THEOREM 2: *For the existence of a basic matrix U satisfying* (4) *it is necessary and sufficient that $j_1 \in \overline{\mathfrak{J}_1}$.*

Proof: Let $U = (u_{ij})$ be a basic matrix satisfying (4). We can find two sequences of indices $i_1, i_2, \ldots, i_s$ and $j_1, j_2, \ldots, j_s$ such that $u_{i_1j_1} = -u_{i_1j_2} = = u_{i_2j_2} = -u_{i_2j_3} = \ldots = -u_{i_{s-1}j_s} = u_{i_sj_s} = -u_{i_sj_1} = -1$.

We have $j_2 \in \mathfrak{J}_1$, $j_2\alpha j_3$, $j_3\alpha j_4$, $\ldots$, $j_s\alpha j_1$ and consequently $j_1 \in \overline{\mathfrak{J}_1}$.

Now let $j_1 \in \overline{\mathfrak{J}_1}$. We may find $j_2, j_3, \ldots, j_s$ such that $j_2 \in \mathfrak{J}_1$, $j_2\alpha j_3$, $j_3\alpha j_4$, $\ldots$, $j_s\alpha j_1$, making s at the same time as small as possible. It follows that $(i_1, j_2) \in \in p_1(B, A)$ and that there are some indices $i_2, i_3, \ldots, i_s$ satisfying $(i_k, j_k) \in \in p_1(A, O)$ and $(i_k, j_{k+1}) \in p_1(B, A)$ for all $k = 2, 3, \ldots, s$ (we put $j_{s+1} = j_1$).

From the fact that s is minimal it follows that $j_1, j_2, \ldots, j_s$ are distinct indices and that $i_k \neq i_{k+1} (k = 1, 2, \ldots, s - 1)$, $i_s \neq i_1$. Now putting $v_{i_kj_k} = -1$ $(k = 1, 2, \ldots, s)$, $v_{i_kj_{k+1}} = 1$ $(k = 1, 2, \ldots, s)$ and $v_{ij} = 0$ in all other cases we get an integer matrix $V = (v_{ij})$ satisfying the conditions (4). Making integer standard decomposition $V = U_1 + U_2 + \ldots + U_r$ in basic matrices and taking $U = U_l$ for suitable l we get a basic matrix U satisfying (4).

REMARK 2: Methods given in this paper may be joined with methods given in [1] for an ordinary transportation problem so that we get methods for solving a transportation problem with given bounds. These methods will be treated in another paper.

O JEDNOM PROBLÉMU SOUVISEJÍCÍM S DOPRAVNÍM PROBLÉMEM

Souhrn

V práci se řeší tento problém: Pro dvě nezáporné matice A, B s racionálními prvky je třeba rozhodnout, zdali existuje matice $O \leq X \leq B$, která by se s maticí A shodovala v řádkových a sloupcových součtech.

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